Residual Based A Posteriori Error Estimates for Convex Optimal Control Problems Governed by Stokes-Darcy Equations

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Abstract. In this paper, we derive a posteriori error estimates for finite element approximations of the optimal control problems governed by the Stokes-Darcy system. We obtain a posteriori error estimators for both the state and the control based on the residual of the finite element approximation. It is proved that the a posteriori error estimate provided in this paper is both reliable and efficient.

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1. Introduction

Flow control problems have received significant attention because of their many engineering applications. Extensive research has been carried out on various theoretical aspects of flow control problems, see, for example, [1–3] and the references therein, for existence results of optimal control, optimality conditions, regularity of optimal solutions and the existence of Lagrange multipliers.

It is obvious that efficient numerical methods are essential to successful applications of control problems. Nowadays, the finite element method is undoubtedly the most widely used numerical method in computing optimal control problems. There exists much literature on the finite element approximation for PDEs and various optimal control problems, see, for example, [4–11].

A posteriori error estimates are computable quantities in terms of the discrete solution or data, allow to measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms with adaptive mesh refinement which

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equi-distribute the computational effort and optimize the approximation efficiency. Ever since the pioneering work of Babuška [12–14], the adaptive finite element method based on a posteriori error estimates has been extensively investigated. The literature in this area is huge. The work related on the a posteriori error estimates for partial differential equations is summarized in [15, 16]. In [17] two a posteriori error estimators for the mini-element discretization of the Stokes equations were presented.

Recently, the residual-based a posteriori error estimators for Stokes-Darcy coupled problems were presented in [18] and [19]. In [18], Bernardi-Raugel and Raviart-Thomas elements for the velocity and piecewise constants for the pressures were considered. In [19], we consider a posteriori error estimate for the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition, where the approximation spaces are the Hood-Taylor element and the piecewise quadratic element for the Stokes and the Darcy regions, respectively.

Concerning the finite element approximation of the distributed optimal control problem governed by partial differential equations, residual based a posteriori error estimates are investigated by [20–23]. Especially, the optimal control problem governed by the Stokes equations was discussed in [3] and [24]. Moreover, we should also mention the dual weighted residual estimates pioneered by R. Becker and R. Rannacher (see, e.g., [25]). Recently, more work on this kind of problems can be found in [26, 28]. To our knowledge, there are still no theoretical results on a posteriori error estimates for the optimal control problem governed by the Stokes-Darcy system.

In this paper, we extend the result of [19] to the optimal control problems, we develop the a posteriori error analysis for the optimal control problem governed by the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition. The approximation spaces for the state equations are the Hood-Taylor element and the piecewise quadratic element for the Stokes and the Darcy regions, respectively, while the control is approximated by piecewise constants space. We obtain the residual based a posteriori error estimators for both state and the control based on the residual of the finite element approximation. It is proved that the a posteriori error estimate provided in this paper is both reliable and efficient. Some techniques used in this paper can be found in, e.g., [3,9,15,16] and [19].

The rest of the paper is organized as follows: in Section 2, we shall construct a weak formulation and finite element approximation for the distributed optimal control problem governed by the Stokes-Darcy system. In Section 3, a posteriori error estimates in H^1-norm are derived for optimal control problems governed by the Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface boundary condition. Reliability and efficiency are obtained in Subsections 3.1 and 3.2, respectively.

2. Finite element approximation of the control problems

Let Ω be an open bounded set in R^2 with piecewise Lipschitz boundary $\partial\Omega$. In this paper, we adopt the notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the standard norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$. For the case of $q = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ and

$\| \cdot \|_{m,\Omega} = \| \cdot \|_{m,2,\Omega}$. Furthermore, we set $W_0^{1,q}(\Omega) = \{v \in W^{1,q}(\Omega) : \gamma v \mid_{\partial\Omega} = 0\}$, where γv is the trace of v on the boundary $\partial\Omega$. In addition c and C denote generic constants.

Firstly, let us consider Stokes-Darcy equation:

$$-\nabla \cdot \mathbb{T}(\mathbf{y}_f, p_f) = \mathbf{f} + B\mathbf{u} \quad \text{in} \ \Omega_f, \tag{2.1}$$

$$\nabla \cdot \mathbf{y}_f = 0 \quad \text{in} \ \Omega_f, \tag{2.2}$$

$$\mathbf{y}_p = -K\nabla\phi_p \quad \text{in} \ \Omega_p, \tag{2.3}$$

$$\nabla \cdot \mathbf{y}_p = 0 \quad \text{in} \ \Omega_p, \tag{2.4}$$

where $\bar{\Omega} = \bar{\Omega}_p \bigcup \bar{\Omega}_f$, Ω_p is the porous media region, Ω_f is the fluid region, $\mathbf{y}_f$ is the fluid velocity, p_f is the kinematic pressure, $\mathbf{f}$ is the external body force, K is a symmetric and uniformly positive definite tensor in Ω_p representing the permeability of the porous media divided by the viscosity, $\mathbb{T}(\mathbf{y}_f, p_f) = 2\nu\mathbb{D}(\mathbf{y}_f) - p_f\mathbb{I}$ is the stress tensor, ν is the kinematic viscosity of the fluid, $\mathbb{D}(\mathbf{y}_f) = (\nabla\mathbf{y}_f + \nabla^T\mathbf{y}_f)/2$ is the deformation tensor, $\mathbf{y}_p$ is the fluid velocity in Ω_p and ϕ_p is the hydraulic head in Ω_p. Along the interface $\Gamma = \bar{\Omega}_p \cap \bar{\Omega}_f$, the Beavers-Joseph-Saffman-Jones interface boundary condition is imposed (see, e.g., [29, 30]):

$$\mathbf{y}_f \cdot \mathbf{n}_f = -\mathbf{y}_p \cdot \mathbf{n}_p, \tag{2.5}$$

$$-\mathbf{t}_f \cdot (\mathbb{T}(\mathbf{y}_f, p_f) \cdot \mathbf{n}_f) = \alpha\mathbf{t}_f \cdot \mathbf{y}_f, \tag{2.6}$$

$$-\mathbf{n}_f \cdot (\mathbb{T}(\mathbf{y}_f, p_f) \cdot \mathbf{n}_f) = g\phi_p, \tag{2.7}$$

where $\mathbf{t}_f$ is the unit tangent vector to Γ, $\mathbf{n}_f$ and $\mathbf{n}_p$ are unit outward normal to Γ from Ω_f and Ω_p, respectively, and g, α are positive constants. On the exterior boundary,

$$\mathbf{y}_f = 0, \qquad \text{on} \quad \partial\Omega \cap \partial\Omega_f, \tag{2.8}$$

$$\mathbf{y}_p \cdot \mathbf{n}_p = 0, \qquad \text{on} \quad \partial\Omega \cap \partial\Omega_p. \tag{2.9}$$

Introduce the spaces:

$$\mathbf{X}_f = \left\{ \mathbf{v}_f \in [H^1(\Omega_f)]^2 : \mathbf{v}_f = 0 \ \text{on} \ \partial\Omega_f \setminus \Gamma \right\},$$
$$Q_f = L^2(\Omega_f),$$
$$X_p = H^1(\Omega_p).$$

For simplicity, in the following part of this paper, in Ω_f, we replace $\mathbf{y}_f$, p_f by $\mathbf{y}$, p respectively and in Ω_p we replace ϕ_p by ϕ. Set

$$\tilde{A}\Big((\mathbf{y}, p, \phi); (\mathbf{v}, q, \psi)\Big) \triangleq a_f(\mathbf{y}, \mathbf{v}) + b_f(\mathbf{v}, p) - b_f(\mathbf{y}, q) + ga_p(\phi, \psi) + g\langle\phi, \mathbf{v} \cdot \mathbf{n}_f\rangle$$
$$- g\langle\mathbf{y} \cdot \mathbf{n}_f, \psi\rangle + \alpha\langle P_\tau\mathbf{y}, P_\tau\mathbf{v}\rangle,$$
$$(\mathbf{y}, p, \phi), (\mathbf{v}, q, \psi) \in \mathbf{X}_f \times Q_f \times X_p, \tag{2.10}$$

where the bilinear forms are defined as

$$a_p(\phi, \psi) = (K\nabla\phi, \nabla\psi)_{\Omega_p},$$
$$a_f(\mathbf{y}, \mathbf{v}) = 2\nu(\mathbb{D}(\mathbf{y}), \mathbb{D}(\mathbf{v}))_{\Omega_f},$$
$$b_f(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)_{\Omega_f},$$

where $(\cdot, \cdot)_{\Omega_f}$ and $(\cdot, \cdot)_{\Omega_p}$ denote the L^2-inner products on the domains Ω_f and Ω_p, respectively, $\langle\cdot, \cdot\rangle$ the L^2-inner products on the interface boundary Γ, P_τ denotes the projection onto the tangent space on Γ, i.e., $P_\tau\mathbf{y} = (\mathbf{y}_f \cdot \mathbf{t}_f)\mathbf{t}_f$. Then, the standard weak formulation for Stokes-Darcy equation (2.1)-(2.9) can be rewritten to: find $(\mathbf{y}, p, \phi) \in \mathbf{X}_f \times Q_f \times X_p$ such that

$$\tilde{A}\Big((\mathbf{y}, p, \phi); (\mathbf{v}, q, \psi)\Big) = (\mathbf{f} + B\mathbf{u}, \mathbf{v})_{\Omega_f}, \qquad \forall(\mathbf{v}, q, \psi) \in \mathbf{X}_f \times Q_f \times X_p. \tag{2.11}$$

For the above problem, it is well known (see [32]) that the following $Babu\check{s}ka - Brezzi$ condition holds:

$$\inf_{q\in Q_f} \sup_{\mathbf{v}\in\mathbf{X}_f} \frac{|b_f(\mathbf{v}, q)|}{\|\mathbf{v}\|_{1,\Omega_f}\|q\|_{0,\Omega_f}} \geq C.$$

Moreover, it can be proved that the above $Babu\check{s}ka - Brezzi$ condition is equivalent to the following inf-sup condition:

Lemma 2.1. [19] *Let* $\mathscr{B} = \mathbf{X}_f \times Q_f \times X_p$, *the bilinear form* $\tilde{A}$ *defined by* (2.10) *from* $\mathscr{B} \times \mathscr{B}$ *satisfies*

$$\inf_{(\mathbf{y},p,\phi)\in\mathscr{B}} \sup_{(\mathbf{v},q,\psi)\in\mathscr{B}} \frac{\tilde{A}\Big((\mathbf{y}, p, \phi); (\mathbf{v}, q, \psi)\Big)}{\Big(\|\mathbf{y}\|_{1,\Omega_f} + \|p\|_{0,\Omega_f} + \|\phi\|_{1,\Omega_p}\Big)\Big(\|\mathbf{v}\|_{1,\Omega_f} + \|q\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p}\Big)} \geq C,$$

where C is a constant independent of $\mathbf{y}$, $\mathbf{v}$, p, q, ϕ, ψ. *Similarly,*

$$\inf_{(\mathbf{y},p,\phi)\in\mathscr{B}} \sup_{(\mathbf{v},q,\psi)\in\mathscr{B}} \frac{\tilde{A}\Big((\mathbf{v}, q, \psi); (\mathbf{y}, p, \phi)\Big)}{\Big(\|\mathbf{y}\|_{1,\Omega_f} + \|p\|_{0,\Omega_f} + \|\phi\|_{1,\Omega_p}\Big)\Big(\|\mathbf{v}\|_{1,\Omega_f} + \|q\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p}\Big)} \geq C.$$

Let Ω_U be a bounded open sets in R^2 with Lipschitz boundary $\partial\Omega_U$, where Ω_U can be a subdomain of Ω_f, and $\Omega_U = \Omega_f$ in some special cases. Let the state space be $\mathbf{X}_f \times Q_f \times X_p$, and the control space be $\mathbf{U} = [L^2(\Omega_U)]^2$. Let B be the linear continuous operator from $\mathbf{U}$ to $\mathbf{H} = [L^2(\Omega_f)]^2$, and $\mathbf{K}$ be a closed convex subset of $\mathbf{U}$. In this paper, we set

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{U} : \mathbf{v} \geqslant 0\},$$

where $\mathbf{v} \geqslant 0$ means $v_i \geqslant 0$, $i = 1, 2$, $\mathbf{v} = (v_1, v_2)$.

Using the weak formulation (2.11), the optimal control problem governed by Stokes-Darcy equation (2.1)-(2.9) can be stated as:

$$\min_{\mathbf{u}\in\mathbf{K}\subset U} \{g_1(\mathbf{y}) + g_2(\phi) + j(\mathbf{u})\} \tag{2.12}$$

subject to

$$\tilde{A}\Big((\mathbf{y},p,\phi);(\mathbf{v},q,\psi)\Big) = (\mathbf{f}+B\mathbf{u},\mathbf{v})_{\Omega_f}, \qquad \forall(\mathbf{v},q,\psi)\in\mathbf{X}_f\times Q_f\times X_p. \tag{2.13}$$

where g_1, g_2 and j are differential convex functionals, g_1 and g_2 are bounded below, $j(\mathbf{w}) \to +\infty$ as $\|\mathbf{w}\|_U \to \infty$. In this paper, we further assume that the strictly convex condition:

$$\Big(j'(\mathbf{w}),\mathbf{w}-\mathbf{v}\Big)_U - \Big(j'(\mathbf{v}),\mathbf{w}-\mathbf{v}\Big)_U \geq c\|\mathbf{w}-\mathbf{v}\|^2_{0,\Omega_U} \tag{2.14}$$

is valid for the functional $j(\cdot)$, and $j'(\cdot)$, $g_1'(\cdot)$ and $g_2'(\cdot)$ are locally Lipschitz continuous.

By means of the standard technique (see, e.g., [2]), it can be proved that the control problem (2.12)-(2.13) has a unique solution $(\mathbf{y},p,\phi,\mathbf{u})$, and that $(\mathbf{y},p,\phi,\mathbf{u})$ is the solution of (2.12)-(2.13) if and only if there is a co-state $(\mathbf{z},s,\xi)\in\mathbf{X}_f\times Q_f\times X_p$ such that $(\mathbf{y},p,\phi,\mathbf{z},s,\xi,\mathbf{u})$ satisfies the following optimality conditions:

$$\tilde{A}((\mathbf{y},p,\phi);(\mathbf{v},q,\psi)) = (\mathbf{f}+B\mathbf{u},\mathbf{v})_{\Omega_f}, \qquad\qquad \forall(\mathbf{v},q,\psi)\in\mathbf{X}_f\times Q_f\times X_p, \tag{2.15}$$

$$\tilde{A}\Big((\mathbf{w},t,\eta);(\mathbf{z},s,\xi)\Big) = \Big(g_1'(\mathbf{y}),\mathbf{w}\Big)_{\Omega_f} + \Big(g_2'(\phi),\eta\Big)_{\Omega_p}, \quad \forall(\mathbf{w},t,\eta)\in\mathbf{X}_f\times Q_f\times X_p, \tag{2.16}$$

$$\Big(j'(\mathbf{u})+B^*\mathbf{z},\mathbf{v}-\mathbf{u}\Big)_U \geq 0, \qquad\qquad \forall\mathbf{v}\in\mathbf{K}\subset\mathbf{U}, \tag{2.17}$$

where B^* is the adjoint operator of B, $(\cdot,\cdot)_U$ is the inner product of $\mathbf{U}$ and $\tilde{A}((\cdot,\cdot,\cdot);(\cdot,\cdot,\cdot))$ is defined by (2.10).

Next, we consider the finite element approximation of the control problem (2.12)-(2.13). For simplicity, we will assume that Ω_f, Ω_p and Ω_U are all polygons in the following. The results can be extended to the general domains with smooth boundaries. Let $\mathcal{T}_{f,h}$ and $\mathcal{T}_{p,h}$ be regular meshes on Ω_f and Ω_p, respectively.

We assume that the meshes $\mathcal{T}_{f,h}$ and $\mathcal{T}_{p,h}$ are compatible such that their nodes on Γ are the same. Let $\mathbf{X}_f^h\times Q_f^h$ be the Hood-Taylor element on $\mathcal{T}_{f,h}$ and X_p^h be conforming piecewise quadratic element space on $\mathcal{T}_{p,h}$, such that

$$V_f^h = \Big\{v\in C(\Omega_f):v|_\tau\in P_2(\tau),\ \forall\tau\in\mathcal{T}_{f,h},\ v=0\ \text{on}\ \partial\Omega_f\backslash\Gamma\Big\},$$
$$\mathbf{X}_f^h = (V_f^h)^2,$$
$$Q_f^h = \Big\{q\in C(\Omega_f):q|_\tau\in P_1(\tau),\ \forall\tau\in\mathcal{T}_{f,h}\Big\},$$
$$X_p^h = \Big\{v\in C(\Omega_p):v|_\tau\in P_2(\tau),\ \forall\tau\in\mathcal{T}_{p,h}\Big\},$$

where P_i denotes the space of polynomials of order i.

