# The Bases of the Non-Uniform Cubic Spline Space $S_3^{1,2}(\Delta_{mn}^{(2)})$

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**Abstract.** In this paper, the dimension of the nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ is discussed based on the theory of multivariate spline space. Moreover, by means of the Conformality of Smoothing Cofactor Method, the basis of $S_3^{1,2}(\Delta_{mn}^{(2)})$ composed of two sets of splines are worked out in the form of the values at ten domain points in each triangular cell, both of which possess distinct local supports. Furthermore, the explicit coefficients in terms of B-net are obtained for the two sets of splines respectively.

## 1. Introduction

In the past years a vast amount of work has been done on the subject of multivariate approximation that is nowadays an increasingly active research area, see, e.g., [7]. The field is both fascinating and intellectually stimulating since much classical univariate theory does not straightforwardly generalize to the multivariate one. As a result, new tools have been developed, such as wavelets, multivariate splines [22, 23], radial-basis functions [2] and so on. Among these many developments are results on multivariate splines which are applied widely in approximation theory, computer aided geometric design and finite element method.

As known, the nonuniform rational B-splines (NURBS) scheme has become a de facto standard in Computer Aided Geometric Design (CAGD), which is based on algebraic polynomials [9, 15, 18, 20]. It is a powerful tool for constructing free-form curves and surfaces. Recently, some new alternatives to the rational model have been proposed for constructing fair-shape-preserving approximations that inherit major geometric properties such as positivity, monotonicity and convexity [16, 21], whose limiting cases are B-splines. However, both B-spline surfaces and the new alternatives are constructed in the form of tensor

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product, which implies that the degrees of the surfaces are the multiplication of that of the parameters in two directions. Take bicubic B-spline surface for in stance, it is a surface of degree six. As a result, due to the high degree, there may be some inflection points on the surface. Moreover, the bivariate function can not reproduce any polynomial of nearly best degree. To keep away the shortcomings, we are now going to directly generalize the bivariate splines and compute the bases by using the Conformality of Smoothing Cofactor Method [23] in this paper and then make a further study of cubic spline quasi-interpolation in another one.

As a matter of fact, the Conformality of Smoothing Cofactor Method has played a great role on the dimension of multivariate splines and the computation of the bases in multivariate spline. In much detail, the Global Conformality Condition at the grid points of a partition $\Delta$ provides a basic tool in determining the dimension of $S_k^{\mu}(\Delta)$, i.e, the multivariate spline space with degree $k$ and smoothness $\mu$ over the domain $D$ with respect to the partition $\Delta$ [13, 22, 23, 29]. It is a fundamental problem in theory of multivariate splines to determine the dimension [1, 5, 6, 10, 17]. As mentioned in [19], the dimension depends heavily on the geometry of the partition $\Delta$. By means of the Conformality of Smoothing Cofactor Method, important theory on the dimension of $S_k^{\mu}(\Delta)$ has been introduced in next section. Besides this, the Conformality of Smoothing Cofactor Method has played a good role on the computation of the bases of the spline space $S_k^{\mu}(\Delta)$ [27, 28]. Recently, Liu, et al. [14] determined the dimension and construct a local support basis of the space $S^{1,d}(\Delta(2))$, for $d = 0$, 1 of the spline functions over the type-2 nonuniform triangulation. The bases of two bivariate cubic and quartic spline spaces on uniform type-2 triangulation $\Delta_{mn}^{(2)}$ were worked out in [12, 24], respectively. As an application, spline quasi-interpolation operators have been presented. On uniform type-2 triangulation, spline quasi-interpolation has been investigated thoroughly in [12, 24]. On nonuniform type-2 triangulation, a class of quasi-interpolation operators were proposed [3], and their convergence results and error estimates were discussed in [4, 11, 25, 26].

However, the results on spline quasi-interpolation on nonuniform type-2 triangulations in above are almost restricted in the bivariate quadratic B-splines. The reason for it may be the lack of computation of the bases. Since nonuniform triangulation may be more useful than the uniform ones, and bivariate nonuniform splines are important but difficult, we focus on the study of the bases of some nonuniform cubic spline space in this paper. Moreover, we have completed the construction of the corresponding spline quasi-interpolation formula in another paper.

A brief outline of this article is as follows. In Section 2, we discuss the dimension of nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$. Secondly, by using the Conformality of Smoothing Cofactor Method, we work out the basis composed of two sets of splines $B_{ij}^1$ and $B_{ij}^2$ with distinct local supports in Section 3, and then discuss their properties. Finally, in section 4, by means of the barycentric coordinates expressions of the two sets of splines, the explicit coefficients in terms of B-net are worked out. As known, the computation of multiple integrals can be converted into the sum of the coefficients in terms of B-net over triangular domain.

## 2. The dimension of nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$

In this section, we shall study the dimension of nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$. Hence, we would like to introduce the basic frame of multivariate spline spaces.

**Definition 2.1** ( [22, 23]). *For a simply connected domain $D \subset \mathbb{R}^2$, let $\Delta$ be a partition of the domain $D$ given by finite irreducible curves, and $D_i$, $i = 1, \cdots, N$ be the cells. For integers $k$ and $\mu$ with $k > \mu \geq 0$, we define:*

$$S_k^{\mu}(\Delta) = \left\{ s(x,y) \in C^{\mu}(D) \mid s(x,y)_{|D_i} \in \mathbb{P}_k, \quad \forall D_i, i = 1,2,\cdots,N \right\}.$$

*It is called multivariate spline space with degree $k$ and smoothness $\mu$ over the domain $D$ with respect to the partition $\Delta$, where $\mathbb{P}_k$ denotes the collection of all bivariate polynomials of real coefficients with total degree $k$.*

**Theorem 2.1** ( [22, 23]). *Let the representation of $z = s(x,y)$ on the two arbitrary adjacent cells $D_i$ and $D_j$ be*

$$z_i = p_i(x,y), \qquad z_j = p_j(x,y),$$

*where $p_i(x,y)$, $p_j(x,y) \in \mathbb{P}_k$. Then $s(x,y) \in C^{\mu}(\overline{D_i \bigcup D_j})$ if and only if there is a polynomial $q_{ij} \in \mathbb{P}_{k-(\mu+1)d}$ such that*

$$p_i(x,y) - p_j(x,y) = [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y), \tag{2.1}$$

*where $l_{ij}(x,y) \in \mathbb{P}_d$ is an irreducible algebraic polynomial and $\Gamma_{ij} : l_{ij} = 0$ is the common interior edge of $\overline{D_i}$ and $\overline{D_j}$.*

We call the polynomial factor $q_{ij}(x,y)$ in Theorem 2.1 the smoothing cofactor on the interior edge $\Gamma_{ij} : l_{ij} = 0$ from $D_i$ to $D_j$. For any interior vertex $A$, define the Conformality Condition at $A$ by

$$\sum_A [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y) \equiv 0, \tag{2.2}$$

where $\sum_A$ represents the summation of all interior edges around $A$ counterclockwise.

Let $A_1, A_2, \cdots, A_M$ be the interior vertices in $\Delta$. Then the Global Conformality Condition is

$$\sum_{A_v} [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y) \equiv 0, \qquad v = 1,\cdots,M. \tag{2.3}$$

**Theorem 2.2** ( [22, 23]). *Let $\Delta$ be a partition of $D$. The multivariate spline $s(x,y) \in S_k^{\mu}(\Delta)$ exist if and only if for every interior edge there exists a smoothing cofactor of the $s(x,y)$ satisfying the Global Conformality Condition.*

The domain $\Omega = [a,b] \times [c,d]$ is partitioned into $mn$ rectangular cells $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,\cdots,m-1$ and $j = 0,\cdots,n-1$, where $m,n$ are given positive integers, and $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$. Thus, we obtain

a type-2 triangulation $\Delta_{mn}^{(2)}$ by adding the two diagonals in each $\Omega_{ij}$. Let $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, respectively. In the particular case of $h_i = h_{i-1}$ and $k_j = k_{j-1}$, such a triangulation is called uniform, otherwise, nonuniform.

A nonuniform bivariate spline $s(x, y) \in S_3^{1,2}(\Delta_{mn}^{(2)})$ is a piecewise polynomial of degree three satisfying two continuous condition.

Firstly, $s(x, y)$ is $C^1$ continuous on the horizontal and vertical grid segments $x = x_i$ and $y = y_j$, where $i = 0, \cdots, m$ and $j = 0, \cdots, n$.

Secondly, $s(x, y)$ is $C^2$ continuous on the diagonal grid segments

$$y - y_j - \frac{k_{j+1}}{h_{i+1}}(x - x_i) = 0 \quad \text{and} \quad y - y_j + \frac{k_{j+1}}{h_{i+1}}(x - x_{i+1}) = 0,$$

where $i = 0, \cdots, m-1$ and $j = 0, \cdots, n-1$.