Then the discrete weak formulation of the state equations (2.11) reads: find $(\mathbf{y}_h,p_h,\phi_h)\in\mathbf{X}_f^h\times Q_f^h\times X_p^h$, such that

$$\tilde{A}\Big((\mathbf{y}_h,p_h,\phi_h);(\mathbf{v}_h,q_h,\psi_h)\Big) = (\mathbf{f}+B\mathbf{u}_h,\mathbf{v}_h)_{\Omega_f}, \quad \forall(\mathbf{v}_h,q_h,\psi_h)\in\mathbf{X}_f^h\times Q_f^h\times X_p^h.$$

Let $\mathcal{T}_U^h$ be a regular partition of Ω_U. Associated with $\mathcal{T}_U^h$ is a finite dimensional subspace W_U^h of $L^2(\Omega_U)$ such that $\chi|_{\tau_U}$ is a constant for all $\chi \in W_U^h$ and $\tau_U \in \mathcal{T}_U^h$. Let $\mathbf{U}^h = (W_U^h)^2$, $\mathbf{K}^h = \mathbf{U}^h \cap \mathbf{K}$. It is easy to see that $\mathbf{U}^h \subset \mathbf{U}$ and $\mathbf{K}^h \subset \mathbf{K}$.

Then the discretized weak formulation of the state equations (2.12)-(2.13) reads:

$$\min_{\mathbf{u}_h \in \mathbf{K}^h \subset \mathbf{U}^h} \left\{ g_1(\mathbf{y}_h) + g_2(\phi_h) + j(\mathbf{u}_h) \right\} \tag{2.18}$$

subject to

$$\tilde{A}\Big((\mathbf{y}_h, p_h, \phi_h); (\mathbf{v}_h, q_h, \psi_h)\Big) = (\mathbf{f} + B\mathbf{u}_h, \mathbf{v}_h)_{\Omega_f}, \quad \forall (\mathbf{v}_h, q_h, \psi_h) \in \mathbf{X}_f^h \times Q_f^h \times X_p^h. \tag{2.19}$$

Similarly, it can be shown that the control problem (2.18)-(2.19) has a unique solution $(\mathbf{y}_h, p_h, \phi_h, \mathbf{u}_h)$ and that $(\mathbf{y}_h, p_h, \phi_h, \mathbf{u}_h) \in \mathbf{X}_f^h \times Q_f^h \times X_p^h \times \mathbf{K}^h$ is the solution of (2.18)-(2.19) if and only if there is a co-state $(\mathbf{z}_h, s_h, \xi_h) \in \mathbf{X}_f^h \times Q_f^h \times X_p^h$ such that $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ satisfying the following optimal conditions:

$$\tilde{A}\Big((\mathbf{y}_h, p_h, \phi_h); (\mathbf{v}_h, q_h, \psi_h)\Big) = (\mathbf{f} + B\mathbf{u}_h, \mathbf{v}_h)_{\Omega_f}, \quad \forall (\mathbf{v}_h, q_h, \psi_h) \in \mathbf{X}_f^h \times Q_f^h \times X_p^h, \tag{2.20}$$

$$\tilde{A}\big((\mathbf{w}_h, t_h, \eta_h); (\mathbf{z}_h, s_h, \xi_h)\big) = \Big(g_1'(\mathbf{y}_h), \mathbf{w}_h\Big)_{\Omega_f} + \Big(g_2'(\phi_h), \eta_h\Big)_{\Omega_p},$$
$$\forall (\mathbf{w}_h, t_h, \eta_h) \in \mathbf{X}_f^h \times Q_f^h \times X_p^h, \tag{2.21}$$

$$\Big(j'(\mathbf{u}_h) + B^* \mathbf{z}_h, \mathbf{v}_h - \mathbf{u}_h\Big)_U \geqslant 0, \qquad \forall \mathbf{v}_h \in \mathbf{K}^h \subset \mathbf{U}. \tag{2.22}$$

3. A posteriori error estimates for the Stokes-Darcy control

In order to obtain a numerical solution of the control problem with acceptable accuracy, the finite element meshes should be refined or adjusted according to a mesh refinement scheme. A widely used approach in engineering is adaptive finite element approximation. The a posteriori error estimates is essential in designing algorithms with adaptive mesh refinement which equi-distribute the computational effort and optimize the approximation efficiency.

In this section, we will derive a posteriori error estimators for the optimal control of Stokes-Darcy flows. The upper bound is derived in Subsection 3.1, while the lower bound is obtained in Subsection 3.2.

3.1. The upper bound

In this subsection, the following well known error estimate for average interpolation will be useful.

Lemma 3.1 ([33]). *Let* $\mathbf{I} : W^{1,q}(\Omega_f) \to V_f^h$ *(or* $W^{1,q}(\Omega_p) \to X_p^h$*),* $1 \leq q \leq \infty$*, be the average interpolation operator defined in [33]. For* $m = 0$ *or* 1*, and* $v \in W^{1,q}(\Omega)$*,*

$$\|v - \mathbf{I}v\|_{m,q,\tau} \leq \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} Ch_\tau^{1-m} |v|_{1,q,\tau'},$$

where h_τ is the diameter of the element τ.

Moreover, using the well known trace Theorem and the standard scaling technique (see, e.g., [7]), the following lemma can be deduced.

Lemma 3.2. *For $v \in W^{1,q}(\Omega)$, $1 \le q \le \infty$,*

$$\|v\|_{0,q,\partial\tau} \le C\left(h_\tau^{-\frac{1}{q}} \|v\|_{0,q,\tau} + h_\tau^{1-\frac{1}{q}} |v|_{1,q,\tau} \right).$$

In order to obtain sharp a posteriori error estimators, we will divide Ω_U into three subsets:

$$\Omega^- = \{x \in \Omega_U : (B^*z_h)(x) + j'(0) \le 0\},$$
$$\Omega^0 = \{x \in \Omega_U : (B^*z_h)(x) + j'(0) > 0, \mathbf{u}_h(x) = 0\},$$
$$\Omega^+ = \{x \in \Omega_U : (B^*z_h)(x) + j'(0) > 0, \mathbf{u}_h(x) > 0\}.$$

Next, we will derive an upper error bound for the finite element approximation of the control.

Lemma 3.3. *Let $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ and $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ be the solutions of (2.15)-(2.17) and (2.20)-(2.22), respectively. Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \le C\eta_1^2 + C\|\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)\|_{0,\Omega_f}^2, \tag{3.1}$$

where

$$\eta_1^2 = \int_{\Omega^- \cup \Omega^+} \left| j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right|^2,$$

and $\mathbf{z}(\mathbf{u}_h)$ is the solution of the equations:

$$\tilde{A}\Big((\mathbf{y}(\mathbf{u}_h), p(\mathbf{u}_h), \phi(\mathbf{u}_h)); (\mathbf{v}, q, \psi) \Big) = (\mathbf{f} + B\mathbf{u}_h, \mathbf{v})_{\Omega_f},$$
$$\forall (\mathbf{v}, q, \psi) \in \mathbf{X}_f \times Q_f \times X_p, \tag{3.2}$$
$$\tilde{A}\Big((\mathbf{w}, t, \eta); (\mathbf{z}(\mathbf{u}_h), s(\mathbf{u}_h), \xi(\mathbf{u}_h)) \Big) = \Big(g_1'(\mathbf{y}(\mathbf{u}_h)), \mathbf{w} \Big)_{\Omega_f} + \Big(g_2'(\phi(\mathbf{u}_h)), \eta \Big)_{\Omega_p},$$
$$\forall (\mathbf{w}, t, \eta) \in \mathbf{X}_f \times Q_f \times X_p. \tag{3.3}$$

Moreover, we have that

$$e^2 \le C\eta_1^2 + C\left(\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \right), \tag{3.4}$$

where

$$e^2 = \int_{\Omega^*} \left((j'(\mathbf{u}) + B^*\mathbf{z}) - \mathscr{P}_h(j'(\mathbf{u}) + B^*\mathbf{z}) \right)^2,$$

$\mathscr{P}_h$ is the L^2−projection operator from $(L^2(\Omega_U))^2$ to $\mathbf{U}^h$, and

$$\Omega^* = \{x \in \Omega^+ : \mathbf{u}(x) = 0, \mathbf{u}_h(x) > 0\}.$$

Proof. Note that $\mathbf{u}_h \in \mathbf{K}$. It follows from (2.14) and (2.17) that

$$C\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \leq \left(j'(\mathbf{u}), \mathbf{u} - \mathbf{u}_h\right)_U - \left(j'(\mathbf{u}_h), \mathbf{u} - \mathbf{u}_h\right)_U$$

$$\leq -\left(B^*\mathbf{z}, \mathbf{u} - \mathbf{u}_h\right)_U - \left(j'(\mathbf{u}_h), \mathbf{u} - \mathbf{u}_h\right)_U$$

$$= \left(B^*\mathbf{z}_h - B^*\mathbf{z}(\mathbf{u}_h), \mathbf{u} - \mathbf{u}_h\right)_U + \left(B^*\mathbf{z}(\mathbf{u}_h) - B^*\mathbf{z}, \mathbf{u} - \mathbf{u}_h\right)_U$$

$$+ \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h, \mathbf{u}_h - \mathbf{u}\right)_U. \tag{3.5}$$

It is easy to see that

$$\left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h, \mathbf{u}_h - \mathbf{u}\right)_U$$

$$= \int_{\Omega^- \cup \Omega^+} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h\right)(\mathbf{u}_h - \mathbf{u}) + \int_{\Omega^0} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h\right)(\mathbf{u}_h - \mathbf{u}), \tag{3.6}$$

and

$$\int_{\Omega^- \cup \Omega^+} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h\right)(\mathbf{u}_h - \mathbf{u})$$

$$\leq C(\delta) \int_{\Omega^- \cup \Omega^+} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h\right)^2 + C\delta\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_U}^2$$

$$= C(\delta)\eta_1^2 + C\delta\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_U}^2, \tag{3.7}$$

where δ is a small positive constant, $C(\delta)$ is a constant dependent on δ. Note that $j'(\mathbf{u}_h) + B^*\mathbf{z}_h \geq j'(0) + B^*\mathbf{z}_h > 0$ and $\mathbf{u}_h - \mathbf{u} = 0 - \mathbf{u} \leq 0$ on the domain Ω_0. Therefore,

$$\int_{\Omega^0} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h\right)(\mathbf{u}_h - \mathbf{u}) \leq 0. \tag{3.8}$$

Then it follows from (3.6)-(3.8) that

$$\left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h, \mathbf{u}_h - \mathbf{u}\right)_U \leq C(\delta)\eta_1^2 + C\delta\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_U}^2. \tag{3.9}$$

Moreover, the Schwartz's inequality implies that

$$\left(B^*\mathbf{z}_h - B^*\mathbf{z}(\mathbf{u}_h), \mathbf{u} - \mathbf{u}_h\right)_U \leq C(\delta)\|B^*(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h))\|_{0,\Omega_U}^2 + C\delta\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_U}^2$$

$$\leq C(\delta)\|\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)\|_{0,\Omega_f}^2 + C\delta\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_U}^2. \tag{3.10}$$

Furthermore, it follows from (2.16)-(2.17) and (3.2)-(3.3) that

$$\left(B^*\mathbf{z}(\mathbf{u}_h) - B^*\mathbf{z}, \mathbf{u} - \mathbf{u}_h\right)_U$$

$$= \left(g_1'(\mathbf{y}(\mathbf{u}_h)) - g_1'(\mathbf{y}), \mathbf{y} - \mathbf{y}(\mathbf{u}_h)\right) + \left(g_2'(\phi(\mathbf{u}_h)) - g_2'(\phi), \phi - \phi(\mathbf{u}_h)\right) \leq 0, \tag{3.11}$$

where we have used the fact that g_1 and g_2 are all convex. Summing up, (3.1) follows from (3.5) and (3.9)-(3.11) where we set δ to be small enough.

Next, we consider the estimate for e. Note that $\mathbf{u}_h > 0$ on Ω^*. Then (2.22) implies that $\mathscr{P}_h(j'(\mathbf{u}_h) + B^*\mathbf{z}_h) = 0$ on Ω^*. Therefore,

$$
\begin{aligned}
e^2 &= \int_{\Omega^*} \left(j'(\mathbf{u}) + B^*\mathbf{z} \right) - \mathscr{P}_h\left(j'(\mathbf{u}) + B^*\mathbf{z} \right)^2 \\
&\leq C \int_{\Omega^*} \left((j'(\mathbf{u}) + B^*\mathbf{z}) - (j'(\mathbf{u}_h) + B^*\mathbf{z}_h) \right)^2 + C \int_{\Omega^*} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right)^2 \\
&\quad + C \int_{\Omega^*} \left(\mathscr{P}_h(j'(\mathbf{u}_h) + B^*\mathbf{z}_h) \right)^2 + C \int_{\Omega^*} \left(\mathscr{P}_h(j'(\mathbf{u}_h) + B^*\mathbf{z}_h) - \mathscr{P}_h(j'(\mathbf{u}) + B^*\mathbf{z}) \right)^2 \\
&\leq C \left(\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \right) + C \int_{\Omega^*} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right)^2 \\
&\leq C \eta_1^2 + C \left(\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \right).
\end{aligned}
$$

This proves (3.4). $\square$

Using the above lemmas, we can derive the following upper bound of error estimates for both control and state in the control problem governed by the Stokes-Darcy system. In this part, techniques used in the proof are standard as in [19]. In the following, we will use τ_f and τ_p to denote the elements in $\mathscr{T}_{f,h}$ and $\mathscr{T}_{p,h}$, l_f and l_p to denote the edges in $\mathscr{T}_{f,h}$ and $\mathscr{T}_{p,h}$, respectively. Moreover, l_b denotes the edge on $\partial\Omega_p \setminus \Gamma$, and l could be any edge on $\mathscr{T}_{f,h}$ or $\mathscr{T}_{p,h}$. For convenience, we first introduce the notations $[A_l]$ and $[D_l]$ for jumps on the edge $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ defined respectively by

$$
[D_l] = \left((2\nu\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I})|_{\tau_{l,f}^1} - (2\nu\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I})|_{\tau_{l,f}^2} \right) \cdot \mathbf{n}_f,
$$

$$
[A_l] = \left((2\nu\mathbb{D}(\mathbf{z}_h) + s_h\mathbb{I})|_{\tau_{l,f}^1} - (2\nu\mathbb{D}(\mathbf{z}_h) + s_h\mathbb{I})|_{\tau_{l,f}^2} \right) \cdot \mathbf{n}_f,
$$

while

$$
[K\nabla\phi_h] \cdot \mathbf{n}_p|_l = \left(K\nabla\phi_h|_{\tau_{l,p}^1} - K\nabla\phi_h|_{\tau_{l,p}^2} \right) \cdot \mathbf{n}_p,
$$

$$
[K^*\nabla\xi_h] \cdot \mathbf{n}_p|_l = \left(K^*\nabla\xi_h|_{\tau_{l,p}^1} - K^*\nabla\xi_h|_{\tau_{l,p}^2} \right) \cdot \mathbf{n}_p,
$$

where $\mathbf{n}_f$ and $\mathbf{n}_p$ are unit outward normal to $\tau_{l,f}^1$ and $\tau_{l,p}^1$ in $\mathscr{T}_{f,h}$ and $\mathscr{T}_{p,h}$, respectively. Moreover, we set the residuals:

$$
\begin{aligned}
\eta_2^2 = &\sum_{\tau_f \in \mathscr{T}_{f,h}} \eta_{2\tau_f}^2 + \sum_{\tau_p \in \mathscr{T}_{p,h}} \eta_{2\tau_p}^2 + \sum_{\tau_f \in \mathscr{T}_{f,h}} \eta_{2\mathrm{div}\tau_f}^2 + \sum_{l_f \cap \partial\Omega_f = \emptyset} \eta_{2l_f}^2 \\
&+ \sum_{l_p \cap \partial\Omega_p = \emptyset} \eta_{2l_p}^2 + \sum_{l_b \subset \partial\Omega_p \setminus \Gamma} \eta_{2l_b}^2 + \sum_{l \subset \Gamma} \tilde{\eta}_{2l}^2 + \sum_{l \subset \Gamma} \hat{\eta}_{2l}^2,
\end{aligned} \tag{3.12}
$$