We are now going to prove that the dimension of the nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ is the same as that of the uniform case, the latter of which has been calculated by Wang and Li in [25]. Along the way we shall introduce some important results on the dimension of the bivariate spline space $S_k^\mu(\Delta)$.

We denote by $\Delta_c$ a crosscut partition, where the grid lines are crosscuts. And we denote by $\Delta_{qc}$ a quasi-crosscut partition, where each mesh segments is either a cross-cut or a part of rays in partition.

**Lemma 2.1** ( [23]).

$$\dim V_N =: d_k^\mu(N)$$
$$= \frac{1}{2}\left(k - \mu - \left[\frac{\mu+1}{N-1}\right]\right)_+ \cdot \left((N-1)k - (N+1)\mu + (N-3) + (N-1)\left[\frac{\mu+1}{N-1}\right]\right),$$

*where $[x]$ is the integers part of $x$.*

**Theorem 2.3** ( [23]). *Let $D$ be a simply connected domain in $\mathbf{R}^2$ and $\Delta_c$ be a crosscut partition of $D$ with $L$ crosscuts and $V$ interior vertices $A_1, \cdots, A_V$ in $D$ such that $e_i$ crosscuts intersect at $A_i, i = 1, \cdots, V$. Then the dimension of the bivariate spline space $S_k^\mu(\Delta_c)$, $0 \le \mu \le k-1$, is given by*

$$\dim S_k^\mu(\Delta) = \binom{k+2}{2} + L\binom{k-\mu+1}{2} + \sum_{i=1}^V d_k^\mu(e_i), \tag{2.4}$$

*where $d_k^\mu(e_i)$ is defined in Lemma 2.1.*

**Theorem 2.4** ( [23]). *Let $D$ be a simply connected domain in $\mathbb{R}^2$ and $\Delta_{qc}$ be a quasi-crosscut partition of $D$ with $L_1$ crosscuts, $L_2$ rays and $V$ interior vertices $A_1, \cdots, A_V$ in $D$. If $\Delta_{qc}$ has $e_i$ crosscuts and rays passing through $A_i, i = 1, \cdots, V$, then the dimension of the bivariate spline space $S_k^\mu(\Delta_{qc})$, $0 \le \mu \le k-1$, is given by*

$$\dim S_k^\mu(\Delta_{qc}) = \binom{k+2}{2} + L_1\binom{k-\mu+1}{2} + \sum_{i=1}^V d_k^\mu(e_i), \tag{2.5}$$

*where $d_k^\mu(e_i)$ is defined in Lemma 2.1.*

It is easily seen that $e_i$ denotes the number of edges with different slopes intersecting at $A_i$, i.e., the summation of the number of crosscuts and rays. Moreover, let $c_i$ denote the number of grid lines crosscutting at $A_i$. Then we define the degree of $A_i$ as $\deg A_i = e_i + c_i$.

Since the uniform partition can be regarded as a special case of the nonuniform partition, it is necessary to know the changes of the dimension of a spline space as the partition changes between uniform and nonuniform. There are some results on the stability of the dimensions.

**Definition 2.2** ( [13]). *Given a spline space $S_k^\mu(\Delta)$ defined on a partition $\Delta$ including $V$ interior vertices, for each interior vertex $A_i$, $e_i$ denotes the number of edges with different slopes intersecting at $A_i$, $i = 1, \cdots, V$. Then we define*

1) *the dimension of $S_k^\mu(\Delta)$ is (strong) stable if it depends only on $k, \mu$ and $\deg A_i$, $i = 1, \cdots, V$;*

2) *the dimension of $S_k^\mu(\Delta)$ is weak stable if it depends only on $k, \mu$ and $e_i$, $i = 1, \cdots, V$.*

We call a chain of grid lines connected consecutively without self-intersection as a piecewise grid line, or a generalized grid line (GGL). They can be classified into three kinds by their endpoints similarly.

**Definition 2.3** ( [13]). *For each generalized grid line, there are three cases:*

1) *Generalized crosscut line (GCL): both of its endpoints lie on the boundary of D.*

2) *Generalized ray line (GRL): only one of its endpoints lies on the boundary of D.*

3) *Generalized truncated-line (GTL): neither of its endpoints lie on the boundary of D.*

In fact, the above three kinds of generalized grid lines are transformed by crosscuts, rays, and T-lines topologically. Furthermore, we call a partition as generalized crosscut partition (denoted by $\Delta_{gc}$) or generalized quasi-crosscut partition (denoted by $\Delta_{gqc}$), if it can be transformed from a crosscuts partition or a quasi-crosscut partition topologically.

**Theorem 2.5** ( [13]). *Let $\Delta_{gc}$ (or $\Delta_{gqc}$) be a generalized (quasi-) crosscut partition of domain D with E interior edges and V interior vertices $A_1, \cdots, A_V$. If and only if for each $A_i$ the degree $\deg A_i \geq 2\mu + 3$, then the dimension of the spline space $S_k^\mu(\Delta_{gc})$ (or $S_k^\mu(\Delta_{gqc})$), $0 \leq \mu \leq k - 1$, is stable and satisfies*

$$
\dim S_k^\mu\left(\Delta_{gc}(or \Delta_{gqc})\right)
$$
$$
= \binom{k+2}{2} + E \binom{k-\mu+1}{2} - V \left( \binom{k+2}{2} - \binom{\mu+2}{2} \right). \qquad (2.6)
$$

By starting with Theorem 2.5, one can testify that the degree of each interior vertices $\deg A_i = 8 > \max\{5, 7\}$ in the domain $\Delta_{mn}^{(2)}$ for $S_3^{1,2}(\Delta_{mn}^{(2)})$. Hence, the dimension of the
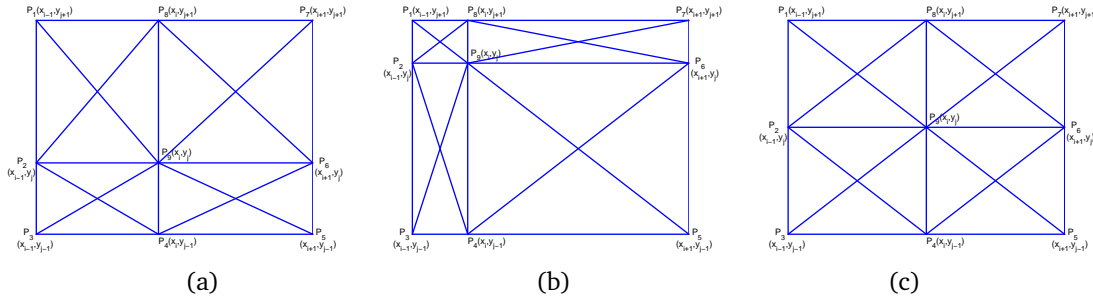
Figure 1: 6, 5 and 4 edges with different slopes intersecting at one interior vertex, respectively.

cubic spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ is invariant, no matter that the partition $\Delta_{mn}^{(2)}$ is uniform or non-uniform, that is ( [12]),

$$\dim S_3^{1,2}(\Delta_{mn}^{(2)}) = 2mn + 3m + 3n + 4. \tag{2.7}$$

For the sake of illustration, denote the vertices by

$$P_1(x_{i-1}, y_{j+1}), \quad P_2(x_{i-1}, y_j), \quad P_3(x_{i-1}, y_{j-1}), \quad P_4(x_i, y_{j-1}), \quad P_5(x_{i+1}, y_{j-1}),$$
$$P_6(x_{i+1}, y_j), \quad P_7(x_{i+1}, y_{j+1}), \quad P_8(x_i, y_{j+1}), \quad P_9(x_i, y_j) \tag{2.8}$$

as shown in Fig. 1(a), (b), and (c). Then we shall discuss the dimension of the cubic spline space $S_3^{1,2}(\Delta_{22}^{(2)})$, where there exists only one interior vertex.