where

$$\eta^2_{2\tau_f} = h^2_{\tau_f} \int_{\tau_f} \left(2v\nabla \cdot \mathbb{D}(\mathbf{y}_h) - \nabla p_h + \mathbf{f} + B\mathbf{u}_h\right)^2,$$

$$\eta^2_{2\tau_p} = h^2_{\tau_p} \int_{\tau_p} \left(g\nabla \cdot (K\nabla\phi_h)\right)^2,$$

$$\eta^2_{2\text{div}\tau_f} = \|\text{div}\mathbf{y}_h\|^2_{0,\tau_f}, \qquad \eta^2_{2_{lp}} = h_{lp} \int_{lp} (g[K\nabla\phi_h] \cdot \mathbf{n}_p)^2,$$

$$\eta^2_{2l_f} = h_{l_f} \int_{l_f} [D_l]^2, \qquad \eta^2_{2l_b} = h_{l_b} \int_{l_b} (gK\nabla\phi_h \cdot \mathbf{n}_p)^2,$$

$$\tilde{\eta}^2_{2l} = h_l \int_l \left((2v\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I}) \cdot \mathbf{n}_f + g\phi_h\mathbf{n}_f + \alpha P_\tau\mathbf{y}_h\right)^2,$$

$$\hat{\eta}^2_{2l} = h_l \int_l (gK\nabla\phi_h \cdot \mathbf{n}_p - g\mathbf{y}_h \cdot \mathbf{n}_f)^2,$$

and

$$\eta^2_3 = \sum_{\tau_f \in \mathscr{T}_{f,h}} \eta^2_{3\tau_f} + \sum_{\tau_p \in \mathscr{T}_{p,h}} \eta^2_{3\tau_p} + \sum_{\tau_f \in \mathscr{T}_{f,h}} \eta^2_{3\text{div}\tau_f} + \sum_{l_f \cap \partial\Omega_f = \emptyset} \eta^2_{3l_f}$$
$$+ \sum_{l_p \cap \partial\Omega_p = \emptyset} \eta^2_{3l_p} + \sum_{l_b \subset \partial\Omega_p \setminus \Gamma} \eta^2_{3l_b} + \sum_{l \subset \Gamma} \tilde{\eta}^2_{3l} + \sum_{l \subset \Gamma} \hat{\eta}^2_{3l}, \qquad (3.13)$$

where

$$\eta^2_{3\tau_f} = h^2_{\tau_f} \int_{\tau_f} \left(2v\nabla \cdot \mathbb{D}(\mathbf{z}_h) + \nabla s_h + g'_1(\mathbf{y}_h)\right)^2,$$

$$\eta^2_{3\tau_p} = h^2_{\tau_p} \int_{\tau_p} \left(g\nabla \cdot (K^*\nabla\xi_h) + g'_2(\phi_h)\right)^2,$$

$$\eta^2_{3\text{div}\tau_f} = \|\text{div}\mathbf{z}_h\|^2_{0,\tau_f}, \qquad \eta^2_{3_{lp}} = h_{lp} \int_{lp} (g[K^*\nabla\xi_h] \cdot \mathbf{n}_p)^2,$$

$$\eta^2_{3l_f} = h_{l_f} \int_{l_f} [A_l]^2, \qquad \eta^2_{3l_b} = h_{l_b} \int_{l_b} (gK^*\nabla\xi_h \cdot \mathbf{n}_p)^2,$$

$$\tilde{\eta}^2_{3l} = h_l \int_l \left((2v\mathbb{D}(\mathbf{z}_h) + s_h\mathbb{I}) \cdot \mathbf{n}_f - g\xi_h \cdot \mathbf{n}_f + \alpha P_\tau\mathbf{z}_h\right)^2,$$

$$\hat{\eta}^2_{3l} = h_l \int_l (gK^*\nabla\xi_h \cdot \mathbf{n}_p + g\mathbf{z}_h \cdot \mathbf{n}_f)^2,$$

where h_{τ_f} and h_{τ_p} are diameters of the elements $\tau_f \in \mathscr{T}_{f,h}$ and $\tau_p \in \mathscr{T}_{p,h}$, respectively, h_{l_f} and h_{l_p} are the sizes of the edges l_f in Ω_f and l_p in Ω_p, respectively, h_{l_b} is the size of the edge l_b on $\partial\Omega_p \setminus \Gamma$ and h_l is the size of the edge l.

Then we have the following a posteriori error estimate.

Theorem 3.1. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-
(2.17) *and* (2.20)-(2.22), *respectively. Assume that all conditions in the Lemma 3.3 are valid.
Moreover, assume that* $g_1'(\cdot)$ *and* $g_2'(\cdot)$ *are locally Lipschitz continuous. Then*

$$e^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2$$
$$+ \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 \leq C\eta^2, \qquad (3.14)$$

where

$$\eta^2 = \eta_1^2 + \eta_2^2 + \eta_3^2,$$

e and η_1 *are defined in Lemma 3.3, while* η_2 *and* η_3 *are defined by* (3.12) *and* (3.13),
respectively.

Proof. Considering Lemma 3.3, we need to estimate $\|\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)\|_{1,\Omega_f}^2$ in order to obtain
the a posteriori error estimate for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2$.

We use the standard technique for residual based a posteriori error estimates in the
following proof, similar details can be found in [19]. Let $E_{\mathbf{z}} = \mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)$, $E_s = s_h - s(\mathbf{u}_h)$,
$E_\xi = \xi_h - \xi(\mathbf{u}_h)$. It follows from Lemma 2.1 that there exists $(\mathbf{w}, t, \eta) \in \mathbf{X}_f \times Q_f \times X_p$ such
that

$$C\left(\|E_{\mathbf{z}}\|_{1,\Omega_f} + \|E_s\|_{0,\Omega_f} + \|E_\xi\|_{1,\Omega_p}\right)\left(\|\mathbf{w}\|_{1,\Omega_f} + \|t\|_{0,\Omega_f} + \|\eta\|_{1,\Omega_p}\right)$$
$$\leq \tilde{A}\left((\mathbf{w}, t, \eta); (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h), s_h - s(\mathbf{u}_h), \xi_h - \xi(\mathbf{u}_h))\right). \qquad (3.15)$$

Let $\mathbf{w}_I = \mathbf{I}\mathbf{w} \in \mathbf{X}_f^h$, $t_I = \mathbf{I}t \in Q_f^h$, $\eta_I = \mathbf{I}\eta \in X_p^h$ be the interpolations of $\mathbf{w}$, t and η defined
in Lemma 3.1. Then it follows from (2.20)-(2.22), (3.2)-(3.3) and (3.15) that

$$C\left(\|E_{\mathbf{z}}\|_{1,\Omega_f} + \|E_s\|_{0,\Omega_f} + \|E_\xi\|_{1,\Omega_p}\right)\left(\|\mathbf{w}\|_{1,\Omega_f} + \|t\|_{0,\Omega_f} + \|\eta\|_{1,\Omega_p}\right)$$

$$\leq \tilde{A}\left((\mathbf{w}, t, \eta); (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h), s_h - s(\mathbf{u}_h), \xi_h - \xi(\mathbf{u}_h))\right)$$

$$= \tilde{A}\left((\mathbf{w} - \mathbf{w}_I, t - t_I, \eta - \eta_I); (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h), s_h - s(\mathbf{u}_h), \xi_h - \xi(\mathbf{u}_h))\right)$$
$$\quad + \tilde{A}\left((\mathbf{w}_I, t_I, \eta_I); (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h), s_h - s(\mathbf{u}_h), \xi_h - \xi(\mathbf{u}_h))\right)$$
$$= 2\nu\left(\mathbb{D}(\mathbf{w} - \mathbf{w}_I), \mathbb{D}(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h))\right)_{\Omega_f} - \left(\text{div}(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)), t - t_I\right)_{\Omega_f}$$
$$\quad + \left(\text{div}(\mathbf{w} - \mathbf{w}_I), s_h - s(\mathbf{u}_h)\right)_{\Omega_f} + g\left(K\nabla(\eta - \eta_I), \nabla(\xi_h - \xi(\mathbf{u}_h))\right)_{\Omega_p}\nu$$
$$\quad - \left\langle g(\mathbf{w} - \mathbf{w}_I)\cdot\mathbf{n}_f, (\xi_h - \xi(\mathbf{u}_h))\right\rangle + \left\langle g(\eta - \eta_I), (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h))\cdot\mathbf{n}_f\right\rangle\nu$$
$$\quad + \alpha\left\langle P_\tau(\mathbf{w} - \mathbf{w}_I), P_\tau(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h))\right\rangle + 2\nu\left(\mathbb{D}(\mathbf{w}_I), \mathbb{D}(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h))\right)_{\Omega_f}$$

$$- \Big(\mathrm{div}(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)), t_I \Big)_{\Omega_f} + \Big(\mathrm{div}(\mathbf{w}_I), s_h - s(\mathbf{u}_h) \Big)_{\Omega_f} + g \Big(K\nabla\eta_I, \nabla(\xi_h - \xi(\mathbf{u}_h)) \Big)_{\Omega_p}$$

$$- \Big\langle g\mathbf{w}_I \cdot \mathbf{n}_f, (\xi_h - \xi(\mathbf{u}_h)) \Big\rangle + \Big\langle g\eta_I, (\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)) \cdot \mathbf{n}_f \Big\rangle + \alpha \Big\langle P_\tau \mathbf{w}_I, P_\tau(\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)) \Big\rangle$$

$$= - \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} 2\nu(\nabla \cdot \mathbb{D}(\mathbf{z}_h))(\mathbf{w} - \mathbf{w}_I) + \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\partial\tau_f} 2\nu \mathbb{D}(\mathbf{z}_h) \cdot \mathbf{n}_f (\mathbf{w} - \mathbf{w}_I)$$

$$- (\mathrm{div}\mathbf{z}_h, t - t_I)_{\Omega_f} - \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} \nabla s_h(\mathbf{w} - \mathbf{w}_I) + \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\partial\tau_f} s_h(\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{n}_f$$

$$- \sum_{\tau_p \in \mathscr{T}_{p,h}} \int_{\tau} g\nabla \cdot (K^*\nabla\xi_h)(\eta - \eta_I) + \sum_{\tau_p \in \mathscr{T}_{p,h}} \int_{\partial\tau_p} gK^*\nabla\xi_h \cdot \mathbf{n}_p(\eta - \eta_I)$$

$$- \Big\langle g(\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{n}_f, \xi_h \Big\rangle + \Big\langle g(\eta - \eta_I), \mathbf{z}_h \cdot \mathbf{n}_f \Big\rangle + \alpha \Big\langle P_\tau(\mathbf{w} - \mathbf{w}_I), P_\tau \mathbf{z}_h \Big\rangle$$

$$- \Big(g_1'(\mathbf{y}(\mathbf{u}_h)), \mathbf{w} - \mathbf{w}_I \Big)_{\Omega_f} - \Big(g_2'(\phi(\mathbf{u}_h)), \eta - \eta_I \Big)_{\Omega_p}$$

$$+ \Big(g_1'(\mathbf{y}_h) - g_1'(\mathbf{y}(\mathbf{u}_h)), \mathbf{w}_I \Big)_{\Omega_f} + \Big(g_2'(\phi_h) - g_2'(\phi(\mathbf{u}_h)), \eta_I \Big)_{\Omega_p}.$$

By rearranging the terms, we obtain

$$C \Big(\|E_\mathbf{z}\|_{1,\Omega_f} + \|E_s\|_{0,\Omega_f} + \|E_\xi\|_{1,\Omega_p} \Big) \Big(\|\mathbf{w}\|_{1,\Omega_f} + \|t\|_{0,\Omega_f} + \|\eta\|_{1,\Omega_p} \Big)$$

$$\leq \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} \Big(-2\nu\nabla \cdot \mathbb{D}(\mathbf{z}_h) - \nabla s_h - g_1'(\mathbf{y}_h) \Big)(\mathbf{w} - \mathbf{w}_I)$$

$$+ \sum_{\tau_p \in \mathscr{T}_{p,h}} \int_{\tau_p} \Big(-g\nabla \cdot (K^*\nabla\xi_h) - g_2'(\phi_h) \Big)(\eta - \eta_I) - (\mathrm{div}\mathbf{z}_h, t - t_I)_{\Omega_f}$$

$$+ \sum_{l_f \cap \partial\Omega_f = \emptyset} \int_{l_f} [A_l](\mathbf{w} - \mathbf{w}_I) + \sum_{l_p \cap \partial\Omega_p = \emptyset} \int_{l_p} g[K^*\nabla\xi_h] \cdot \mathbf{n}_p(\eta - \eta_I)$$

$$+ \sum_{l_b \subset \partial\Omega_p \backslash \Gamma} \int_{l_b} gK^*\nabla\xi_h \cdot \mathbf{n}_p(\eta - \eta_I)$$

$$+ \sum_{l \in \Gamma} \int_l \Big((2\nu\mathbb{D}(\mathbf{z}_h) + s_h\mathbb{I}) \cdot \mathbf{n}_f - g\xi_h \cdot \mathbf{n}_f + \alpha P_\tau \mathbf{z}_h \Big)(\mathbf{w} - \mathbf{w}_I)$$

$$+ \sum_{l \in \Gamma} \int_l (gK^*\nabla\xi_h \cdot \mathbf{n}_p + g\mathbf{z}_h \cdot \mathbf{n}_f)(\eta - \eta_I)$$

$$+ \Big(g_1'(\mathbf{y}_h) - g_1'(\mathbf{y}(\mathbf{u}_h)), \mathbf{w}_I \Big)_{\Omega_f} + \Big(g_2'(\phi_h) - g_2'(\phi(\mathbf{u}_h)), \eta_I \Big)_{\Omega_p}$$

$$+ \Big(g_1'(\mathbf{y}_h) - g_1'(\mathbf{y}(\mathbf{u}_h)), \mathbf{w} - \mathbf{w}_I \Big)_{\Omega_f} + \Big(g_2'(\phi_h) - g_2'(\phi(\mathbf{u}_h)), \eta - \eta_I \Big)_{\Omega_p}.$$

Using Lemma 3.1, we have that

$$
C\left(\|E_{\mathbf{z}}\|_{1,\Omega_f} + \|E_s\|_{0,\Omega_f} + \|E_\xi\|_{1,\Omega_p}\right)\left(\|\mathbf{w}\|_{1,\Omega_f} + \|t\|_{0,\Omega_f} + \|\eta\|_{1,\Omega_p}\right)
$$

$$
\leq C\Bigg(\sum_{\tau_f\in\mathscr{T}_{f,h}} h_{\tau_f}^2\int_{\tau_f}\left(2\nu\nabla\cdot\mathbb{D}(\mathbf{z}_h)+\nabla s_h+g_1'(\mathbf{y}_h)\right)^2
$$

$$
+\sum_{\tau_p\in\mathscr{T}_{p,h}} h_{\tau_p}^2\int_{\tau_p}(g\nabla\cdot(K^*\nabla\xi_h)+g_2'(\phi_h))^2 + \sum_{l_f\cap\partial\Omega_f=\emptyset} h_{l_f}\int_{l_f}[A_l]^2
$$

$$
+\sum_{l_p\cap\partial\Omega_p=\emptyset} h_{l_p}\int_{l_p}(g[K^*\nabla\xi_h]\cdot\mathbf{n}_p)^2 + \sum_{l_b\subset\partial\Omega_p\setminus\Gamma} h_{l_b}\int_{l_b}(gK^*\nabla\xi_h\cdot\mathbf{n}_p)^2
$$

$$
+\sum_{l\in\Gamma} h_l\int_l\left((2\nu\mathbb{D}(\mathbf{z}_h)+s_h\mathbb{I})\cdot\mathbf{n}_f - g\xi_h\cdot\mathbf{n}_f + \alpha P_\tau\mathbf{z}_h\right)^2
$$

$$
+\sum_{l\in\Gamma} h_l\int_l(gK^*\nabla\xi_h\cdot\mathbf{n}_p+g\mathbf{z}_h\cdot\mathbf{n}_f)^2 + \|\mathrm{div}\mathbf{z}_h\|_{0,\Omega_f}^2
$$

$$
+\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\phi_h-\phi(\mathbf{u}_h)\|_{0,\Omega_p}^2\Bigg)^{\frac12}\left(\|\mathbf{w}\|_{1,\Omega_f}+\|t\|_{0,\Omega_f}+\|\eta\|_{1,\Omega_p}\right).
$$