**Case 1:** consider the case of the non-uniform cubic spline space where six edges with different slopes intersect at the interior vertex, that is, none of $k_j/h_i$, $k_j/h_{i+1}$, $k_{j+1}/h_i$, and $k_{j+1}/h_{i+1}$ are equal as shown in Fig. 1(a). According to the dimension of bivariate spline space, we denote by $d_3^{1,2}(6)$ the corresponding last term in (2.5), which equals the dimension of the conformality condition at the interior vertex of $\Delta_{22}^{(2)}$ as follows:

$$c_1 \left[ y - y_j - \frac{k_j}{h_i}(x - x_i) \right]^3 + c_2 \left[ y - y_j + \frac{k_j}{h_{i+1}}(x - x_i) \right]^3 + c_3 \left[ y - y_j - \frac{k_{j+1}}{h_{i+1}}(x - x_i) \right]$$

$$+ c_4 \left[ y - y_j + \frac{k_{j+1}}{h_i}(x - x_i) \right]^3 + \left[ \alpha_5(x - x_i) + \beta_5(y - y_j) + \gamma_5 \right](x - x_i)^2$$

$$+ \left[ \alpha_6(x - x_i) + \beta_6(y - y_j) + \gamma_6 \right](y - y_j)^2 \equiv 0, \tag{2.9}$$

where denote by

$$E_1 : y - y_j - \frac{k_j}{h_i}(x - x_i) = 0, \qquad E_2 : y - y_j + \frac{k_j}{h_{i+1}}(x - x_i) = 0,$$

$$E_3 : y - y_j - \frac{k_{j+1}}{h_{i+1}}(x - x_i) = 0, \qquad E_4 : y - y_j + \frac{k_{j+1}}{h_i}(x - x_i) = 0$$

the four rays $P_9P_3$, $P_9P_5$, $P_9P_7$, $P_9P_1$, respectively, while denote by $c_l \in \mathbb{R}(l = 1, 2, 3, 4)$ and $E_l : \alpha_l(x - x_i) + \beta_l(y - y_j) + \gamma_l(l = 5, 6)$ the $C^2$ and $C^1$ smoothing cofactors of the six

grid edges with different slopes intersecting at $P_9(x_i, y_j)$, respectively. The conformality condition is equivalent to the system of homogeneous linear equations with $c_1$, $c_2$, $c_3$, $c_4$, $\alpha_1$, $\beta_1$, $\gamma_1$, $\alpha_2$, $\beta_2$, $\gamma_2$ as the unknown, that is,

$$
\begin{pmatrix}
-\dfrac{k_j^3}{h_i^3} & \dfrac{k_j^3}{h_{i+1}^3} & -\dfrac{k_{j+1}^3}{h_{i+1}^3} & \dfrac{k_{j+1}^3}{h_i^3} & 1 & 0 & 0 & 0 & 0 & 0 \\[2mm]
\dfrac{3k_j^2}{h_i^2} & \dfrac{3k_j^2}{h_{i+1}^2} & \dfrac{3k_{j+1}^2}{h_{i+1}^2} & \dfrac{3k_{j+1}^2}{h_i^2} & 0 & 1 & 0 & 0 & 0 & 0 \\[2mm]
-\dfrac{3k_j}{h_i} & \dfrac{3k_j}{h_{i+1}} & -\dfrac{3k_{j+1}}{h_{i+1}} & \dfrac{3k_{j+1}}{h_i} & 0 & 0 & 0 & 1 & 0 & 0 \\[2mm]
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\[2mm]
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\[2mm]
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \alpha_2 \\ \beta_2 \\ \gamma_2
\end{pmatrix}
= 0, \qquad (2.10)
$$

which is apparently equivalent to $\gamma_1 = 0 = \gamma_2$ and the following system of homogeneous linear equations

$$
\begin{pmatrix}
-\dfrac{k_j^3}{h_i^3} & \dfrac{k_j^3}{h_{i+1}^3} & -\dfrac{k_{j+1}^3}{h_{i+1}^3} & \dfrac{k_{j+1}^3}{h_i^3} & 1 & 0 & 0 & 0 \\[2mm]
\dfrac{3k_j^2}{h_i^2} & \dfrac{3k_j^2}{h_{i+1}^2} & \dfrac{3k_{j+1}^2}{h_{i+1}^2} & \dfrac{3k_{j+1}^2}{h_i^2} & 0 & 1 & 0 & 0 \\[2mm]
-\dfrac{3k_j}{h_i} & \dfrac{3k_j}{h_{i+1}} & -\dfrac{3k_{j+1}}{h_{i+1}} & \dfrac{3k_{j+1}}{h_i} & 0 & 0 & 1 & 0 \\[2mm]
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2
\end{pmatrix}
= 0. \qquad (2.11)
$$

Thus, one can easily obtain $d_3^{1,2}(6) = 8 - 4 = 4$. Therefore from (2.5), the dimension of the non-uniform cubic spline space is

$$
\dim S_3^{1,2}\left(\Delta_{22}^{(2)}\right)
$$
$$
= \binom{3+2}{2} + 2 \times \binom{3-1+1}{2} + 4 \times \binom{3-2+1}{2} + d_3^{1,2}(6) = 24. \qquad (2.12)
$$

**Case 2:** consider the case of the non-uniform cubic spline space where five edges with different slopes intersect at the interior vertex. Without loss of generality, we suppose

$$
\frac{k_{j+1}}{h_i} = \frac{k_j}{h_i + 1}, \qquad \frac{k_j}{h_i} \neq \frac{k_{j+1}}{h_i + 1}
$$

as shown in Fig. 1(b). In a similar way, we denote by $d_3^{1,2}(5)$ the dimension of the confor-

mality condition at the interior vertex of $\Delta_{22}^{(2)}$ as follows:

$$
c_1 \left[ y - y_j - \frac{k_j}{h_i}(x - x_i) \right]^3 + c_2 \left[ y - y_j + \frac{k_j}{h_{i+1}}(x - x_i) \right]^3 + c_3 \left[ y - y_j - \frac{k_{j+1}}{h_{i+1}}(x - x_i) \right]
$$

$$
+ \left[ \alpha_5(x - x_i) + \beta_5(y - y_j) + \gamma_5 \right] (x - x_i)^2 + \left[ \alpha_6(x - x_i) + \beta_6(y - y_j) + \gamma_6 \right] (y - y_j)^2
$$

$$
\equiv 0, \tag{2.13}
$$

which is apparently equivalent to $\gamma_1 = 0 = \gamma_2$, and the following system of homogeneous linear equations

$$
\begin{pmatrix}
-\frac{k_j^3}{h_i^3} & \frac{k_j^3}{h_{i+1}^3} & -\frac{k_{j+1}^3}{h_{i+1}^3} & 1 & 0 & 0 & 0 \\[2mm]
\frac{3k_j^2}{h_i^2} & \frac{3k_j^2}{h_{i+1}^2} & \frac{3k_{j+1}^2}{h_{i+1}^2} & 0 & 1 & 0 & 0 \\[2mm]
-\frac{3k_j}{h_i} & \frac{3k_j}{h_{i+1}} & -\frac{3k_{j+1}}{h_{i+1}} & 0 & 0 & 1 & 0 \\[2mm]
1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2
\end{pmatrix}
= 0. \tag{2.14}
$$

Thus, we obtain $d_3^{1,2}(5) = 7 - 4 = 3$, and

$$
\dim S_3^{1,2}\left(\Delta_{22}^{(2)}\right)
$$
$$
= \binom{3+2}{2} + 2 \times \binom{3-1+1}{2} + 5 \times \binom{3-2+1}{2} + d_3^{1,2}(5) = 24. \tag{2.15}
$$

**Case 3:** Consider the case of the uniform cubic spline space where four edges with different slopes intersect at the interior vertex, i.e.,

$$
h_i = h_{i+1}, \qquad k_j = k_{j+1}
$$

as shown in Fig. 1(c). By using (2.5), we can calculate directly $d_3^{1,2}(4) = 2$, and

$$
\dim S_3^{1,2}\left(\Delta_{22}^{(2)}\right)
$$
$$
= \binom{3+2}{2} + 2 \times \binom{3-1+1}{2} + 6 \times \binom{3-2+1}{2} + d_3^{1,2}(4) = 24. \tag{2.16}
$$

Apart from this, We shall point out that the basis of the cubic spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ is composed of two sets of splines with distinct local support for the coefficient of the term $mn$ above is 2. And we shall calculate the two splines in next section.
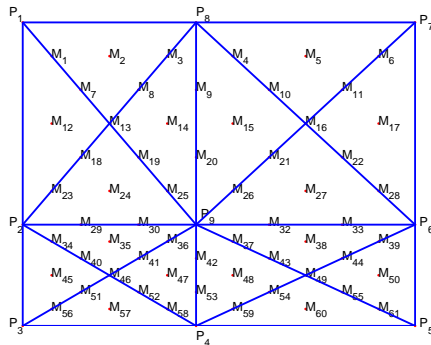
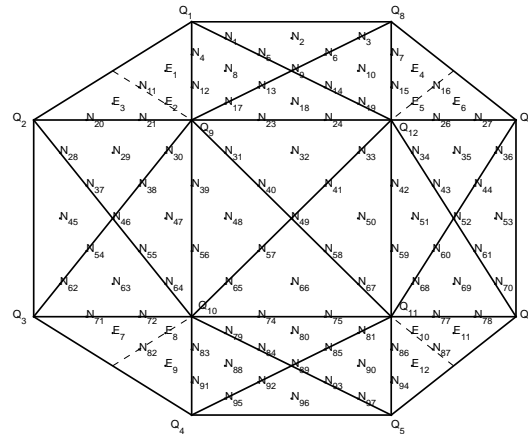Figure 2: The support of the basis element $B^1_{ij}$ in $S^{1,2}_3(\Delta^{(2)}_{mn})$.

Figure 3: The support of the basis element $B^2_{ij}$ in $S^{1,2}_3(\Delta^{(2)}_{mn})$.

## 3. The bases of the non-uniform spline space $S^{1,2}_3(\Delta^{(2)}_{mn})$

In this section, one can see that the Conformality of Smoothing Cofactor Method [23] plays a good role on the computation of the basis of $S^{1,2}_3(\Delta^{(2)}_{mn})$. Whereas the basis in the uniform case have been calculated in [12], we shall be concerned with the basis of the non-uniform cubic spline space $S^{1,2}_3(\Delta^{(2)}_{mn})$.

There exist two sets of the splines $B^1_{ij}$ and $B^2_{ij}$ with local supports as shown in Fig. 2 and Fig. 3, respectively. On one hand, given the vertices $P_i (i = 1, 2, \cdots, 9)$ as (2.8) in the basis element $B^1_{ij}(x, y)$ with the center $(x_i, y_j)$ as shown in Fig. 2, the system of global conformality condition equations at all the vertices can be obtained by means of the continuity of $C^1$ on the horizontal and vertical grid segments, and $C^2$ on the diagonal grid segments. On the other hand, given the vertices

$$Q_1(x_i, y_{j+2}), \qquad Q_2(x_{i-1}, y_{j+1}), \qquad Q_3(x_{i-1}, y_j), \qquad Q_4(x_i, y_{j-1}),$$
$$Q_5(x_{i+1}, y_{j-1}), \qquad Q_6(x_{i+2}, y_j), \qquad Q_7(x_{i+2}, y_{j+1}), \qquad Q_8(x_{i+1}, y_{j+2}),$$
$$Q_9(x_i, y_{j+1}), \qquad Q_{10}(x_i, y_j), \qquad Q_{11}(x_{i+1}, y_j), \qquad Q_{12}(x_{i+1}, y_{j+1})$$

in the basis $B^2_{ij}(x, y)$ with the center $((x_i + x_{i+1})/2, (y_j + y_{j+1})/2)$ as shown in Fig. 3, the system of global conformality condition equations at all the vertices can be obtained in a similar way. Moreover, by letting the two splines satisfy the partition of unity on each triangle, we can complete the computation of the spline basis functions. The two types of the cubic bivariate bases are represented in the form of the values at the trisected points on each edge of each triangle and at its barycenter as shown in Fig. 2 (Table 1), and Fig. 3 (Tables 2, 3 and 4), respectively, where

$$A_i = \frac{h_i}{h_i + h_{i+1}}, \qquad A'_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \qquad B_j = \frac{k_j}{k_j + k_{j+1}}, \qquad B'_j = \frac{k_{j+1}}{k_j + k_{j+1}}, \qquad (3.1)$$

Table 1: The spline $B_{ij}^1(x,y)$ $(\lambda \neq -1, 0)$ with local support.

| Points | Values | Points | Values | Points | Values |
|--------|--------|--------|--------|--------|--------|
| $M_1$ | $\frac{1+\lambda}{108}$ | $M_{23}$ | $\frac{7(1+\lambda)}{108}$ | $M_{45}$ | $\frac{1+\lambda}{27}$ |
| $M_2$ | $\frac{1+\lambda}{27}$ | $M_{24}$ | $\frac{25(1+\lambda)}{54}$ | $M_{46}$ | $\frac{1+\lambda}{4}$ |
| $M_3$ | $\frac{7(1+\lambda)}{108}$ | $M_{25}$ | $\frac{31(1+\lambda)}{36}$ | $M_{47}$ | $\frac{25(1+\lambda)}{54}$ |
| $M_4$ | $\frac{7(1+\lambda)}{108}$ | $M_{26}$ | $\frac{31(1+\lambda)}{36}$ | $M_{48}$ | $\frac{25(1+\lambda)}{54}$ |
| $M_5$ | $\frac{1+\lambda}{27}$ | $M_{27}$ | $\frac{25(1+\lambda)}{54}$ | $M_{49}$ | $\frac{1+\lambda}{4}$ |
| $M_6$ | $\frac{1+\lambda}{108}$ | $M_{28}$ | $\frac{7(1+\lambda)}{108}$ | $M_{50}$ | $\frac{1+\lambda}{27}$ |
| $M_7$ | $\frac{2(1+\lambda)}{27}$ | $M_{29}$ | $\frac{7(1+\lambda)}{27}$ | $M_{51}$ | $\frac{2(1+\lambda)}{27}$ |
| $M_8$ | $\frac{5(1+\lambda)}{27}$ | $M_{30}$ | $\frac{20(1+\lambda)}{27}$ | $M_{52}$ | $\frac{5(1+\lambda)}{27}$ |
| $M_9$ | $\frac{7(1+\lambda)}{27}$ | $P_9$ | $1+\lambda$ | $M_{53}$ | $\frac{7(1+\lambda)}{27}$ |
| $M_{10}$ | $\frac{5(1+\lambda)}{27}$ | $M_{32}$ | $\frac{20(1+\lambda)}{27}$ | $M_{54}$ | $\frac{5(1+\lambda)}{27}$ |
| $M_{11}$ | $\frac{2(1+\lambda)}{27}$ | $M_{33}$ | $\frac{7(1+\lambda)}{27}$ | $M_{55}$ | $\frac{2(1+\lambda)}{27}$ |
| $M_{12}$ | $\frac{1+\lambda}{27}$ | $M_{34}$ | $\frac{7(1+\lambda)}{108}$ | $M_{56}$ | $\frac{1+\lambda}{108}$ |
| $M_{13}$ | $\frac{1+\lambda}{4}$ | $M_{35}$ | $\frac{25(1+\lambda)}{54}$ | $M_{57}$ | $\frac{1+\lambda}{27}$ |
| $M_{14}$ | $\frac{25(1+\lambda)}{54}$ | $M_{36}$ | $\frac{31(1+\lambda)}{36}$ | $M_{58}$ | $\frac{7(1+\lambda)}{108}$ |
| $M_{15}$ | $\frac{25(1+\lambda)}{54}$ | $M_{37}$ | $\frac{31(1+\lambda)}{36}$ | $M_{59}$ | $\frac{7(1+\lambda)}{108}$ |
| $M_{16}$ | $\frac{1+\lambda}{4}$ | $M_{38}$ | $\frac{25(1+\lambda)}{54}$ | $M_{60}$ | $\frac{1+\lambda}{27}$ |
| $M_{17}$ | $\frac{1+\lambda}{27}$ | $M_{39}$ | $\frac{7(1+\lambda)}{108}$ | $M_{61}$ | $\frac{1+\lambda}{108}$ |
| $M_{18}$ | $\frac{5(1+\lambda)}{27}$ | $M_{40}$ | $\frac{5(1+\lambda)}{27}$ | | |
| $M_{19}$ | $\frac{5(1+\lambda)}{9}$ | $M_{41}$ | $\frac{5(1+\lambda)}{9}$ | | |
| $M_{20}$ | $\frac{20(1+\lambda)}{27}$ | $M_{42}$ | $\frac{20(1+\lambda)}{27}$ | | |
| $M_{21}$ | $\frac{5(1+\lambda)}{9}$ | $M_{43}$ | $\frac{5(1+\lambda)}{9}$ | | |
| $M_{22}$ | $\frac{5(1+\lambda)}{27}$ | $M_{44}$ | $\frac{5(1+\lambda)}{27}$ | | |

and $\forall \lambda(\neq 0, -1) \in \mathbb{R}$. Both of the bases vanish on the boundary of the supports in Fig. 2 and Fig. 3, respectively. And the explicit figs of the two bases with special $\lambda$'s are shown as Figs. 4-11.