Therefore,

$$
\|\mathbf{z}_h-\mathbf{z}(\mathbf{u}_h)\|_{1,\Omega_f} + \|s_h-s(\mathbf{u}_h)\|_{0,\Omega_f} + \|\xi_h-\xi(\mathbf{u}_h)\|_{1,\Omega_p}
$$

$$
\leq C\Bigg(\sum_{\tau_f\in\mathscr{T}_{f,h}} h_{\tau_f}^2\int_{\tau_f}\left(2\nu\nabla\cdot\mathbb{D}(\mathbf{z}_h)+\nabla s_h+g_1'(\mathbf{y}_h)\right)^2
$$

$$
+\sum_{\tau_p\in\mathscr{T}_{p,h}} h_{\tau_p}^2\int_{\tau_p}\left(g\nabla\cdot(K^*\nabla\xi_h)+g_2'(\phi_h)\right)^2 + \sum_{l_f\cap\partial\Omega_f=\emptyset} h_{l_f}\int_{l_f}[A_l]^2
$$

$$
+\sum_{l_p\cap\partial\Omega_p=\emptyset} h_{l_p}\int_{l_p}(g[K^*\nabla\xi_h]\cdot\mathbf{n}_p)^2 + \sum_{l_b\subset\partial\Omega_p\setminus\Gamma} h_{l_b}\int_{l_b}(gK^*\nabla\xi_h\cdot\mathbf{n}_p)^2
$$

$$
+\sum_{l\in\Gamma} h_l\int_l\left((2\nu\mathbb{D}(\mathbf{z}_h)+s_h\mathbb{I})\cdot\mathbf{n}_f - g\xi_h\cdot\mathbf{n}_f + \alpha P_\tau\mathbf{z}_h\right)^2
$$

$$
+\sum_{l\in\Gamma} h_l\int_l(gK^*\nabla\xi_h\cdot\mathbf{n}_p+g\mathbf{z}_h\cdot\mathbf{n}_f)^2
$$

$$
+\|\mathrm{div}\mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\phi_h-\phi(\mathbf{u}_h)\|_{0,\Omega_p}^2\Bigg)^{\frac12}
$$

$$
= C\left(\eta_3^2 + \|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\phi_h-\phi(\mathbf{u}_h)\|_{0,\Omega_p}^2\right)^{\frac12}. \tag{3.16}
$$

Then in order to have the error estimate $\|\mathbf{z}_h-\mathbf{z}(\mathbf{u}_h)\|_{1,\Omega_f}^2$, we still need to estimate $\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{1,\Omega_f}$ and $\|\phi_h-\phi(\mathbf{u}_h)\|_{1,\Omega_p}$. Similarly as above, let $E_y = \mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)$, $E_p = p_h -$

$p(\mathbf{u}_h)$, $E_\phi = \phi_h - \phi(\mathbf{u}_h)$. Let $\mathbf{v}_I = \mathbf{I}\mathbf{v} \in \mathbf{X}_f^h$, $q_I = \mathbf{I}q \in Q_f^h$, $\psi_I = \mathbf{I}\psi \in X_p^h$ be the interpolations of $\mathbf{v} \in \mathbf{X}_f$, $q \in Q_f$ and $\psi \in X_f$ defined in Lemma 3.1, respectively. We have

$$C\Big(\|E_y\|_{1,\Omega_f} + \|E_p\|_{0,\Omega_f} + \|E_\phi\|_{1,\Omega_p}\Big)\Big(\|\mathbf{v}\|_{1,\Omega_f} + \|q\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p}\Big)$$

$$\leq \tilde{A}\Big((\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h), p_h - p(\mathbf{u}_h), \phi_h - \phi(\mathbf{u}_h)); (\mathbf{v}, q, \psi)\Big)$$

$$= \tilde{A}\Big((\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h), p_h - p(\mathbf{u}_h), \phi_h - \phi(\mathbf{u}_h)); (\mathbf{v} - \mathbf{v}_I, q - q_I, \psi - \psi_I)\Big)$$

$$= 2\nu\Big(\mathbb{D}(\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h)), \mathbb{D}(\mathbf{v} - \mathbf{v}_I)\Big)_{\Omega_f} - \Big(\text{div}(\mathbf{v} - \mathbf{v}_I), p_h - p(\mathbf{u}_h)\Big)_{\Omega_f} + (\text{div}\mathbf{y}_h, q - q_I)_{\Omega_f}$$

$$+ g\Big(K\nabla(\phi_h - \phi(\mathbf{u}_h)), \nabla(\psi - \psi_I)\Big)_{\Omega_p} - g\Big\langle(\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h)) \cdot \mathbf{n}_f, \psi - \psi_I\Big\rangle$$

$$+ g\Big\langle(\phi_h - \phi(\mathbf{u}_h)), (\mathbf{v} - \mathbf{v}_I) \cdot \mathbf{n}_f\Big\rangle + \alpha\Big\langle P_\tau(\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h)), P_\tau(\mathbf{v} - \mathbf{v}_I)\Big\rangle$$

$$= \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} \Big(-2\nu\nabla \cdot \mathbb{D}(\mathbf{y}_h) + \nabla p_h - \mathbf{f} - B\mathbf{u}_h\Big)(\mathbf{v} - \mathbf{v}_I) + (\text{div}\mathbf{y}_h, q - q_I)_{\Omega_f}$$

$$+ \sum_{\tau_p \in \mathscr{T}_{p,h}} \int_{\tau_p} -g\Big(\nabla \cdot (K\nabla\phi_h)\Big)(\psi - \psi_I) + \sum_{l_f \cap \partial\Omega_f = \emptyset} \int_{l_f} [D_l](\mathbf{v} - \mathbf{v}_I)$$

$$+ \sum_{l \in \Gamma} \int_l \Big(2\nu\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I}\Big) \cdot \mathbf{n}_f(\mathbf{v} - \mathbf{v}_I) + \sum_{l_p \cap \partial\Omega_p = \emptyset} \int_{l_p} g[K\nabla\phi_h] \cdot \mathbf{n}_p(\psi - \psi_I)$$

$$+ \sum_{l_b \subset \partial\Omega_p \backslash \Gamma} h_{l_b} \int_{l_b} gK\nabla\phi_h \cdot \mathbf{n}_p(\psi - \psi_I) + \sum_{l \in \Gamma} \int_l gK\nabla\phi_h \cdot \mathbf{n}_p(\psi - \psi_I)$$

$$- g\Big\langle\mathbf{y}_h \cdot \mathbf{n}_f, \psi - \psi_I\Big\rangle + g\Big\langle\phi_h, (\mathbf{v} - \mathbf{v}_I) \cdot \mathbf{n}_f\Big\rangle + \alpha\Big\langle P_\tau\mathbf{y}_h, P_\tau(\mathbf{v} - \mathbf{v}_I)\Big\rangle,$$

which gives

$$C\Big(\|E_y\|_{1,\Omega_f} + \|E_p\|_{0,\Omega_f} + \|E_\phi\|_{1,\Omega_p}\Big)\Big(\|\mathbf{v}\|_{1,\Omega_f} + \|q\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p}\Big)$$

$$\leq C\bigg(\sum_{\tau_f \in \mathscr{T}_{f,h}} h_{\tau_f}^2 \int_{\tau_f} \Big(2\nu\nabla \cdot \mathbb{D}(\mathbf{y}_h) - \nabla p_h + \mathbf{f} + B\mathbf{u}_h\Big)^2 + \sum_{\tau_p \in \mathscr{T}_{p,h}} h_{\tau_p}^2 \int_{\tau_p} \Big(g\nabla \cdot (K\nabla\phi_h)\Big)^2$$

$$+ \sum_{l_f \cap \partial\Omega_f = \emptyset} h_{l_f} \int_{l_f} [D_l]^2 + \sum_{l_p \cap \partial\Omega_p = \emptyset} h_{l_p} \int_{l_p} \Big(g[K\nabla\phi_h] \cdot \mathbf{n}_p\Big)^2$$

$$+ \sum_{l_b \subset \partial\Omega_p \backslash \Gamma} h_{l_b} \int_{l_b} (gK\nabla\phi_h \cdot \mathbf{n}_p)^2 + \|\text{div}\mathbf{y}_h\|_{0,\Omega_f}^2$$

$$+ \sum_{l \in \Gamma} h_l \int_l \Big((2\nu\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I}) \cdot \mathbf{n}_f + g\phi_h\mathbf{n}_f + \alpha P_\tau\mathbf{y}_h\Big)^2$$

$$+ \sum_{l \in \Gamma} h_l \int_l (gK\nabla\phi_h \cdot \mathbf{n}_p - g\mathbf{y}_h \cdot \mathbf{n}_f)^2 \bigg)^{\frac{1}{2}} \left(\|\mathbf{v}\|_{1,\Omega_f} + \|q\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p} \right). \qquad (3.17)$$

Therefore,

$$\|\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h)\|_{1,\Omega_f}^2 + \|p_h - p(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\phi_h - \phi(\mathbf{u}_h)\|_{1,\Omega_p}^2$$

$$\leq C \bigg(\sum_{\tau_f \in \mathscr{T}_{f,h}} h_{\tau_f}^2 \int_{\tau_f} \left(2\nu\nabla\cdot\mathbb{D}(\mathbf{y}_h) - \nabla p_h + \mathbf{f} + B\mathbf{u}_h \right)^2 + \sum_{\tau_p \in \mathscr{T}_{p,h}} h_{\tau_p}^2 \int_{\tau_p} \left(g\nabla\cdot(K\nabla\phi_h) \right)^2$$

$$+ \sum_{l_f \cap \partial\Omega_f = \emptyset} h_{l_f} \int_{l_f} [D_l]^2 + \sum_{l_p \cap \partial\Omega_p = \emptyset} h_{l_p} \int_{l_p} \left(g[K\nabla\phi_h]\cdot\mathbf{n}_p \right)^2$$

$$+ \sum_{l_b \subset \partial\Omega_p \backslash \Gamma} h_{l_b} \int_{l_b} (gK\nabla\phi_h \cdot \mathbf{n}_p)^2 + \|\text{div}\mathbf{y}_h\|_{0,\Omega_f}^2$$

$$+ \sum_{l \in \Gamma} h_l \int_l \left((2\nu\mathbb{D}(\mathbf{y}_h) - p_h\mathbb{I})\cdot\mathbf{n}_f + g\phi_h\mathbf{n}_f + \alpha P_\tau\mathbf{y}_h \right)^2$$

$$+ \sum_{l \in \Gamma} h_l \int_l (gK\nabla\phi_h \cdot \mathbf{n}_p - g\mathbf{y}_h \cdot \mathbf{n}_f)^2 \bigg) \leq C\eta_2^2. \qquad (3.18)$$

Summing up, it follows from (3.16) and (3.18) that

$$\|\mathbf{z}_h - \mathbf{z}(\mathbf{u}_h)\|_{1,\Omega_f}^2 \leq C(\eta_2^2 + \eta_3^2). \qquad (3.19)$$

Thus, Lemma 3.3 and (3.19) imply the a posteriori error estimate for the control:

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \leq C(\eta_1^2 + \eta_2^2 + \eta_3^2) = C\eta^2. \qquad (3.20)$$

Next, let us consider the a posteriori error estimate for state and costate. Note that

$$\|\mathbf{y} - \mathbf{y}(\mathbf{u}_h)\|_{1,\Omega_f}^2 + \|p - p(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\phi - \phi(\mathbf{u}_h)\|_{1,\Omega_p}^2 \leq C\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \qquad (3.21)$$

and

$$\|\mathbf{z} - \mathbf{z}(\mathbf{u}_h)\|_{1,\Omega_f}^2 + \|s - s(\mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\xi - \xi(\mathbf{u}_h)\|_{1,\Omega_p}^2$$

$$\leq C \left(\|\mathbf{y} - \mathbf{y}(\mathbf{u}_h)\|_{1,\Omega_f}^2 + \|\phi - \phi(\mathbf{u}_h)\|_{0,\Omega_p}^2 \right)$$

$$\leq C\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2. \qquad (3.22)$$

Then the inequalities (3.16)-(3.18) and (3.20)-(3.22) imply that

$$\|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2$$

$$+ \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|s - s_h\|_{0,\Omega_f}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 \leq C\eta^2. \qquad (3.23)$$

Moreover, it is easy to be deduced from (3.4), (3.20) and (3.23) that

$$e^2 \leq C \left(\eta_1^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 \right) \leq C\eta^2. \qquad (3.24)$$

Summing up, (3.14) follows from (3.20), (3.23) and (3.24). $\qquad\qquad\square$

3.2. The lower bound

We will prove the lower bounds for the error of the optimal control problem governed by the Stokes-Darcy system so that to show the a posteriori estimators provided in Theorem 3.1 are not only reliable but also efficient. In this subsection, we set the functional $j(\cdot) = \int_{\Omega_U} \tilde{j}(\cdot)$, where $\tilde{j}(\cdot)$ is a function. Hence $(j'(w), v) = \int_{\Omega_U} \tilde{j}'(w)v$. For simplicity, we still denote $\tilde{j}(\cdot)$ by $j(\cdot)$.

Theorem 3.2. *Let $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ and $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ be the solutions of (2.15)-(2.17) and (2.20)-(2.22), respectively. Assume that all the conditions in the above Theorem 3.1 hold. Then*

$$
\begin{aligned}
\eta^2 \leq C \Big(& \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 \\
& + \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 + e^2 \Big) + C(\epsilon_2^2 + \epsilon_3^2),
\end{aligned}
\tag{3.25}
$$

where e is defined in Lemma 3.3, η is defined in Theorem 3.1, and

$$
\begin{aligned}
\epsilon_2^2 = & \sum_{\tau_f \in \mathcal{T}_{f,h}} h_{\tau_f}^2 \int_{\tau_f} \left(\mathbf{f} + B\mathbf{u}_h - \overline{\mathbf{f} + B\mathbf{u}_h} \right)^2 \\
& + \sum_{\tau_p \in \mathcal{T}_{p,h}} h_{\tau_p}^2 \int_{\tau_p} \left(\overline{g \nabla \cdot (K \nabla \phi_h)} - g \nabla \cdot (K \nabla \phi_h) \right)^2 \\
& + \sum_{l_p \cap \partial \Omega_p = \emptyset} h_{l_p} \int_{l_p} \left(\overline{g [K \nabla \phi_h] \cdot \mathbf{n}_p} - g [K \nabla \phi_h] \cdot \mathbf{n}_p \right)^2 \\
& + \sum_{l_b \subset \partial \Omega_p \backslash \Gamma} h_{l_b} \int_{l_b} \left(\overline{g K \nabla \phi_h \cdot \mathbf{n}_p} - g K \nabla \phi_h \cdot \mathbf{n}_p \right)^2 \\
& + \sum_{l \in \Gamma} h_l \int_l \left(\overline{g K \nabla \phi_h \cdot \mathbf{n}_p} - g K \nabla \phi_h \cdot \mathbf{n}_p \right)^2,
\end{aligned}
\tag{3.26a}
$$

$$
\begin{aligned}
\epsilon_3^2 = & \sum_{\tau_f \in \mathcal{T}_{f,h}} h_{\tau_f}^2 \int_{\tau_f} \left(g_1'(\mathbf{y}_h) - \overline{g_1'(\mathbf{y}_h)} \right)^2 + \sum_{\tau_p \in \mathcal{T}_{p,h}} h_{\tau_p}^2 \int_{\tau_p} \left(g_2'(\phi_h) - \overline{g_2'(\phi_h)} \right)^2 \\
& + \sum_{\tau_p \in \mathcal{T}_{p,h}} h_{\tau_p}^2 \int_{\tau_p} \left(\overline{g \nabla \cdot (K^* \nabla \xi_h)} - g \nabla \cdot (K^* \nabla \xi_h) \right)^2 \\
& + \sum_{l_p \cap \partial \Omega_p = \emptyset} h_{l_p} \int_{l_p} \left(\overline{g [K^* \nabla \xi_h] \cdot \mathbf{n}_p} - g [K^* \nabla \xi_h] \cdot \mathbf{n}_p \right)^2 \\
& + \sum_{l_b \subset \partial \Omega_p \backslash \Gamma} h_{l_b} \int_{l_b} \left(\overline{g K^* \nabla \xi_h \cdot \mathbf{n}_p} - g K^* \nabla \xi_h \cdot \mathbf{n}_p \right)^2 \\
& + \sum_{l \in \Gamma} h_l \int_l \left(\overline{g K^* \nabla \xi_h \cdot \mathbf{n}_p} - g K^* \nabla \xi_h \cdot \mathbf{n}_p \right)^2,
\end{aligned}
\tag{3.26b}
$$

where $\overline{\mathbf{f} + B\mathbf{u}_h}$ *and* $\overline{g_1'(\mathbf{y}_h)}$ *are the linear approximations of* $\mathbf{f} + B\mathbf{u}_h$ *and* $g_1'(\mathbf{y}_h)$ *on the element* τ_f *in* $\mathcal{T}_{f,h}$, $\overline{g_2'(\phi_h)}$, $\overline{g\nabla \cdot (K\nabla\phi^h)}$ *and* $\overline{g\nabla \cdot (K^*\nabla\xi^h)}$ *are the linear approximations of* $g_2'(\phi_h)$, $g\nabla \cdot (K\nabla\phi^h)$ *and* $g\nabla \cdot (K^*\nabla\xi^h)$ *on the element* τ_p *in* $\mathcal{T}_{p,h}$, $\overline{g[K\nabla\phi^h] \cdot \mathbf{n}_p}$ *and* $\overline{g[K^*\nabla\xi^h] \cdot \mathbf{n}_p}$ *are the linear approximations of* $g[K\nabla\phi^h] \cdot \mathbf{n}_p$ *and* $g[K^*\nabla\xi^h] \cdot \mathbf{n}_p$ *on the interior edges* l_p *in* Ω_p, $\overline{gK\nabla\phi^h \cdot \mathbf{n}_p}$ *and* $\overline{gK^*\nabla\xi^h \cdot \mathbf{n}_p}$ *are the linear approximations of* $gK\nabla\phi^h \cdot \mathbf{n}_p$ *and* $gK^*\nabla\xi^h \cdot \mathbf{n}_p$ *on the edges located on the interface* Γ *or Neumann boundary* $\partial\Omega_p \setminus \Gamma$ *(see, e.g.,* [15], *for more details on these approximations).*

In order to prove the above theorem and derive the lower bound of the error, we will use the following well known properties on the interior bubble functions for elements and edges (see, e.g., Theorems 2.2 and 2.4 in [15]).