In view of the translation of the bases, $B_{ij}^1(x,y)$'s and $B_{ij}^2(x,y)$'s do not vanish identically on the domain $\Omega$ defined in Section 2 for the index set

$$I_1 = \left\{ (i,j) = (\alpha, \beta) : 0 \leq \alpha \leq m, 0 \leq \beta \leq n \right\},$$
$$I_2 = \left\{ (i,j) = (\alpha, \beta) : -1 \leq \alpha \leq m, -1 \leq \beta \leq n \right\},$$

respectively. Then it follows from the cardinality of $I_1$ and $I_2$ that the total number of the linear independent functions $B_{ij}^1(x,y)$'s and $B_{ij}^2(x,y)$'s which do not vanish identically on the domain $\Omega$ is

$$2mn + 3m + 3n + 5,$$

which is more than the dimension of $S_3^{1,2}(\Delta_{mn}^{(2)})$ by comparing with (2.7). As a result,
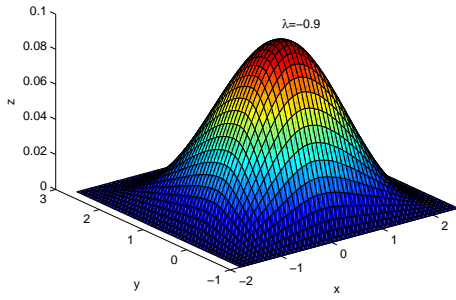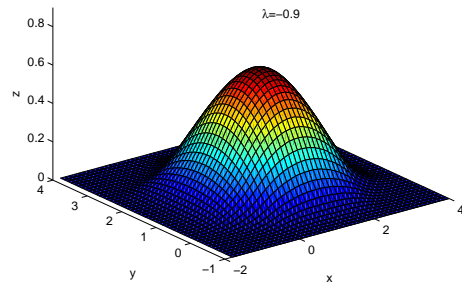
Figure 4: $B_{ij}^1(x,y)$ with $\lambda = -0.9$.
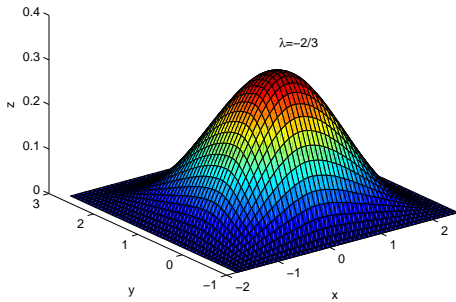


Figure 5: $B_{ij}^2(x,y)$ with $\lambda = -0.9$.
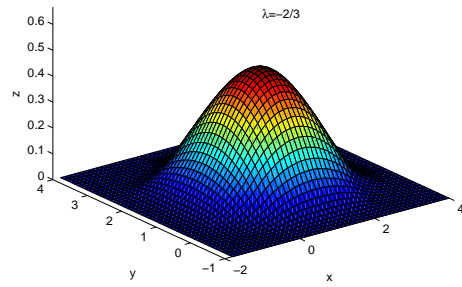


Figure 6: $B_{ij}^1(x,y)$ with $\lambda = -2/3$.
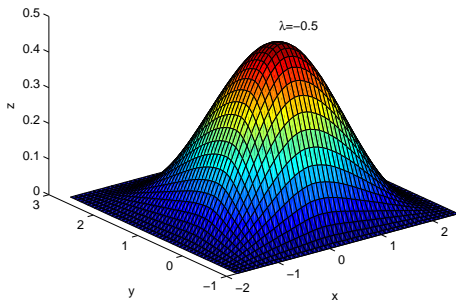


Figure 7: $B_{ij}^2(x,y)$ with $\lambda = -2/3$.



Figure 8: $B_{ij}^1(x,y)$ with $\lambda = -0.5$.
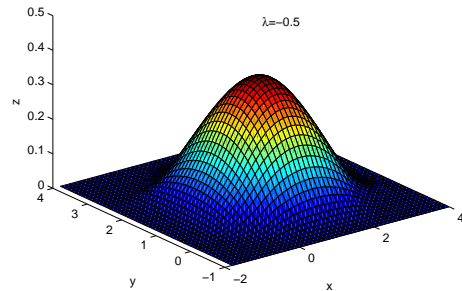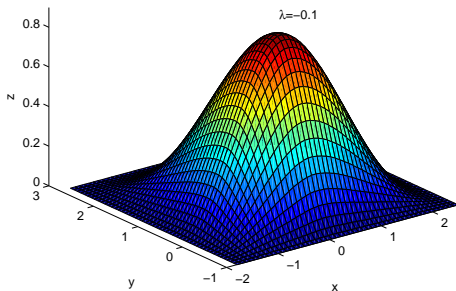


Figure 9: $B_{ij}^2(x,y)$ with $\lambda = -0.5$.
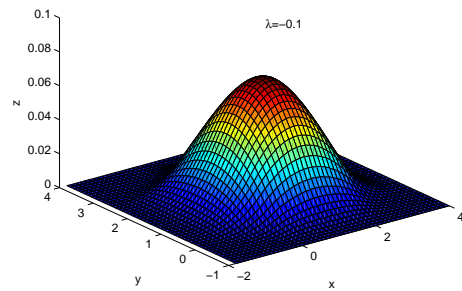


Figure 10: $B_{ij}^1(x,y)$ with $\lambda = -0.1$.



Figure 11: $B_{ij}^2(x,y)$ with $\lambda = -0.1$.

Table 2: The spline $B_{ij}^2(x,y)$ $(\lambda \neq -1, 0)$ with local support (Part I).

| Points | Values | Points | Values |
|---|---|---|---|
| $N_1$ | $-\frac{\lambda}{216}B'_{j+1}$ | $N_{24}$ | $\lambda(-1+\frac{A'_i}{27}+\frac{8A_{i+1}}{27})B'_{j+1}$ |
| $N_2$ | $-\frac{\lambda}{216}B'_{j+1}$ | $Q_{12}$ | $-\lambda A'_{i+1}B'_{j+1}$ |
| $N_3$ | $-\frac{\lambda}{216}B'_{j+1}$ | $N_{26}$ | $-\frac{8\lambda}{27}A'_{i+1}B'_{j+1}$ |
| $N_4$ | $-\frac{\lambda}{27}A_iB'_{j+1}$ | $N_{27}$ | $-\frac{\lambda}{27}A'_{i+1}B'_{j+1}$ |
| $N_5$ | $-\frac{\lambda}{27}B'_{j+1}$ | $N_{28}$ | $-\frac{\lambda A_i}{216}$ |
| $N_6$ | $-\frac{\lambda}{27}B'_{j+1}$ | $N_{29}$ | $\lambda A_i(-\frac{1}{8}+\frac{B_{j+1}}{27})$ |
| $N_7$ | $-\frac{\lambda}{27}A'_{i+1}B'_{j+1}$ | $N_{30}$ | $\lambda A_i(-\frac{125}{216}+\frac{8B_{j+1}}{27})$ |
| $N_8$ | $\lambda(-\frac{1}{8}+\frac{A'_i}{27})B'_{j+1}$ | $N_{34}$ | $\lambda A'_{i+1}(-\frac{125}{216}+\frac{8B_{j+1}}{27})$ |
| $N_9$ | $-\frac{\lambda}{8}B'_{j+1}$ | $N_{35}$ | $\lambda A'_{i+1}(-\frac{1}{8}+\frac{B_{j+1}}{27})$ |
| $N_{10}$ | $\lambda(-\frac{1}{8}+\frac{A_{i+1}}{27})B'_{j+1}$ | $N_{36}$ | $-\frac{\lambda A'_{i+1}}{216}$ |
| $N_{11}$ | $-\frac{\lambda}{27}A_iB'_{j+1}$ | $N_{37}$ | $-\frac{\lambda A_i}{27}$ |
| $N_{12}$ | $-\frac{8\lambda}{27}A_iB'_{j+1}$ | $N_{38}$ | $\lambda A_i(-\frac{8}{27}+\frac{B_{j+1}}{27})$ |
| $N_{13}$ | $\lambda(-\frac{8}{27}+\frac{A'_i}{27})B'_{j+1}$ | $N_{39}$ | $\lambda A_i(-1+\frac{B'_j}{27}+\frac{8B_{j+1}}{27})$ |
| $N_{14}$ | $\lambda(-\frac{8}{27}+\frac{A_{i+1}}{27})B'_{j+1}$ | $N_{42}$ | $\lambda A'_{i+1}(-1+\frac{B'_j}{27}+\frac{8B_{j+1}}{27})$ |
| $N_{15}$ | $-\frac{8\lambda}{27}A'_{i+1}B'_{j+1}$ | $N_{43}$ | $\lambda A'_{i+1}(-\frac{8}{27}+\frac{B_{j+1}}{27})$ |
| $N_{16}$ | $-\frac{\lambda}{27}A'_{i+1}B'_{j+1}$ | $N_{44}$ | $-\frac{\lambda A'_{i+1}}{27}$ |
| $N_{17}$ | $\lambda(-\frac{125}{216}+\frac{8A'_i}{27})B'_{j+1}$ | $N_{45}$ | $-\frac{\lambda}{216}A_i$ |
| $N_{18}$ | $\lambda(-\frac{125}{216}+\frac{A'_i+A_{i+1}}{27})B'_{j+1}$ | $N_{46}$ | $-\frac{\lambda}{8}A_i$ |
| $N_{19}$ | $\lambda(-\frac{125}{216}+\frac{8A_{i+1}}{27})B'_{j+1}$ | $N_{47}$ | $\lambda A_i(-\frac{125}{216}+\frac{B'_j+B_{j+1}}{27})$ |
| $N_{20}$ | $-\frac{\lambda}{27}A_iB'_{j+1}$ | $N_{51}$ | $\lambda A'_{i+1}(-\frac{125}{216}+\frac{B'_j}{27}+\frac{B_{j+1}}{27})$ |
| $N_{21}$ | $-\frac{8\lambda}{27}A_iB'_{j+1}$ | $N_{52}$ | $-\frac{\lambda}{8}A'_{i+1}$ |
| $Q_9$ | $-\lambda A_iB'_{j+1}$ | $N_{53}$ | $-\frac{\lambda}{216}A'_{i+1}$ |
| $N_{23}$ | $\lambda(-1+\frac{8A'_i}{27}+\frac{A_{i+1}}{27})B'_{j+1}$ | $N_{54}$ | $-\frac{\lambda A_i}{27}$ |