Lemma 3.4 ([15]). *For any regular element* τ, *there exists a constant C independent of* v *and* h_τ *such that for all* $v \in P$

$$C^{-1}\|v\|_{0,\tau}^2 \leq \int_\tau \psi_\tau v^2 dx \leq C\|v\|_{0,\tau}^2, \tag{3.27a}$$

$$C^{-1}\|v\|_{0,\tau} \leq \|\psi_\tau v\|_{0,\tau} + h_\tau|\psi_\tau v|_{1,\tau} \leq C\|v\|_{0,\tau}, \tag{3.27b}$$

where P is the subspace of polynomials of order n on the element τ, $n < \infty$, ψ_τ *is the triangular bubble function on the element* τ *such that* $\psi_\tau|_{\partial\tau} = 0$.

Lemma 3.5 ([15]). *Let* $l = \partial\tau_l^1 \cap \partial\tau_l^2$ *be an interior edge and let* χ_l *be the corresponding edge bubble function such that* $\chi_l = 0$ *on* $\partial(\bar\tau_l^1 \cup \bar\tau_l^2)$. *Let* $P(l)$ *be the space of polynomials of order n*, $n < \infty$. *There exists a constant C, independent of function* v *and the edge size* h_l, *such that for all* $v \in P(l)$,

$$C^{-1}\|v\|_{0,l}^2 \leq \int_l \chi_l v^2 ds \leq C\|v\|_{0,l}^2 \tag{3.28a}$$

$$h_l^{-1/2}\|\chi_l v\|_{0,\tau_l^1 \cup \tau_l^2} + h_l^{1/2}|\chi_l v|_{1,\tau_l^1 \cup \tau_l^2} \leq C\|v\|_{0,l}. \tag{3.28b}$$

Remark 3.1. In this subsection, we will use the similar edge bubble functions for the edges located on the interface Γ or Neumann boundary $\partial\Omega_p \setminus \Gamma$ instead of on the interior edges as in Lemma 3.5. When l is located on the boundary, the l is an edge of the element $\tau_{l,f} \in \mathcal{T}_{f,h}$ or $\tau_{l,p} \in \mathcal{T}_{p,h}$ (not edges of two elements as in Lemma 3.5). Then, the bubble functions $\chi_{l,f}$ and $\chi_{l,p}$ for l are defined on the elements $\tau_{l,f}$ and $\tau_{l,p}$ such that $\chi_{l,f} = 0$ on $\partial\tau_{l,f} \setminus l$ and $\chi_{l,p} = 0$ on $\partial\tau_{l,p} \setminus l$, respectively. Although in the boundary edge cases, there are a few differences on the definitions of the bubble functions, all results of Lemma 3.5 are still valid and will be applied in a posteriori error estimates in the following.

Next, we will use the bubble function technique [15, 16] to deal with η_2 and η_3 provided in Theorem 3.1.

Firstly, we consider the estimate for η_2. We will make use of the following equation of the residual (see, e.g., the proof of (3.18)) :

$$
\begin{aligned}
&\tilde{A}\Big((\mathbf{y}-\mathbf{y}_h, p-p_h, \phi-\phi_h);(\mathbf{v},q,\psi)\Big) \\
&= \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} (\mathbf{f}+B\mathbf{u}_h + \nabla\cdot\mathbb{T}(\mathbf{y}_h,p_h))\mathbf{v} + \sum_{\tau_f \in \mathscr{T}_{f,h}} \int_{\tau_f} (B\mathbf{u}-B\mathbf{u}_h)\mathbf{v} \\
&\quad + \sum_{\tau_p \in \mathscr{T}_{p,h}} \int_{\tau_p} g\big(\nabla\cdot(K\nabla\phi_h)\big)\psi - (\mathrm{div}\,\mathbf{y}_h,q)_{\Omega_f} \\
&\quad - \sum_{l_f \cap \partial\Omega_f = \emptyset} \int_{l_f} [D_l]\mathbf{v} - \sum_{l_p \cap \partial\Omega_p = \emptyset} \int_{l_p} g[K\nabla\phi_h]\cdot\mathbf{n}_p\psi - \sum_{l_b \subset \partial\Omega_p \backslash \Gamma} \int_{l_b} gK\nabla\phi_h\cdot\mathbf{n}_p\psi \\
&\quad + \sum_{l\in\Gamma} \int_l \Big(\big(-2\nu\mathbb{D}(\mathbf{y}_h)+p_h\mathbb{I}\big)\cdot\mathbf{n}_f - g\phi_h\mathbf{n}_f - \alpha P_\tau\mathbf{y}_h\Big)\mathbf{v} \\
&\quad + \sum_{l\in\Gamma} \int_l (-gK\nabla\phi_h\cdot\mathbf{n}_p + g\mathbf{y}_h\cdot\mathbf{n}_f)\psi.
\end{aligned}
\tag{3.29}
$$

Lemma 3.6. *Let* $(\mathbf{y},p,\phi,\mathbf{z},s,\xi,\mathbf{u})$ *and* $(\mathbf{y}_h,p_h,\phi_h,\mathbf{z}_h,s_h,\xi_h,\mathbf{u}_h)$ *be the solutions of (2.15)-(2.17) and (2.20)-(2.22) respectively. Then there exists a constant* $C>0$, *independent of* h_{τ_f}, *such that for each* $\tau_f \in \mathscr{T}_{f,h}$ *there holds*

$$
\begin{aligned}
\eta_{2\tau_f}^2 \leq C\Big(&\|\mathbf{y}-\mathbf{y}_h\|_{1,\tau_f}^2 + \|p-p_h\|_{0,\tau_f}^2 + \|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau_f}^2 \\
&+ h_{\tau_f}^2 \|\mathbf{f}+B\mathbf{u}_h - \overline{\mathbf{f}+B\mathbf{u}_h}\|_{0,\tau_f}^2 \Big)
\end{aligned}
\tag{3.30}
$$

where $\eta_{2\tau_f}$ *is defined in (3.12), and* $\overline{\mathbf{f}+B\mathbf{u}_h}$ *is defined in Theorem 3.2.*

Proof. Denote the interior residual $r_f = \nabla\cdot\mathbb{T}(\mathbf{y}_h,p_h)+\mathbf{f}+B\mathbf{u}_h$. Let ψ_{τ_f} be the interior bubble functions for the element τ_f in $\mathscr{T}_{f,h}$ defined in Lemma 3.4. Let $\bar{\mathbf{r}}_f$ be a piecewise discontinuous approximation to the interior residual $\mathbf{r}_f$ on element τ_f. Then Lemma 3.4 implies that

$$
C\|\bar{\mathbf{r}}_f\|_{0,\tau_f}^2 \leq \int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f^2 = \int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f(\bar{\mathbf{r}}_f - \mathbf{r}_f) + \int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f\mathbf{r}_f.
\tag{3.31}
$$

The first term on the right side of (3.31) can be bounded with the aid of the Cauchy-Schwartz inequality and Lemma 3.4,

$$
\begin{aligned}
\int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f(\bar{\mathbf{r}}_f - \mathbf{r}_f) &\leq C\|\psi_{\tau_f}\bar{\mathbf{r}}_f\|_{0,\tau_f}\|\bar{\mathbf{r}}_f - \mathbf{r}_f\|_{0,\tau_f} \\
&\leq C\|\bar{\mathbf{r}}_f\|_{0,\tau_f}\|\bar{\mathbf{r}}_f - \mathbf{r}_f\|_{0,\tau_f}.
\end{aligned}
\tag{3.32}
$$

Let $\psi_{\tau_f}\bar{\mathbf{r}}_f$ denote its continuous extension to Ω_f. By the residual equation (3.29) and choosing $\mathbf{v} = \psi_{\tau_f}\bar{\mathbf{r}}_f$, $q = 0$, $\psi = 0$, we have that

$$\tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (\mathbf{v}, 0, 0)\Big) = \int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f \mathbf{r}_f + (B\mathbf{u} - B\mathbf{u}_h, \psi_{\tau_f}\bar{\mathbf{r}}_f).$$

Thus by Lemma 3.4, we have

$$\begin{aligned}
&\int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f \mathbf{r}_f \\
&= \tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (\psi_{\tau_f}\bar{\mathbf{r}}_f, 0, 0)\Big) - (B\mathbf{u} - B\mathbf{u}_h, \psi_{\tau_f}\bar{\mathbf{r}}_f)_{\tau_f} \\
&= 2\nu\Big(\mathbb{D}(\mathbf{y} - \mathbf{y}_h), \mathbb{D}(\psi_{\tau_f}\bar{\mathbf{r}}_f)\Big)_{\tau_f} - \Big(\operatorname{div}(\psi_{\tau_f}\bar{\mathbf{r}}_f), p - p_h\Big)_{\tau_f} - (B\mathbf{u} - B\mathbf{u}_h, \psi_{\tau_f}\bar{\mathbf{r}}_f)_{\tau_f} \\
&\le C\Big((\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f} + \|p - p_h\|_{0,\tau_f})\|\psi_{\tau_f}\bar{\mathbf{r}}_f\|_{1,\tau_f} + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f}\|\psi_{\tau_f}\bar{\mathbf{r}}_f\|_{0,\tau_f}\Big) \\
&\le Ch_{\tau_f}^{-1}\Big(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f} + \|p - p_h\|_{0,\tau_f}\Big)\|\bar{\mathbf{r}}_f\|_{0,\tau_f} + C\|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f}\|\bar{\mathbf{r}}_f\|_{0,\tau_f}. \quad (3.33)
\end{aligned}$$

It follows from (3.31)-(3.33) that

$$\begin{aligned}
\|\bar{\mathbf{r}}_f\|_{0,\tau_f}^2 &\le C\left(\int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f(\bar{\mathbf{r}}_f - \mathbf{r}_f) + \int_{\tau_f} \psi_{\tau_f}\bar{\mathbf{r}}_f\mathbf{r}_f\right) \\
&\le C\|\bar{r}_f\|_{0,\tau_f}\|\bar{\mathbf{r}}_f - \mathbf{r}\|_{0,\tau_f} + Ch_{\tau_f}^{-1}\Big(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f} + \|p - p_h\|_{0,\tau_f}\Big)\|\bar{\mathbf{r}}_f\|_{0,\tau_f} \\
&\quad + C\|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f}\|\bar{\mathbf{r}}_f\|_{0,\tau_f}, \quad (3.34)
\end{aligned}$$

and therefore we have

$$\|\bar{\mathbf{r}}_f\|_{0,\tau_f} \le C\Big(h_{\tau_f}^{-1}(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f} + \|p - p_h\|_{0,\tau_f}) + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f} + \|\bar{\mathbf{r}}_f - \mathbf{r}\|_{0,\tau_f}\Big). \quad (3.35)$$

Then it follows from (3.35) and an application of triangle inequality that

$$\begin{aligned}
\|\mathbf{r}_f\|_{0,\tau_f} &\le \|\bar{\mathbf{r}}_f - \mathbf{r}_f\|_{0,\tau_f} + \|\bar{\mathbf{r}}_f\|_{0,\tau_f} \\
&\le C\Big(h_{\tau_f}^{-1}(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f} + \|p - p_h\|_{0,\tau_f}) + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f} + \|\bar{\mathbf{r}}_f - \mathbf{r}\|_{0,\tau_f}\Big). \quad (3.36)
\end{aligned}$$

The perturbation term $\bar{\mathbf{r}}_f - \mathbf{r}_f$ reduces to $\overline{\mathbf{f} + B\mathbf{u}_h} - (\mathbf{f} + B\mathbf{u}_h)$. Thus the desired bound for the residual η_{τ_f} follows:

$$\begin{aligned}
\eta_{2\tau_f}^2 &= h_{\tau_f}^2 \|\mathbf{r}_f\|_{0,\tau_f}^2 \\
&\le C\Big(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f}^2 + \|p - p_h\|_{0,\tau_f}^2 + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_f}^2 + h_{\tau_f}^2\|\overline{\mathbf{f} + B\mathbf{u}_h} - (\mathbf{f} + B\mathbf{u}_h)\|_{0,\tau_f}^2\Big).
\end{aligned}$$

This completes the proof. $\square$

Lemma 3.7. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C > 0$*, independent of* h_{τ_p}*, such that for each* $\tau_p \in \mathscr{T}_{p,h}$ *there holds*

$$\eta^2_{2\tau_p} \le C\left(\|\phi - \phi_h\|^2_{1,\tau_p} + \|\overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)\|^2_{0,\tau_p}\right), \tag{3.37}$$

where $\eta_{2\tau_p}$ *is defined in* (3.12)*, and* $\overline{g\nabla\cdot(K\nabla\phi_h)}$ *is defined in Theorem 3.2.*

Proof. The proof is similar to Lemma 3.6. Denote $r_\phi = g\nabla\cdot(K\nabla\phi_h)$. Let ψ_{τ_p} be the interior bubble function for the element τ_p in $\mathscr{T}_{p,h}$. Let $\bar{r}_\phi$ be a piecewise discontinuous approximation to the interior residual r_ϕ on the element τ_p.