we can construct the bases of $S_3^{1,2}(\Delta_{mn}^{(2)})$ by getting rid of arbitrary one function in either $B_{ij}^1(x,y)$ where $(i,j) \in I_1$, or $B_{ij}^2(x,y)$ where $(i,j) \in I_2$.

**Theorem 3.1.** *For arbitrary chosen $(i_0, j_0) \in I_1$, $(i_1, j_1) \in I_2$,, let*

$$\mathbb{B}^1 = \left\{ B_{ij}^1 : (i,j) \in I_1 \backslash \{(i_0,j_0)\} \right\} \bigcup \left\{ B_{ij}^2 : (i,j) \in I_2 \right\},$$
$$\mathbb{B}^2 = \left\{ B_{ij}^2 : (i,j) \in I_2 \backslash \{(i_1,j_1)\} \right\} \bigcup \left\{ B_{ij}^1 : (i,j) \in I_1 \right\}.$$

*Then, either $\mathbb{B}^1$ or $\mathbb{B}^2$ is a basis of the non-uniform cubic spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$.*

Table 3: The spline $B_{ij}^2(x,y)$ $(\lambda \neq -1, 0)$ with local support (Part II).

| Points | Values | Points | Values |
|--------|--------|--------|--------|
| $N_{55}$ | $\lambda A_i(-\frac{8}{27} + \frac{B_j'}{27})$ | $N_{79}$ | $\lambda(-\frac{125}{216} + \frac{8A_i'}{27})B_j$ |
| $N_{56}$ | $\lambda A_i(-1 + \frac{8B_j'}{27} + \frac{B_{j+1}'}{27})$ | $N_{80}$ | $\lambda(-\frac{125}{216} + \frac{A_i'+A_{i+1}'}{27})B_j$ |
| $N_{59}$ | $\lambda A_{i+1}'(-1 + \frac{8B_j'}{27} + \frac{B_{j+1}'}{27})$ | $N_{81}$ | $\lambda(-\frac{125}{216} + \frac{8A_{i+1}'}{27})B_j$ |
| $N_{60}$ | $\lambda A_{i+1}'(-\frac{8}{27} + \frac{B_j'}{27})$ | $N_{82}$ | $-\frac{\lambda}{27}A_iB_j$ |
| $N_{61}$ | $-\frac{\lambda}{27}A_{i+1}'$ | $N_{83}$ | $-\frac{8\lambda}{27}A_iB_j$ |
| $N_{62}$ | $-\frac{\lambda}{216}A_i$ | $N_{84}$ | $\lambda(-\frac{8}{27} + \frac{A_i'}{27})B_j$ |
| $N_{63}$ | $\lambda A_i(-\frac{1}{8} + \frac{B_j'}{27})$ | $N_{85}$ | $\lambda(-\frac{8}{27} + \frac{A_{i+1}'}{27})B_j$ |
| $N_{64}$ | $\lambda A_i(-\frac{125}{216} + \frac{8B_j'}{27})$ | $N_{86}$ | $-\frac{8\lambda}{27}A_{i+1}'B_j$ |
| $N_{68}$ | $\lambda A_{i+1}'(-\frac{125}{216} + \frac{8B_j'}{27})$ | $N_{87}$ | $-\frac{\lambda}{27}A_{i+1}'B_j$ |
| $N_{69}$ | $\lambda A_{i+1}'(-\frac{1}{8} + \frac{B_j'}{27})$ | $N_{88}$ | $\lambda(-\frac{1}{8} + \frac{A_i'}{27})B_j$ |
| $N_{70}$ | $-\frac{\lambda}{216}A_{i+1}'$ | $N_{89}$ | $-\frac{\lambda}{8}B_j$ |
| $N_{71}$ | $-\frac{\lambda}{27}A_iB_j$ | $N_{90}$ | $\lambda(-\frac{1}{8} + \frac{A_{i+1}'}{27})B_j$ |
| $N_{72}$ | $-\frac{8\lambda}{27}A_iB_j$ | $N_{91}$ | $-\frac{\lambda}{27}A_iB_j$ |
| $Q_{10}$ | $-\lambda A_iB_j$ | $N_{92}$ | $-\frac{\lambda}{27}B_j$ |
| $N_{74}$ | $\lambda(-1 + \frac{8A_i'}{27} + \frac{A_{i+1}'}{27})B_j$ | $N_{93}$ | $-\frac{\lambda}{27}B_j$ |
| $N_{75}$ | $\lambda(-1 + \frac{A_i'}{27} + \frac{8A_{i+1}'}{27})B_j$ | $N_{94}$ | $-\frac{\lambda}{27}A_{i+1}'B_j$ |
| $Q_{11}$ | $-\lambda A_{i+1}'B_j$ | $N_{95}$ | $-\frac{\lambda}{216}B_j$ |
| $N_{77}$ | $-\frac{8\lambda}{27}A_{i+1}'B_j$ | $N_{96}$ | $-\frac{\lambda}{216}B_j$ |
| $N_{78}$ | $-\frac{\lambda}{27}A_{i+1}'B_j$ | $N_{97}$ | $-\frac{\lambda}{216}B_j$ |
| $E_1$ | $-\frac{\lambda}{27}A_iB_{j+1}'$ | $E_7$ | $-\frac{\lambda}{27}A_iB_j$ |
| $E_2$ | $-\frac{8\lambda}{27}A_iB_{j+1}'$ | $E_8$ | $-\frac{8\lambda}{27}A_iB_j$ |
| $E_3$ | $-\frac{\lambda}{27}A_iB_{j+1}'$ | $E_9$ | $-\frac{\lambda}{27}A_iB_j$ |
| $E_4$ | $-\frac{\lambda}{27}A_{i+1}'B_{j+1}'$ | $E_{10}$ | $-\frac{8\lambda}{27}A_{i+1}'B_j$ |
| $E_5$ | $-\frac{8\lambda}{27}A_{i+1}'B_{j+1}'$ | $E_{11}$ | $-\frac{\lambda}{27}A_{i+1}'B_j$ |
| $E_6$ | $-\frac{\lambda}{27}A_{i+1}'B_{j+1}'$ | $E_{12}$ | $-\frac{\lambda}{27}A_{i+1}'B_j$ |

By means of translation of the bases and the values at the ten points on each triangle, it follows that the bases with minimal local support satisfy

**Theorem 3.2.** *For all* $(x, y) \in \Omega$,

$$\sum_{(i,j)\in I_1} B_{ij}^1(x, y) = 1 + \lambda, \tag{3.2}$$

$$\sum_{(i,j)\in I_2} B_{ij}^2(x, y) = -\lambda, \tag{3.3}$$

Table 4: The spline $B_{ij}^2(x,y)$ $(\lambda \neq -1,0)$ with local support (Part III).