Similar to Lemma 3.6, we have that

$$\begin{aligned}
\int_{\tau_p} \psi_{\tau_p}\bar{r}_\phi r_\phi &= \tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (0, 0, \psi_{\tau_p}\bar{r}_\phi)\Big) \\
&= ga_p(\phi - \phi_h, \psi_{\tau_p}\bar{r}_\phi) = g\left(K\nabla(\phi - \phi_h), \nabla(\psi_{\tau_p}\bar{r}_\phi)\right)_{\tau_p} \\
&\le C\|\phi - \phi_h\|_{1,\tau_p}\|\psi_{\tau_p}\bar{r}_\phi\|_{1,\tau_p} \le Ch^{-1}_{\tau_p}\|\phi - \phi_h\|_{1,\tau_p}\|\bar{r}_\phi\|_{0,\tau_p}.
\end{aligned} \tag{3.38}$$

Lemma 3.4 and (3.38) imply that

$$\begin{aligned}
\|\bar{r}_\phi\|^2_{0,\tau_p} &\le C\int_{\tau_p}\psi_{\tau_p}\bar{r}^2_\phi = C\left(\int_{\tau_p}\psi_{\tau_p}\bar{r}_\phi(\bar{r}_\phi - r_\phi) + \int_{\tau_p}\psi_{\tau_p}\bar{r}_\phi r_\phi\right) \\
&\le C\|\bar{r}_\phi\|_{0,\tau_p}\left(h^{-1}_{\tau_p}\|\phi - \phi_h\|_{1,\tau_p} + \|\bar{r}_\phi - r_\phi\|_{0,\tau_p}\right),
\end{aligned} \tag{3.39}$$

and hence,

$$\|\bar{r}_\phi\|_{0,\tau_p} \le C\left(h^{-1}_{\tau_p}\|\phi - \phi_h\|_{1,\tau_p} + \|\bar{r}_\phi - r_\phi\|_{0,\tau_p}\right). \tag{3.40}$$

The Schwartz's inequality implies that

$$\begin{aligned}
\|r_\phi\|_{0,\tau_p} &\le \|\bar{r}_\phi - r_\phi\|_{0,\tau_p} + \|\bar{r}_\phi\|_{0,\tau_p} \\
&\le C\left(h^{-1}_{\tau_p}\|\phi - \phi_h\|_{1,\tau_p} + \|\bar{r}_\phi - r_\phi\|_{0,\tau_p}\right).
\end{aligned} \tag{3.41}$$

Note that the perturbation term $\bar{r}_\phi - r_\phi = \overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)$. Thus for $\tau_p \in \mathscr{T}_{p,h}$,

$$\eta^2_{2\tau_p} = h^2_{\tau_p}\|r_\phi\|^2_{0,\tau_p} \le C\left(\|\phi - \phi_h\|^2_{1,\tau_p} + h^2_{\tau_p}\|\overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)\|^2_{0,\tau_p}\right).$$

This completes the proof. $\qquad\qquad\square$

Lemma 3.8. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C > 0$*, such that*

$$\begin{aligned}
\eta^2_{2l_f} \le C\Big(&\|\mathbf{y} - \mathbf{y}_h\|^2_{1,\tilde{l}_f} + \|p - p_h\|^2_{0,\tilde{l}_f} + \|B(\mathbf{u} - \mathbf{u}_h)\|^2_{0,\tilde{l}_f} \\
&+ h^2_{l_f}\|\mathbf{f} + B\mathbf{u}_h - \overline{\mathbf{f} + B\mathbf{u}_h}\|^2_{0,\tilde{l}_f}\Big),
\end{aligned} \tag{3.42}$$

where η_{2l_f} is defined in (3.12), and $\overline{\mathbf{f} + B\mathbf{u}_h}$ is defined in Theorem 3.2, $\tilde{l}_f = \bar{\tau}_{l_f}^1 \cup \bar{\tau}_{l_f}^2$ denotes the subdomain of Ω_f consisting of the union of the elements such that l_f is one of the element's edges.

Proof. For convenience, let $R_f = -[D_{l_f}]$ be the edge residual on the interior edge. Let χ_{l_f} be the corresponding bubble function for the interior edges l_f defined by Lemma 3.5 such that $\chi_{l_f} = 0$ on $\partial \tilde{l}_f = \partial(\bar{\tau}_{l_f}^1 \cup \bar{\tau}_{l_f}^2)$. Then we have

$$C\|\bar{R}_f\|_{0,l_f}^2 \le \int_{l_f} \chi_{l_f} \bar{R}_f^2 = \int_{l_f} \chi_{l_f} \bar{R}_f(\bar{R}_f - R_f) + \int_{l_f} \chi_{l_f} \bar{R}_f R_f. \tag{3.43}$$

It should be pointed that $\bar{R}_f$ is only defined on l_f. In the following and later, we will still use it (or other edge residuals) to denote their continuous extension to $\tilde{l}_f$, and use $\chi_{l_f} \bar{R}_f$ to denote its continuous extension to Ω_f. Then $\chi_{l_f} \bar{R}_f \in \mathbf{X}_f$. Note that

$$\tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (\chi_{l_f} \bar{R}_f, 0, 0)\Big)$$
$$= \int_{\tilde{l}_f} \Big(\mathbf{f} + B\mathbf{u} + \nabla \cdot \mathbb{T}(\mathbf{y}_h, p_h)]\Big)(\chi_{l,f} \bar{R}_f) dx - \int_{l_f} [D_{l_f}](\chi_{l,f} \bar{R}_f) ds$$
$$= \int_{\tilde{l}_f} \Big(\mathbf{f} + B\mathbf{u}_h + \nabla \cdot \mathbb{T}(\mathbf{y}_h, p_h)\Big)(\chi_{l,f} \bar{R}_f) dx$$
$$+ \int_{\tilde{l}_f} (B\mathbf{u} - B\mathbf{u}_h)\chi_{l,f} \bar{R}_f dx - \int_{l_f} [D_l](\chi_{l,f} \bar{R}_f) ds. \tag{3.44}$$

Then, using Lemma 3.5, we have that

$$\int_{l_f} \chi_{l_f} \bar{R}_f R_f ds$$
$$= \tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (\chi_{l_f} \bar{R}_f, 0, 0)\Big)$$
$$\quad - \int_{\tilde{l}_f} \Big(\mathbf{f} + B\mathbf{u}_h + \nabla \cdot \mathbb{T}(\mathbf{u}^h, p_h)\Big)(\chi_{l_f} \bar{R}_f) dx - \int_{\tilde{l}_f} (B\mathbf{u} - B\mathbf{u}_h)\chi_{l,f} \bar{R}_f dx$$
$$= \Big(2\nu \mathbb{D}(\mathbf{y} - \mathbf{y}_h), \mathbb{D}(\chi_{l,f} \bar{R}_f)\Big)_{\Omega_f} - \Big(\text{div}(\chi_{l,f} \bar{R}_f), p - p_h\Big)_{\Omega_f}$$
$$\quad - \int_{\tilde{l}_f} \mathbf{r}_f \chi_{l,f} \bar{R}_f dx - \int_{\tilde{l}_f} (B\mathbf{u} - B\mathbf{u}_h)\chi_{l,f} \bar{R}_f dx$$
$$\le C \Big((\|\mathbf{y} - \mathbf{y}_h\|_{1,\tilde{l}_f} + \|p - p_h\|_{0,\tilde{l}_f})\|\chi_{l,f} \bar{R}_f\|_{1,\tilde{l}_f}$$
$$\quad + \|\mathbf{r}_f\|_{0,\tilde{l}_f}\|\chi_{l,f} \bar{R}_f\|_{0,\tilde{l}_f} + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tilde{l}_f}\|\chi_{l,f} \bar{R}_f\|_{0,\tilde{l}_f} \Big)$$

$$
\begin{aligned}
\leq C\Big(& h_{l_f}^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tilde{l}_f}+\|p-p_h\|_{0,\tilde{l}_f}\big)\\
& +h_{l_f}^{1/2}\big(\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tilde{l}_f}+\|\mathbf{r}_f\|_{0,\tilde{l}_f}\big)\Big)\|\bar{R}_f\|_{0,l_f}.
\end{aligned}
\tag{3.45}
$$

Combining (3.43)-(3.45), we have

$$
\begin{aligned}
\|\bar{R}_f\|_{0,l_f}^2 \leq & C\left(\int_{l_f}\chi_{l_f}\bar{R}_f(\bar{R}_f-R_f)ds+\int_{l_f}\chi_{l_f}\bar{R}_f R_f ds\right)\\
\leq & C\|\bar{R}_f\|_{0,l_f}\|\bar{R}_f-R_f\|_{0,l_f}+C\Big(h_{l_f}^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tilde{l}_f}+\|p-p_h\|_{0,\tilde{l}_f}\big)\\
& +h_{l_f}^{1/2}\big(\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tilde{l}_f}+\|\mathbf{r}_f\|_{0,\tilde{l}_f}\big)\Big)\|\bar{R}_f\|_{0,l_f}.
\end{aligned}
\tag{3.46}
$$

Then

$$
\begin{aligned}
\|\bar{R}_f\|_{0,l_f} \leq & C\Big(h_{l_f}^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tilde{l}_f}+\|p-p_h\|_{0,\tilde{l}_f}\big)+h_{l_f}^{1/2}\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tilde{l}_f}\\
& +h_{l_f}^{1/2}\|\mathbf{r}_f\|_{0,\tilde{l}_f}+\|\bar{R}_f-R_f\|_{0,l_f}\Big).
\end{aligned}
\tag{3.47}
$$

Noting that $\bar{\mathbf{r}}_f-\mathbf{r}_f=\overline{\mathbf{f}+B\mathbf{u}_h}-(\mathbf{f}+B\mathbf{u}_h)$ and using the estimate of the interior residual (3.36), it follows that

$$
\begin{aligned}
\|R_f\|_{0,l_f} \leq & C\Big(h_{l_f}^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tilde{l}_f}+\|p-p_h\|_{0,\tilde{l}_f}\big)+h_{l_f}^{1/2}\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tilde{l}_f}\\
& +h_{l_f}^{1/2}\|\mathbf{f}+B\mathbf{u}_h-\overline{\mathbf{f}+B\mathbf{u}_h}\|_{0,\tilde{l}_f}+\|\bar{R}_f-R_f\|_{0,l_f}\Big).
\end{aligned}
\tag{3.48}
$$

Note that on the interior edge $l_f\in\mathscr{T}_{f,h}\backslash\partial\Omega_f$, the perturbation term $\bar{R}_f-R_f=0$. Then we obtain the estimate:

$$
\begin{aligned}
\eta_{2l_f}^2 = & h_{l_f}\|R_f\|_{0,l_f}^2\\
\leq & C\Big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tilde{l}_f}^2+\|p-p_h\|_{0,\tilde{l}_f}^2+\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tilde{l}_f}^2+h_{l_f}^2\|\mathbf{f}+B\mathbf{u}_h-\overline{\mathbf{f}+B\mathbf{u}_h}\|_{0,\tilde{l}_f}^2\Big).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Lemma 3.9. *Let* $(\mathbf{y},p,\phi,\mathbf{z},s,\xi,\mathbf{u})$ *and* $(\mathbf{y}_h,p_h,\phi_h,\mathbf{z}_h,s_h,\xi_h,\mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C>0$, *such that*

$$
\begin{aligned}
\eta_{2l_p}^2 \leq & C\Big(\|\phi-\phi_h\|_{1,\tilde{l}_p}^2+h_{l_p}^2\|\overline{g\nabla\cdot(K\nabla\phi_h)}-g\nabla\cdot(K\nabla\phi_h)\|_{0,\tilde{l}_p}^2\\
& +h_{l_p}\|g[K\nabla\phi_h]\cdot\mathbf{n}_p-\overline{g[K\nabla\phi_h]\cdot\mathbf{n}_p}\|_{0,l_p}^2\Big),
\end{aligned}
\tag{3.49}
$$

where η_{2l_p} *is defined in* (3.12), $\overline{g\nabla\cdot(K\nabla\phi_h)}$ *and* $\overline{g[K\nabla\phi_h]\cdot\mathbf{n}_p}$ *are defined in Theorem 3.2,* $\tilde{l}_p$ *denotes the subdomain of* Ω_p *consisting of the union of elements with* l_p *as their one edge.*

Proof. The proof is similar to Lemma 3.8. Let $R_\phi = -g[K\nabla\phi_h]\cdot\mathbf{n}_p$ be the edge residual on the interior edge. χ_{l_p} is the corresponding bubble functions for the interior edges l_p. From Lemma 3.5, we have that

$$C\|\bar{R}_\phi\|_{0,l_p}^2 \leq \int_{l_p} \chi_{l_p}\bar{R}_\phi^2 = \int_{l_p} \chi_{l_p}\bar{R}_\phi(\bar{R}_\phi - R_\phi)ds + \int_{l_p} \chi_{l_p}\bar{R}_\phi R_\phi ds. \tag{3.50}$$

Similar to Lemma 3.8, we can estimate $\int_{l_p} \chi_{l_p}\bar{R}_\phi R_\phi ds$ as follows.

$$\int_{l_p} \chi_{l_p}\bar{R}_\phi R_\phi ds$$

$$= \tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (0, 0, \chi_{l_p}\bar{R}_\phi)\Big) - \int_{\tilde{l}_p} g\nabla\cdot(K\nabla\phi_h)(\chi_{l_p}\bar{R}_\phi)dx$$

$$= g(K\nabla(\phi - \phi_h), \nabla(\chi_{l_p}\bar{R}_\phi))_{\tilde{l}_p} - \int_{\tilde{l}_p} r_\phi(\chi_{l_p}\bar{R}_\phi)dx$$

$$\leq C\left(h_{l_p}^{-1/2}\|\phi - \phi_h\|_{1,\tilde{l}_p} + h_{l_p}^{1/2}\|r_\phi\|_{0,\tilde{l}_p}\right)\|\bar{R}_\phi\|_{0,l_p}. \tag{3.51}$$

Combining (3.50)-(3.51) and (3.41), we have

$$\|R_\phi\|_{0,l_p} \leq C\left(h_{l_p}^{-1/2}\|\phi - \phi_h\|_{1,\tilde{l}_p} + h_{l_p}^{1/2}\|\bar{r}_\phi - r_\phi\|_{0,\tilde{l}_p} + \|\bar{R}_\phi - R_\phi\|_{0,l_p}\right). \tag{3.52}$$

On the interior edge l_p, the perturbation term

$$\bar{R}_\phi - R_\phi = g[K\nabla\phi_h]\cdot\mathbf{n}_p - \overline{g[K\nabla\phi_h]\cdot\mathbf{n}_p}. \tag{3.53}$$

Then, we have that for interior edge l_p,

$$\eta_{2l_p}^2 = h_{l_p}\int_l \left(g[K\nabla\phi_h]\cdot\mathbf{n}_p\right)^2$$

$$= h_{l_p}\|R_\phi\|_{0,l_p}^2$$

$$\leq C\Big(\|\phi - \phi_h\|_{1,\tilde{l}_p}^2 + h_{l_p}^2\|\overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)\|_{0,\tilde{l}_p}^2$$

$$+ h_{l_p}\|g[K\nabla\phi_h]\cdot\mathbf{n}_p - \overline{g[K\nabla\phi_h]\cdot\mathbf{n}_p}\|_{0,l_p}^2\Big).$$

This completes the proof.   □

The following lemma is about the boundary residual η_{2l_b} on the Neumann boundary $\partial\Omega_p\backslash\Gamma$.