| Points | Values |
|---|---|
| $N_{31}$ | $-\lambda + \frac{\lambda}{216}(B_j' + 125B_{j+1} + 61A_i' + A_{i+1} + 64A_i'B_{j+1}')$ |
| $N_{32}$ | $-\lambda + \frac{\lambda}{216}[B_j' + 125B_{j+1} + 19A_i' + 19A_{i+1} + 8(A_i' + A_{i+1})B_{j+1}']$ |
| $N_{33}$ | $-\lambda + \frac{\lambda}{216}(B_j' + 125B_{j+1} + A_i' + 61A_{i+1} + 64A_{i+1}B_{j+1}')$ |
| $N_{40}$ | $-\lambda + \frac{\lambda}{27}(B_j' + 8B_{j+1} + 7A_i' + A_{i+1} + A_i'B_{j+1}')$ |
| $N_{41}$ | $-\lambda + \frac{\lambda}{27}(B_j' + 8B_{j+1} + A_i' + 7A_{i+1} + A_{i+1}B_{j+1}')$ |
| $N_{48}$ | $-\lambda + \frac{\lambda}{216}[125A_i' + A_{i+1} + 19B_j' + 19B_{j+1} + 8A_i(B_j' + B_{j+1})]$ |
| $N_{49}$ | $-\lambda + \frac{\lambda}{8}(B_j' + B_{j+1} + A_i' + A_{i+1})$ |
| $N_{50}$ | $-\lambda + \frac{\lambda}{216}[A_i' + 125A_{i+1} + 19B_j' + 19B_{j+1} + 8A_{i+1}'(B_j' + B_{j+1})]$ |
| $N_{57}$ | $-\lambda + \frac{\lambda}{27}(8B_j' + B_{j+1} + 7A_i' + A_{i+1} + A_i'B_j)$ |
| $N_{58}$ | $-\lambda + \frac{\lambda}{27}(8B_j' + B_{j+1} + A_i' + 7A_{i+1} + A_{i+1}B_j)$ |
| $N_{65}$ | $-\lambda + \frac{\lambda}{216}(125B_j' + B_{j+1} + 61A_i' + A_{i+1} + 64A_i'B_j)$ |
| $N_{66}$ | $-\lambda + \frac{\lambda}{216}[125B_j' + B_{j+1} + 19A_i' + 19A_{i+1} + 8(A_i' + A_{i+1})B_j]$ |
| $N_{67}$ | $-\lambda + \frac{\lambda}{216}(125B_j' + B_{j+1} + A_i' + 61A_{i+1} + 64A_{i+1}B_j)$ |

Table 5: The explicit coefficients in terms of B-net of the spline $B_{ij}^1(x,y)$ $(\lambda \neq -1,0)$.

| Coefficients | Values | Coefficients | Values | Coefficients | Values | Coefficients | Values |
|---|---|---|---|---|---|---|---|
| $M_8$ | $\frac{1+\lambda}{4}$ | $M_{21}$ | $\frac{1+\lambda}{2}$ | $M_{35}$ | $\frac{1+\lambda}{2}$ | $M_{46}$ | $\frac{1+\lambda}{4}$ |
| $M_{10}$ | $\frac{1+\lambda}{4}$ | $M_{22}$ | $\frac{1+\lambda}{4}$ | $M_{36}$ | $1+\lambda$ | $M_{47}$ | $\frac{1+\lambda}{2}$ |
| $M_{13}$ | $\frac{1+\lambda}{4}$ | $M_{24}$ | $\frac{1+\lambda}{2}$ | $M_{37}$ | $1+\lambda$ | $M_{48}$ | $\frac{1+\lambda}{2}$ |
| $M_{14}$ | $\frac{1+\lambda}{2}$ | $M_{25}$ | $1+\lambda$ | $M_{38}$ | $\frac{1+\lambda}{2}$ | $M_{49}$ | $\frac{1+\lambda}{4}$ |
| $M_{15}$ | $\frac{1+\lambda}{2}$ | $M_{26}$ | $1+\lambda$ | $M_{40}$ | $\frac{1+\lambda}{4}$ | $M_{52}$ | $\frac{1+\lambda}{4}$ |
| $M_{16}$ | $\frac{1+\lambda}{4}$ | $M_{27}$ | $\frac{1+\lambda}{2}$ | $M_{41}$ | $\frac{1+\lambda}{2}$ | $M_{54}$ | $\frac{1+\lambda}{4}$ |
| $M_{18}$ | $\frac{1+\lambda}{4}$ | $M_{30}$ | $1+\lambda$ | $M_{42}$ | $1+\lambda$ | others | $0$ |
| $M_{19}$ | $\frac{1+\lambda}{2}$ | $P_9$ | $1+\lambda$ | $M_{43}$ | $\frac{1+\lambda}{2}$ | | |
| $M_{20}$ | $1+\lambda$ | $M_{32}$ | $1+\lambda$ | $M_{44}$ | $\frac{1+\lambda}{4}$ | | |

$$\sum_{(i,j)\in I_1} B_{ij}^1(x,y) + \sum_{(i,j)\in I_2} B_{ij}^2(x,y) = 1, \tag{3.4}$$

where $\lambda \neq 0, -1$.

**Remark 3.1.** With the choice of $\lambda \in (-1,0)$, all the functions $B_{ij}^1(x,y)$'s and $B_{ij}^2(x,y)$'s range from 0 to 1 as shown in Figs. 4-11.

Table 6: The explicit coefficients in terms of B-net of the spline $B_{ij}^2(x,y)$ $(\lambda \neq -1, 0)$.

| Coefficients | Values | Coefficients | Values |
|---|---|---|---|
| $N_9$ | $-\frac{\lambda}{8}B_{j+1}'$ | $N_{50}$ | $-\lambda + \frac{\lambda}{2}A_{i+1}$ |
| $N_{13}$ | $-\frac{\lambda}{4}B_{j+1}'$ | $N_{51}$ | $-\frac{\lambda}{2}A_{i+1}'$ |
| $N_{14}$ | $-\frac{\lambda}{4}B_{j+1}'$ | $N_{52}$ | $-\frac{\lambda}{8}A_{i+1}'$ |
| $N_{17}$ | $-\frac{\lambda}{2}B_{j+1}'$ | $N_{55}$ | $-\frac{\lambda}{4}A_i$ |
| $N_{18}$ | $-\frac{\lambda}{2}B_{j+1}'$ | $N_{56}$ | $-\lambda A_i$ |
| $N_{19}$ | $-\frac{\lambda}{2}B_{j+1}'$ | $N_{57}$ | $-\lambda + \frac{\lambda}{4}(A_i' + B_j')$ |
| $Q_9$ | $-\lambda A_i B_{j+1}'$ | $N_{58}$ | $-\lambda + \frac{\lambda}{4}(A_{i+1} + B_j')$ |
| $N_{23}$ | $-\lambda B_{j+1}'$ | $N_{59}$ | $-\lambda A_{i+1}'$ |
| $N_{24}$ | $-\lambda B_{j+1}'$ | $N_{60}$ | $-\frac{\lambda}{4}A_{i+1}'$ |
| $Q_{12}$ | $-\lambda A_{i+1}' B_{j+1}'$ | $N_{64}$ | $-\frac{\lambda}{2}A_i$ |
| $N_{30}$ | $-\frac{\lambda}{2}A_i$ | $N_{65}$ | $-\lambda + \frac{\lambda}{2}(A_i' + B_j')$ |
| $N_{31}$ | $-\lambda + \frac{\lambda}{2}(A_i' + B_{j+1})$ | $N_{66}$ | $-\lambda + \frac{\lambda}{2}B_j'$ |
| $N_{32}$ | $-\lambda + \frac{\lambda}{2}B_{j+1}$ | $N_{67}$ | $-\lambda + \frac{\lambda}{2}(A_{i+1} + B_j')$ |
| $N_{33}$ | $-\lambda + \frac{\lambda}{2}(A_{i+1} + B_{j+1})$ | $N_{68}$ | $-\frac{\lambda}{2}A_{i+1}'$ |
| $N_{34}$ | $-\frac{\lambda}{2}A_{i+1}'$ | $Q_{10}$ | $-\lambda A_i B_j$ |
| $N_{38}$ | $-\frac{\lambda}{4}A_i$ | $N_{74}$ | $-\lambda B_j$ |
| $N_{39}$ | $-\lambda A_i$ | $N_{75}$ | $-\lambda B_j$ |
| $N_{40}$ | $-\lambda + \frac{\lambda}{4}(A_i' + B_{j+1})$ | $Q_{11}$ | $-\lambda A_{i+1}' B_j$ |
| $N_{41}$ | $-\lambda + \frac{\lambda}{4}(A_{i+1} + B_{j+1})$ | $N_{79}$ | $-\frac{\lambda}{2}B_j$ |
| $N_{42}$ | $-\lambda A_{i+1}'$ | $N_{80}$ | $-\frac{\lambda}{2}B_j$ |
| $N_{43}$ | $-\frac{\lambda}{4}A_{i+1}'$ | $N_{81}$ | $-\frac{\lambda}{2}B_j$ |
| $N_{46}$ | $-\frac{\lambda}{8}A_i$ | $N_{84}$ | $-\frac{\lambda}{4}B_j$ |
| $N_{47}$ | $-\frac{\lambda}{2}A_i$ | $N_{85}$ | $-\frac{\lambda}{4}B_j$ |
| $N_{48}$ | $-\lambda + \frac{\lambda}{2}A_i'$ | $N_{89}$ | $-\frac{\lambda}{8}B_j$ |
| $N_{49}$ | $-\lambda + \frac{\lambda}{8}(B_j' + B_{j+1} + A_i' + A_{i+1})$ | others | $0$ |

## 4. The explicit coefficients in terms of B-net of the splines in $S_3^{1,2}(\Delta_{mn}^{(2)})$

Many authors have contributed valuable insights into the multivariate integrals in terms of the bivariate B-net method [8], among which is the novel result on the numerical integration based on bivariate quadratic splines on bounded domains [11]. In order to work out the multiple integrals based on bivariate cubic splines effectively and conveniently over each triangular domain in another paper, we make an analysis of the two sets of cubic splines in terms of the bivariate B-net method.