Lemma 3.10. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C > 0$, *independent of* h_{l_b}, *such that for each boundary edge* $l_b \subset \partial\Omega_p \backslash \Gamma$, *there holds*

$$\eta_{2l_b}^2 \leq C \Big(\|\phi - \phi_h\|_{1,\tilde{l}_b}^2 + h_{l_b}^2 \|\overline{g\nabla \cdot (K\nabla\phi_h)} - g\nabla \cdot (K\nabla\phi_h)\|_{0,\tilde{l}_b}^2$$
$$+ h_{l_b} \|\overline{gK\nabla\phi_h \cdot \mathbf{n}_p} - gK\nabla\phi_h \cdot \mathbf{n}_p\|_{0,l_b}^2 \Big), \tag{3.54}$$

where η_{l_b} *is defined in* (3.12), $\overline{g\nabla \cdot (K\nabla\phi_h)}$ *and* $\overline{gK\nabla\phi_h \cdot \mathbf{n}_p}$ *are defined in Theorem* 3.2, $\tilde{l}_b$ *denotes the elements with* l_b *as one of its edges.*

Proof. Again, let $R_{\phi,b} = -gK\nabla\phi_h \cdot \mathbf{n}_p$ be the boundary residual on the Neumann boundary $\partial\Omega_p \backslash \Gamma$. Let χ_{l_b} be the bubble function for the edges $l_b \subset \partial\Omega_p \backslash \Gamma$ such that $\chi_{l_b} = 0$ on $\partial\tilde{l}_b \backslash l_b$. Similar to Lemma 3.9, we have that

$$\|\bar{R}_{\phi,b}\|_{0,l_b}^2 \leq C \int_{l_b} \chi_{l_b} \bar{R}_{\phi,b}^2 ds = \int_{l_b} \chi_{l_b} \bar{R}_{\phi,b}(\bar{R}_{\phi,b} - R_{\phi,b}) ds + \int_{l_b} \chi_{l_b} \bar{R}_{\phi,b} R_{\phi,b} ds. \tag{3.55}$$

Note that

$$\tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (0, 0, \chi_{l_b} \bar{R}_{\phi,b}) \Big)$$
$$= \int_{\tilde{l}_b} g\nabla \cdot (K\nabla\phi_h)(\chi_{l_b} \bar{R}_{\phi,b}) dx + \int_{l_b} -gK\nabla\phi_h \cdot \mathbf{n}_p (\chi_{l_b} \bar{R}_{\phi,b}) ds$$
$$= \int_{\tilde{l}_b} r_\phi \chi_{l_b} \bar{R}_{\phi,b} dx + \int_{l_b} \chi_{l_b} \bar{R}_{\phi,b} R_{\phi,b} ds. \tag{3.56}$$

Then,

$$\int_{l_b} \chi_{l_b} \bar{R}_{\phi,b} R_{\phi,b} ds$$
$$= \tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (0, 0, \chi_{l_b} \bar{R}_{\phi,b}) \Big) - \int_{\tilde{l}_b} r_\phi (\chi_{l_b} \bar{R}_{\phi,b}) dx$$
$$= g\Big(K\nabla(\phi - \phi_h), \nabla(\chi_{l_b} \bar{R}_{\phi,b}) \Big)_{\tilde{l}_b} - \int_{\tilde{l}_b} r_\phi (\chi_{l_b} \bar{R}_{\phi,b}) dx$$
$$\leq C \Big(\|\phi - \phi_h\|_{1,\tilde{l}_b} \|\chi_{l_b} \bar{R}_{\phi,b}\|_{1,\tilde{l}_b} + \|r_\phi\|_{0,\tilde{l}_b} \|\chi_{l_b} \bar{R}_{\phi,b}\|_{0,\tilde{l}_b} \Big)$$
$$\leq C \Big(h_{l_b}^{-1/2} \|\phi - \phi_h\|_{1,\tilde{l}_b} + h_{l_b}^{1/2} \|r_\phi\|_{0,\tilde{l}_b} \Big) \|\bar{R}_{\phi,b}\|_{0,l_b}. \tag{3.57}$$

Therefore, from (3.55)−(3.57) and the estimate for the interior residual (3.41), we have the following estimate:

$$\|R_{\phi,b}\|_{0,l_b} \leq C \Big(h_{l_b}^{-1/2} \|\phi - \phi_h\|_{1,\tilde{l}_b} + h_{l_b}^{1/2} \|\bar{r}_\phi - r_\phi\|_{0,\tilde{l}_b} + \|\bar{R}_{\phi,b} - R_{\phi,b}\|_{0,l_b} \Big).$$

Note that on boundary edge the perturbation term

$$\bar{R}_{\phi,b} - R_{\phi,b} = -\overline{gK\nabla\phi_h \cdot \mathbf{n}_p} + gK\nabla\phi_h \cdot \mathbf{n}_p. \tag{3.58}$$

Then, for $l_b \in \partial\Omega_p\backslash\Gamma$,

$$\begin{aligned}
\eta_{2l_b}^2 =& h_{l_b}\|R_{\phi,b}\|_{0,l_b}^2 \\
\leq& C\Big(\|\phi - \phi_h\|_{1,\tilde{l}_b}^2 + h_{l_b}^2\|\overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)\|_{0,\tilde{l}_b}^2 \\
&+ h_{l_b}\|\overline{gK\nabla\phi_h\cdot\mathbf{n}_p} - gK\nabla\phi_h\cdot\mathbf{n}_p\|_{0,l_b}^2\Big).
\end{aligned}$$

This completes the proof. $\qquad\qquad\square$

Now, estimators $\tilde{\eta}_{2l}^2$ and $\hat{\eta}_{2l}^2$ in η_2 are left to be estimated.

Lemma 3.11. *Let $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ and $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ be the solutions of (2.15)-(2.17) and (2.20)-(2.22) respectively. Then there exists a constant $C > 0$, independent of h_l, such that for $l \subset \Gamma$,*

$$\begin{aligned}
\tilde{\eta}_{2l}^2 + \hat{\eta}_{2l}^2 \leq& C\Big(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_{l,f}}^2 + \|p - p_h\|_{0,\tau_{l,f}}^2 + \|\phi - \phi_h\|_{1,\tau_{l,p}}^2 + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_{l,f}}^2\Big) \\
&+ h_l^2\|\mathbf{f} + B\mathbf{u}_h - \overline{\mathbf{f} + B\mathbf{u}_h}\|_{0,\tau_{l,f}}^2 + h_l^2\|\overline{g\nabla\cdot(K\nabla\phi_h)} - g\nabla\cdot(K\nabla\phi_h)\|_{0,\tau_{l,p}}^2 \\
&+ h_l\|gK\nabla\phi_h\cdot\mathbf{n}_p - \overline{gK\nabla\phi_h\cdot\mathbf{n}_p}\|_l^2, \tag{3.59}
\end{aligned}$$

where $\tilde{\eta}_{2l}$ and $\hat{\eta}_{2l}$ are defined in (3.12), $\overline{\mathbf{f} + B\mathbf{u}_h}$, $\overline{g\nabla\cdot(K\nabla\phi_h)}$ and $\overline{gK\nabla\phi_h\cdot\mathbf{n}_p}$ are defined in Theorem 3.2, $\tau_{l,f} \in \mathscr{T}_{f,h}$ and $\tau_{l,p} \in \mathscr{T}_{p,h}$ are elements with l as their one edge such that $l = \partial\tau_{l,f}\cap\partial\tau_{l,p}$.

Proof. For $l = \bar{\tau}_{l,f} \cap \bar{\tau}_{l,p} \subset \Gamma$, let $\chi_{l,f}$ and $\chi_{l,p}$ be the bubble functions for l on the elements $\tau_{l,f}$ and $\tau_{l,p}$, respectively, such that $\chi_{l,f} = 0$ on $\partial\tau_{l,f}\backslash l$ and $\chi_{l,p} = 0$ on $\partial\tau_{l,p}\backslash l$. Let $R_{l,\phi}$ and $R_{l,f}$ denote the edge residuals on the edge of interface Γ such that

$$R_{l,f} = \big(-2\nu\mathbb{D}(\mathbf{y}_h) + p_h\mathbb{I}\big)\cdot\mathbf{n}_f - g\phi_h\mathbf{n}_f - \alpha P_\tau\mathbf{y}_h, \tag{3.60a}$$

$$R_{l,\phi} = -gK\nabla\phi_h\cdot\mathbf{n}_p + g\mathbf{y}_h\cdot\mathbf{n}_f. \tag{3.60b}$$

Similar to the estimate of the interior edge residual, from Lemma 3.5, we have

$$\|\bar{R}_{l,f}\|_{0,l}^2 \leq C\int_l \chi_{l,f}\bar{R}_{l,f}^2\,ds, \qquad \|\bar{R}_{l,\phi}\|_{0,l}^2 \leq C\int_l \chi_{l,p}\bar{R}_{l,\phi}^2\,ds. \tag{3.61}$$

Making use of the residual equation (3.29), we have

$$\begin{aligned}
&\tilde{A}\Big((\mathbf{y} - \mathbf{y}_h, p - p_h, \phi - \phi_h); (\chi_{l,f}\bar{R}_{l,f}, 0, \chi_{l,p}\bar{R}_{l,\phi})\Big) \\
&= \int_{\tau_{l,f}}\Big(\mathbf{f} + B\mathbf{u}_h + \nabla\cdot\mathbb{T}(\mathbf{u}^h, p_h)\Big)(\chi_{l,f}\bar{R}_{l,f})\,dx + \int_{\tau_{l,f}}(B\mathbf{u} - B\mathbf{u}_h)\chi_{l,f}\bar{R}_{l,f}\,dx \\
&\quad + \int_{\tau_{l,p}}g\nabla\cdot(K\nabla\phi_h)(\chi_{l,p}\bar{R}_{l,\phi})\,dx + \int_l \chi_{l,f}R_{l,f}\bar{R}_{l,f}\,ds + \int_l \chi_{l,p}R_{l,\phi}\bar{R}_{l,\phi}\,ds.
\end{aligned}$$

Then,

$$
\int_l \chi_{l,f} R_{l,f} \bar{R}_{l,f}\, ds + \int_l \chi_{l,p} R_{l,\phi} \bar{R}_{l,\phi}\, ds
$$

$$
= \tilde{A}\Big((\mathbf{y}-\mathbf{y}_h, p - p_h, \phi - \phi_h); (\chi_{l,f}\bar{R}_{l,f}, q, \chi_{l,p}\bar{R}_{l,\phi}) \Big)
$$

$$
- \int_{\tau_{l,f}} \Big(\mathbf{f} + B\mathbf{u}_h + \nabla\cdot \mathbb{T}(\mathbf{u}^h, p_h) \Big)(\chi_{l,f}\bar{R}_{l,f})\, dx - \int_{\tau_{l,f}} (B\mathbf{u} - B\mathbf{u}_h)\chi_{l,f}\bar{R}_{l,f}\, dx
$$

$$
- \int_{\tau_{l,p}} g \nabla\cdot (K\nabla\phi_h)(\chi_{l,p}\bar{R}_{l,\phi})\, dx
$$

$$
= \Big(2\nu(\mathbb{D}(\mathbf{y}-\mathbf{y}_h), \mathbb{D}(\chi_{l,f}\bar{R}_{l,f})\Big)_{\tau_{l,f}} - \Big(\mathrm{div}(\chi_{l,f}\bar{R}_{l,f}), p - p_h \Big)_{\tau_{l,f}}
$$

$$
+ g\Big(K\nabla(\phi - \phi_h), \nabla(\chi_{l,p}\bar{R}_{l,\phi}) \Big)_{\tau_{l,p}} + g\Big\langle \phi - \phi_h, \chi_{l,f}\bar{R}_{l,f}\cdot \mathbf{n}_f \Big\rangle
$$

$$
- g\Big\langle (\mathbf{y}-\mathbf{y}_h)\cdot \mathbf{n}_f, \chi_{l,p}R_{l,\phi} \Big\rangle + \alpha\Big\langle P_\tau(\mathbf{y}-\mathbf{y}_h), P_\tau(\chi_{l,f}\bar{R}_{l,f}) \Big\rangle
$$

$$
- \int_{\tau_{l,f}} \mathbf{r}_f(\chi_{l,f}\bar{R}_{l,f})\, dx - \int_{\tau_{l,f}} B(\mathbf{u}-\mathbf{u}_h)\chi_{l,f}\bar{R}_{l,f}\, dx - \int_{\tau_{l,p}} r_\phi(\chi_{l,p}\bar{R}_{l,\phi})\, dx
$$

$$
\leq C\Big(\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tau_{l,f}} + \|p - p_h\|_{0,\tau_{l,f}} \big)\|\chi_{l,f}\bar{R}_{l,f}\|_{1,\tau_{l,f}}
$$

$$
+ \|\phi - \phi_h\|_{1,\tau_{l,p}}\|\chi_{l,p}\bar{R}_{l,\phi}\|_{1,\tau_{l,p}} + \|\phi - \phi_h\|_{0,l}\|\chi_{l,f}\bar{R}_{l,f}\|_{0,l}
$$

$$
+ \|\mathbf{y}-\mathbf{y}_h\|_{0,l}\|\chi_{l,p}\bar{R}_{l,\phi}\|_{0,l} + \|\mathbf{y}-\mathbf{y}_h\|_{0,l}\|\chi_{l,f}\bar{R}_{l,f}\|_{0,l}
$$

$$
+ \|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau_{l,f}}\|\chi_{l,f}\bar{R}_{l,f}\|_{0,\tau_{l,f}} + \|\mathbf{r}_f\|_{0,\tau_{l,f}}\|\chi_{l,f}\bar{R}_{l,f}\|_{0,\tau_{l,f}}
$$

$$
+ \|r_\phi\|_{0,\tau_{l,p}}\|\chi_{l,p}\bar{R}_{l,\phi}\|_{0,\tau_{l,p}} \Big). \tag{3.62}
$$

Using the above result and Lemma 3.5, we have

$$
\int_l \chi_{l,f} R_{l,f}\bar{R}_{l,f}\, ds + \int_l \chi_{l,p} R_{l,\phi}\bar{R}_{l,\phi}\, ds
$$

$$
\leq C\Big(h_l^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tau_{l,f}} + \|p - p_h\|_{0,\tau_{l,f}} \big) + h_l^{-1/2}\|\phi - \phi_h\|_{1,\tau_{l,p}} + \|\phi - \phi_h\|_{0,l}
$$

$$
+ \|\mathbf{y}-\mathbf{y}_h\|_{0,l} + h_l^{1/2}\|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau_{l,f}} + h_l^{1/2}\|\mathbf{r}_f\|_{0,\tau_{l,f}} + h_l^{1/2}\|r_\phi\|_{0,\tau_{l,p}} \Big)
$$

$$
\times (\|\bar{R}_{l,\phi}\|_{0,l} + \|\bar{R}_{l,f}\|_{0,l}). \tag{3.63}
$$

Therefore, it follows from (3.36), (3.41) and (3.61)-(3.63) that

$$
\|R_{l,f}\|_{0,l} + \|R_{l,\phi}\|_{0,l} \leq C\Big(h_l^{-1/2}\big(\|\mathbf{y}-\mathbf{y}_h\|_{1,\tau_{l,f}} + \|p - p_h\|_{0,\tau_{l,f}} + \|\phi - \phi_h\|_{1,\tau_{l,p}} \big)
$$

$$
+ h_l^{1/2}\big(\|\bar{\mathbf{r}}_f - \mathbf{r}_f\|_{0,\tau_{l,f}} + \|\bar{r}_\phi - r_\phi\|_{0,\tau_{l,p}} + \|B(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau_{l,f}} \big)
$$

$$
+ \|\phi - \phi_h\|_{0,l} + \|\mathbf{y}-\mathbf{y}_h\|_{0,l} + \|\bar{R}_{l,f} - R_{l,f}\|_{0,l} + \|\bar{R}_{l,\phi} - R_{l,\phi}\|_{0,l} \Big).
$$

Note that on the edge l, the perturbation term

$$\bar{R}_{l,f} - R_{l,f} = 0, \tag{3.64a}$$

$$\bar{R}_{l,\phi} - R_{l,\phi} = -\overline{gK\nabla\phi_h \cdot \mathbf{n}_p} + gK\nabla\phi_h \cdot \mathbf{n}_p. \tag{3.64b}$$

Moreover, Lemma 3.2 implies that

$$\|\phi - \phi_h\|_{0,l} \le Ch_l^{-1/2}\|\phi - \phi_h\|_{1,\tau_{l,p}}, \tag{3.65a}$$

$$\|\mathbf{y} - \mathbf{y}_h\|_{0,l} \le Ch_l^{-1/2}\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_{l,f}}. \tag{3.65b}$$

Then, for $l \subset \Gamma$, we obtain the estimate on the interface:

$$
\begin{aligned}
\tilde{\eta}_{2l}^2 + \hat{\eta}_{2l}^2 =& h_l\|R_{l,f}\|_{0,l}^2 + h_l\|R_{l,\phi}\|_{0,l}^2 \\
\le& C\Big(\|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_{l,f}}^2 + \|p - p_h\|_{0,\tau_{l,f}}^2 + \|\phi - \phi_h\|_{1,\tau_{l,p}}^2 + \|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau_{l,f}}^2 \\
& + h_l^2\|\mathbf{f} + B\mathbf{u}_h - \overline{\mathbf{f} + B\mathbf{u}_h}\|_{0,\tau_{l,f}}^2 + h_l^2\|\overline{g\nabla\cdot(\mathbb{K}\nabla\phi_h)} - g\nabla\cdot(\mathbb{K}\nabla\phi_h)\|_{0,\tau_{l,p}}^2 \\
& + h_l\|g\mathbb{K}\nabla\phi_h \cdot \mathbf{n}_p - \overline{g\mathbb{K}\nabla\phi_h \cdot \mathbf{n}_p}\|_l^2\Big).
\end{aligned}
\tag{3.66}
$$

Considering Lemmas 3.6 - 3.11, only the div-residual term needs to be estimated for η_2. It is easy to see that

$$\eta_{2\mathrm{div}\tau_f}^2 = \|\mathrm{div}\mathbf{y}_h\|_{0,\tau_f}^2 = \|\mathrm{div}\mathbf{y}_h - \mathrm{div}\mathbf{y}\|_{0,\tau_f}^2 \le \|\mathbf{y} - \mathbf{y}_h\|_{1,\tau_f}^2. \tag{3.67}$$

Then, it follows from Lemmas 3.6 - 3.11 and (3.67) that

$$
\begin{aligned}
\eta_2^2 \le& C\Big(\|B(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_f}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2\Big) + C\epsilon_2^2 \\
\le& C\Big(\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2\Big) + C\epsilon_2^2.
\end{aligned}
\tag{3.68}
$$

Thus we completed the derivation of the lower bound for the term η_2 defined by (3.12).

Similarly; we can obtain the lower bound for η_3. To estimate terms defined in η_3, we will make use of the following equation (which can be obtained similarly to (3.29)):

$$
\begin{aligned}
& \tilde{A}\Big((\mathbf{w}, t, \eta);(\mathbf{z} - \mathbf{z}_h, s - s_h, \xi - \xi_h)\Big) \\
&= \sum_{\tau_f \in \mathcal{T}_{f,h}} \int_{\tau_f} \Big(g_1'(\mathbf{y}_h) + 2\nu\nabla\cdot\mathbb{D}(\mathbf{z}_h) + \nabla s_h\Big)\mathbf{w} + \sum_{\tau_f \in \mathcal{T}_{f,h}} \int_{\tau_p} \Big(g_1'(\mathbf{y}) - g_1'(\mathbf{y}_h)\Big)\mathbf{w} \\
&\quad + \sum_{\tau_p \in \mathcal{T}_{p,h}} \int_{\tau_p} \Big(g\nabla\cdot(K^*\nabla\xi_h) + g_2'(\phi_h)\Big)\eta + \sum_{\tau_p \in \mathcal{T}_{p,h}} \int_{\tau_p} \Big(g_2'(\phi) - g_2'(\phi_h)\Big)\eta
\end{aligned}
$$

$$+ \sum_{l_f \cap \partial \Omega_f = \emptyset} \int_{l_f} -[A_l]\mathbf{w} + \sum_{l_f \cap \partial \Omega_f = \emptyset} \int_{l_p} g[K^*\nabla\xi_h] \cdot \mathbf{n}_p \eta$$

$$+ \sum_{l_b \subset \partial \Omega_p \setminus \Gamma} \int_{l_b} g K^*\nabla\xi_h \cdot \mathbf{n}_p \eta + (\mathrm{div}\mathbf{z}_h, t)_{\Omega_f}$$

$$+ \sum_{l \in \Gamma} \int_l \Big(- \big(2\nu\mathbb{D}(\mathbf{z}_h) + s_h\mathbb{I}\big) \cdot \mathbf{n}_f + g\xi_h \cdot \mathbf{n}_f - \alpha P_\tau \mathbf{z}_h \Big)\mathbf{w}$$

$$+ \sum_{l \in \Gamma} \int_l (-g K^*\nabla\xi_h \cdot \mathbf{n}_p - g\mathbf{z}_h \cdot \mathbf{n}_f)\eta \, . \tag{3.69}$$

Using the above equation, we can prove the following lemma. Because the technique of the proof for the lemma is similar to the proof on η_2, we only display the results and omit the proofs.

Lemma 3.12. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C > 0$ *such that*

$$\eta_{3\tau_f}^2 \leq C\Big(\|\mathbf{z} - \mathbf{z}_h\|_{1,\tau_f}^2 + \|s - s_h\|_{0,\tau_f}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{0,\tau_f} + h_{\tau_f}^2 \|g_1'(\mathbf{y}_h) - \overline{g_1'(\mathbf{y}_h)}\|_{0,\tau_f}^2 \Big),$$

$$\eta_{3\tau_p}^2 \leq C\Big(\|\xi - \xi_h\|_{1,\tau_p}^2 + \|\phi - \phi_h\|_{0,\tau_p}^2 + h_{\tau_p}^2 \|g\nabla\cdot(K^*\nabla\xi_h) - \overline{g\nabla\cdot(K^*\nabla\xi_h)}\|_{0,\tau_p}^2$$
$$+ h_{\tau_p}^2 \|g_2'(\phi_h) - \overline{g_2'(\phi_h)}\|_{0,\tau_p}^2 \Big),$$

$$\eta_{3l_f}^2 \leq C\Big(\|\mathbf{z} - \mathbf{z}_h\|_{1,\tilde{l}_f}^2 + \|s - s_h\|_{0,\tilde{l}_f}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{0,\tilde{l}_f}^2 + h_{l_f}^2 \|g_1'(\mathbf{y}_h) - \overline{g_1'(\mathbf{y}_h)}\|_{0,\tilde{l}_f}^2 \Big),$$

$$\eta_{3l_p}^2 \leq C\Big(\|\xi - \xi_h\|_{1,\tilde{l}_p}^2 + \|\phi - \phi_h\|_{0,\tilde{l}_p}^2 + h_{l_p}^2 \|g\nabla\cdot(K^*\nabla\xi_h) - \overline{g\nabla\cdot(K^*\nabla\xi_h)}\|_{0,\tilde{l}_p}^2$$
$$+ h_{l_p}^2 \|g_2'(\phi_h) - \overline{g_2'(\phi_h)}\|_{0,\tilde{l}_p}^2 + h_{l_p} \|\overline{g[K^*\nabla\xi_h]\cdot\mathbf{n}_p} - g[K^*\nabla\xi_h]\cdot\mathbf{n}_p\|_{0,l_p}^2 \Big),$$

$$\eta_{3l_b}^2 \leq C\Big(\|\xi - \xi_h\|_{1,\tilde{l}_b}^2 + \|\phi - \phi_h\|_{0,\tilde{l}_b}^2 + h_{l_b}^2 \|g\nabla\cdot(K^*\nabla\xi_h) - \overline{g\nabla\cdot(K^*\nabla\xi_h)}\|_{0,\tilde{l}_b}^2$$
$$+ h_{l_b}^2 \|g_2'(\phi_h) - \overline{g_2'(\phi_h)}\|_{0,\tilde{l}_b}^2 + h_{l_b} \|\overline{g(K^*\nabla\xi_h)\cdot\mathbf{n}_p} - g(K^*\nabla\xi_h)\cdot\mathbf{n}_p\|_{0,l_b}^2 \Big),$$

$$\tilde{\eta}_{3l}^2 + \hat{\eta}_{3l}^2 \leq C\Big(\|\mathbf{z} - \mathbf{z}_h\|_{1,\tau_{l,f}}^2 + \|s - s_h\|_{0,\tau_{l,f}}^2 + \|\xi - \xi_h\|_{1,\tau_{l,p}}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{0,\tau_{l,f}}^2$$
$$+ \|\phi - \phi_h\|_{0,\tau_{l,p}}^2 + h_l^2 \|g_1'(\mathbf{y}_h) - \overline{g_1'(\mathbf{y}_h)}\|_{0,\tau_{l,f}}^2 + h_l^2 \|g_2'(\phi_h) - \overline{g_2'(\phi_h)}\|_{0,\tau_{l,p}}^2$$
$$+ h_l^2 \|g\nabla\cdot(K^*\nabla\xi_h) - \overline{g\nabla\cdot(K^*\nabla\xi_h)}\|_{0,\tau_{l,p}}^2 + h_l \|g K^*\nabla\xi_h \cdot \mathbf{n}_p - \overline{g K^*\nabla\xi_h \cdot \mathbf{n}_p}\|_{0,l}^2 \Big),$$

$$\eta_{3\mathrm{div}\tau_f}^2 = \|\mathrm{div}\mathbf{z}_h\|_{0,\tau_f}^2 \leq C\|\mathbf{z} - \mathbf{z}_h\|_{1,\tau_f}^2,$$

where $\eta_{3\tau_f}$, $\eta_{3\tau_p}$, η_{3l_f}, η_{3l_p}, η_{3l_b}, $\tilde{\eta}_{3l}$, $\hat{\eta}_{3l}$ *and* $\eta_{3\mathrm{div}\tau_f}$ *are defined in* (3.13), $\overline{g_1'(\mathbf{y}_h)}$, $\overline{g_2'(\phi_h)}$, $\overline{g\nabla\cdot(K^*\nabla\xi_h)}$, $\overline{g[K^*\nabla\xi_h]\cdot\mathbf{n}_p}$ *and* $\overline{g(K^*\nabla\xi_h)\cdot\mathbf{n}_p}$ *are defined in Theorem* 3.2.

Summing up, we have that

$$\eta_3^2 \leq C \left(\|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 \right.$$
$$\left. + \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 \right) + C\epsilon_3^2. \tag{3.70}$$

Thus, we obtain the lower bounds for the term η_3 defined by (3.13). Finally, we consider the estimate for the term η_1.

Lemma 3.13. *Let* $(\mathbf{y}, p, \phi, \mathbf{z}, s, \xi, \mathbf{u})$ *and* $(\mathbf{y}_h, p_h, \phi_h, \mathbf{z}_h, s_h, \xi_h, \mathbf{u}_h)$ *be the solutions of* (2.15)-(2.17) *and* (2.20)-(2.22) *respectively. Then there exists a constant* $C > 0$, *such that*

$$\eta_1^2 \leq C \left(e^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 \right), \tag{3.71}$$

where η_1 *and* e *are defined in Lemma* 3.3.

Proof. Let

$$\Omega_0 = \{x \in \Omega^- : \mathbf{u}(x) = 0\}.$$

Note that $B^*\mathbf{z} + j'(\mathbf{u}) = 0$ when $u > 0$ and $B^*\mathbf{z} + j'(\mathbf{u}) \geqslant 0$ when $\mathbf{u} = 0$. Moreover, considering that $j'(\cdot)$ is locally Lipschitz continuous and $\mathbf{U}^h$ is a piecewise constant finite element space, we have that

$$\int_{\Omega^-} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right)^2$$
$$= \int_{\Omega_0} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h - j'(\mathbf{u}) + j'(0) \right)^2 + \int_{\Omega^- \backslash \Omega_0} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h - j'(\mathbf{u}) - B^*\mathbf{z} \right)^2$$
$$\leq C \left(\|j'(\mathbf{u}) - j'(\mathbf{u}_h)\|_{0,\Omega_U}^2 + \|B^*(\mathbf{z} - \mathbf{z}_h)\|_{0,\Omega_U}^2 \right) + \int_{\Omega_0} \left(B^*\mathbf{z}_h + j'(0) \right)^2$$
$$\leq C \left(\|j'(\mathbf{u}) - j'(\mathbf{u}_h)\|_{0,\Omega_U}^2 + \|B^*(\mathbf{z} - \mathbf{z}_h)\|_{0,\Omega_U}^2 \right) + \int_{\Omega_0} \left(B^*\mathbf{z}_h + j'(0) - B^*\mathbf{z} - j'(0) \right)^2$$
$$\leq C \left(\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 \right), \tag{3.72}$$

where we used the fact that $B^*\mathbf{z}_h + j'(0) \leq 0 \leq B^*\mathbf{z} + j'(0)$ on Ω_0. Therefore, we have

$$\int_{\Omega^-} (j'(\mathbf{u}_h) + B^*\mathbf{z}_h)^2 \leq C \left(\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_U}^2 \right). \tag{3.73}$$

Moreover, we note that $\mathbf{u} > 0$ and hence $B^*\mathbf{z} + j'(\mathbf{u}) = 0$ on $\Omega^+ \backslash \Omega^*$, and $\mathbf{u}_h > 0$ and hence

$\mathscr{P}_h(B^*\mathbf{z}_h + j'(\mathbf{u}_h)) = 0$ on Ω^*, where Ω^* is defined in Lemma 3.3. It can be deduced that

$$\int_{\Omega^+} \left(B^*\mathbf{z}_h + j'(\mathbf{u}_h) \right)^2$$

$$= \int_{\Omega^*} \left(B^*\mathbf{z}_h + j'(\mathbf{u}_h) \right)^2 + \int_{\Omega^+\setminus\Omega^*} \left(B^*\mathbf{z}_h + j'(\mathbf{u}_h) \right)^2$$

$$= \int_{\Omega^*} \left(B^*\mathbf{z}_h + j'(\mathbf{u}_h) - \mathscr{P}_h(B^*\mathbf{z}_h + j'(\mathbf{u}_h)) \right)^2 + \int_{\Omega^+\setminus\Omega^*} \left(B^*\mathbf{z}_h + j'(\mathbf{u}_h) - (B^*\mathbf{z} + j'(\mathbf{u})) \right)^2$$

$$\leq C \int_{\Omega^*} \left(B^*\mathbf{z} + j'(\mathbf{u}) - \mathscr{P}_h(B^*\mathbf{z} + j'(\mathbf{u})) \right)^2 + C \int_{\Omega^*} \left((B^*\mathbf{z}_h + j'(\mathbf{u}_h)) - (B^*\mathbf{z} + j'(\mathbf{u})) \right)^2$$

$$+ C \int_{\Omega^*} \left(\mathscr{P}_h(B^*\mathbf{z} + j'(\mathbf{u})) - \mathscr{P}_h(B^*\mathbf{z}_h + j'(\mathbf{u}_h)) \right)^2 + C \left(\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \right)$$

$$\leq C e^2 + C \left(\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 \right). \tag{3.74}$$

Thus, (3.73) and (3.74) lead that

$$\eta_1^2 = \int_{\Omega^-} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right)^2 + \int_{\Omega^+} \left(j'(\mathbf{u}_h) + B^*\mathbf{z}_h \right)^2$$

$$\leq C \left(e^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega_f}^2 \right). \qquad \square$$

Summing up, Theorem 3.2 is a straightforward consequence of (3.68)-(3.71), and then we completed the derivation of the lower bound.

3.3. Conclusions

It is obtained from Theorem 3.1 and Theorem 3.2 that

$$e^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2$$

$$+ \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 \leq C\eta^2, \tag{3.75a}$$

$$\eta^2 \leq C \left(\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2 \right.$$

$$\left. + \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 + e^2 \right) + C \left(\epsilon_2^2 + \epsilon_3^2 \right), \tag{3.75b}$$

where the constant C is independent of the mesh size h, but dependent on the constant in the strict convexity condition and the Lipschitz constant of the cost functional. It is easy to see that ϵ_2^2 and ϵ_3^2 are all higher order terms, if $\mathbf{f}$, g_1', g_2' and K are smooth enough. This means that the a posteriori estimator η^2 provided in this paper is *equivalent* to the error:

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_U}^2 + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega_f}^2 + \|\mathbf{z} - \mathbf{z}_h\|_{1,\Omega_f}^2 + \|p - p_h\|_{0,\Omega_f}^2$$

$$+ \|s - s_h\|_{0,\Omega_f}^2 + \|\phi - \phi_h\|_{1,\Omega_p}^2 + \|\xi - \xi_h\|_{1,\Omega_p}^2 + e^2, \tag{3.76}$$

if the higher order terms ϵ_2 and ϵ_3 can be ignored.

Moreover, note that

$$e^2 = \int_{\Omega^*} \left(\left(j'(\mathbf{u}) + B^*\mathbf{z} \right) - \mathscr{P}_h\left(\left(j'(\mathbf{u}) + B^*\mathbf{z} \right) \right) \right)^2, \tag{3.77}$$

where $\mathscr{P}_h$ is the L^2-projection operator from $(L^2(\Omega_U))^2$ to $\mathbf{U}^h$, and

$$\Omega^* = \{x \in \Omega^+ : \mathbf{u}(x) = 0, \ \mathbf{u}_h(x) > 0\}. \tag{3.78}$$

Then e^2 at least has the same order with $\|\mathbf{u} - \mathbf{u}_h\|^2_{0,\Omega_U}$. Furthermore, it measures the error between

$$\Omega_U^+ = \{x \in \Omega_U : \mathbf{u}(x) > 0\}, \qquad \Omega_U^{h+} = \{x \in \Omega_U : \mathbf{u}_h(x) > 0\}. \tag{3.79}$$

Especially, e^2 can be a higher order term if

$$meas(\Omega^*) \le meas(\Omega_U^{h+} \setminus \Omega_U^+) = o(1), \tag{3.80}$$

where $meas(\Omega^*)$ is the area of Ω^*.

Based on the above theoretical results, the a posteriori error estimates provided in this paper can be used as the indicators of the adaptive finite element mesh refinement, where η_2 and η_3 defined by (3.12) and (3.13) can be used as the indicators for the state and costate, while η_1 defined in Lemma 3.1 can be used for control if the operator B^* is well defined and can be calculated easily and locally.

In this paper, we discuss the a posteriori error estimate of the finite element approximation for the optimal control problem governed by Stokes-Darcy equations. There are still many important issues to be addressed in this area. Especially, many computational issues have to be addressed for efficient adaptive finite element method of the related problem.

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