Let $P_i$, $i = 1, 2, 3$ be the counter-clockwise ordered vertices of triangle $\triangle := \{P_1, P_2, P_3\}$. Then any $P \in \mathbf{R}^2$ can be expressed as

$$P = uP_1 + vP_2 + wP_3, \qquad (4.1)$$

where $u + v + w = 1$, and it is easy to obtain

$$u = \frac{det(P_2 - P, P_3 - P)}{det(P_2 - P_1, P_3 - P_1)}, \tag{4.2a}$$

$$v = \frac{det(P_1 - P, P_3 - P)}{det(P_1 - P_2, P_3 - P_2)}, \tag{4.2b}$$

$$w = \frac{det(P_1 - P, P_2 - P)}{det(P_1 - P_3, P_2 - P_3)}. \tag{4.2c}$$

Eq. (4.1) is called the barycentric coordinates transformation of $P$ and is also called the barycentric coordinates expression as introduced in [8]. Obviously, this kind of expression is unique. The barycentric coordinates transformation has affine invariance property which is given by G. Farin in [8]. Any polynomial $p(x, y) = \sum_{i+j \leq n} a_{i,j} x^i y^j$ of degree $n$ over $\triangle$ can be uniquely expressed as

$$p(x, y) := p(u, v, w) = \sum_{i+j+k=n} b_{i,j,k} B^n_{i,j,k}(u, v, w), \tag{4.3}$$

where $b_{i,j,k}$'s are the coefficients in terms of the B-net in $\triangle$ and the Bernstein polynomials of degree $n$ over $\triangle$ are defined by

$$B^n_{i,j,k}(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k, \qquad u + v + w = 1, \quad u, v, w \geq 0. \tag{4.4}$$

Thus, using the values in Table 1 and Tables 2-4, we obtain the explicit coefficients in terms of B-net of the splines $B^1_{ij}$ and $B^2_{ij}$ as shown in Fig. 2 (Table 5), and Fig. 3 (Table 6), respectively, where $A_i$'s, $A'_i$'s, $B_j$'s and $B'_j$'s are defined as (3.1). With a choice of $\lambda = -2/3$ in the uniform case, the coefficients in terms of B-net in Tables 5 and 6 coincide with those given in [11] for uniform partitions.

## References

[1] P. ALFELD AND L. L. SCHUMAKER, *The dimension of bivariate spline spaces of smoothness r for degree d ≥ 4r + 1*, Constr. Approx., 3 (1987), pp. 189–197.

[2] M. D. BUHMANN, *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, 2001.

[3] C. DAGNINO AND P. LAMBERTI, *On the approximation power of bivariate quadratic $C^1$ splines*, J. Comput. and Appl. Math., 131 (2001), pp. 321–332.

[4]  C. Dagnino and P. Lamberti, *Some performances of local bivariate quadratic $C^1$ quasi-interpolating splines on nonuniform type-2 triangulations*, J. Comput. and Appl. Math., 173 (2005), pp. 21–37.

[5]  J. S. Deng, Y. Y. Feng and J. Kozak, *A note on the dimension of the bivariate spline space over the Morgan-Scott triangulation*, SIAM J. Numer. Anal., 37(3) (2000), pp. 1021–1028.

[6]  D. Diener, *Instability in the dimension of spaces of bivariate piecewise polynomials of degree $2r$ and smoothness order $r$*, SIAM J. Numer. Anal., 27 (1990), pp. 543–551.

[7]  N. Dyn, D. Leviatan, D. Levin and A. Pinkus, *Multivariate Approximation and Applications*, Cambridge University Press, 2001.

[8]  G. Farin, *Triangular Bernstein-Bézier*, Comput. Aided Geom. Des., 3(2) (1986), pp. 83–127.

[9]  G. Farin, *Curves and Surfaces for Computer Aided Geometric Design, A Practical Guide*, Academic Press, Boston, 1992.

[10]  D. Hong, *Spaces of bivariate spline functions over triangulations*, Approx. Theory Appl., 7 (1991), pp. 56–75.

[11]  P. Lamberti, *Numerical integration based on bivariate quadratic spline quasi-interpolants on bounded domains*, BIT Numer. Math., 49 (2009), pp. 565–588.

[12]  C. J. Li and R. H. Wang,, *Bivariate cubic spline space and bivariate cubic NURBS surfaces*, Proceedings of Geometric Modeling and Processing 2004, IEEE Computer Society Pressvol, pp. 115–123.

[13]  C. J. Li and J. Chen, *On the dimensions of bivariate spline spaces and their stability*, J. Comput. Appl. Math., 236 (2011), pp. 765–774.

[14]  H. W. Liu, D. Hong and D. Q. Cao, *Bivariate $C^1$ Cubic Spline Space Over a Nonuniform Type-2 Triangulation and its Subspaces with Boundary Conditions*, J. Comput. Math. Appl., 49 (2005), pp. 1853–1865.

[15]  L. Piegl and W. Tiller,, *The NURBS Book*, Spring-Verlag, Berlin, 1997.

[16]  J. Qian and Y. H. Tang, *On non-uniform algebraic-hyperbolic (NUAH) B-splines*, Numer. Math. A Journal of Chinese Universities, 15(4) (2006), pp. 320–335.

[17]  L. L. Schumaker, *On the dimension of spaces of piecewise polynomials in two variables*, Multivariate Approximation Theory, W. Schempp and K. Zeller (Eds.), Birkhauser, Basel, (1979), pp. 396–412 .

[18]  L. L. Schumaker, *Spline Functions: Basic Theory*, Krieger Publishing Company, Malabar FL, 1993.

[19]  X. Q. Shi, *The singularity of Morgan-Scott triangulation*, Comput. Aided Geom. Design, 8 (1991), pp. 201–206.

[20]  G. J Wang, G. Z. Wang and J. M. Zheng,, *Computer Aided Geometric Design*, China Higher Education Press/Spring-Verlag Berlin, Beijing/Heidelberg, 2001, (in Chinese).

[21]  G. Z. Wang, Q. Y. Chen and M. H. Zhou, *NUAT B-spline Curves*, Comput. Aided Geom. Design, 21 (2004), pp. 193–205.

[22]  R. H. Wang, *The structural characterization and interpolation for multivariate splines*, Acta Math. Sinica, 18 (1975), pp. 91–106, (in Chinese).

[23]  R. H. Wang, X. Q. Shi, Z. X. Luo and Z. X. Su, *Multivariate Spline Functions and Their Applications*, Science Press/Kluwer Academic Publishers, Beijing, New York, Dordrecht, Boston, London, 2001.

[24]  R. H. Wang and C. J. Li, *Bivariate quartic spline spaces and quasi-interpolation operators*, J. Comput. Appl. Math., 190 (2006), pp. 325–338.

[25]  R. H. Wang and Y. Lu, *Quasi-interpolating operators and their applications in hypersingular integrals*, J. Comput. Math., 16(4) (1998), pp. 337–344.

[26]  R. H. Wang and Y. Lu, *Quasi-interpolating operators in $S_2^1(\Delta_{mn}^2*)$ on non-uniform type-2 tri-*

*angulations*, Numer. Math. A Journal of Chinese Universities, 2 (1999), pp. 97–103.

[27] S. M. WANG, *Spline interpolation over type-2 triangulations*, Appl. Math. Comput., 49 (1992), pp. 299–313.

[28] S. M. WANG AND C. L. WANG, *Smooth interpolation on some triangulations*, Utilitas Math., 41 (1992), pp. 309–317.

[29] Z. Q. XU AND R. H. WANG, *The instability degree in the dimension of spaces of bivariate spline*, Approx. Theory Appl., 18(1) (2002), pp. 68–80.