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A Spectral Method for Neutral Volterra Integro-Differential Equation with Weakly Singular Kernel

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Abstract. This paper is concerned with obtaining an approximate solution and an approximate derivative of the solution for neutral Volterra integro-differential equation with a weakly singular kernel. The solution of this equation, even for analytic data, is not smooth on the entire interval of integration. The Jacobi collocation discretization is proposed for the given equation. A rigorous analysis of error bound is also provided which theoretically justifies that both the error of approximate solution and the error of approximate derivative of the solution decay exponentially in L^∞ norm and weighted L^2 norm. Numerical results are presented to demonstrate the effectiveness of the spectral method.

AMS subject classifications: 65R20, 45J05, 65N12

Key words: Neutral Volterra integro-differential equation, weakly singular kernel, Jacobi collocation discretization, convergence analysis.

1. Introduction

We study the neutral Volterra integro-differential equation (VIDE) of the form

$$
\begin{aligned}
y'(t) = a(t)y(t) + b(t) + \int_0^t (t-s)^{-\frac{1}{2}}[K_0(t,s)y(s) \\
+ K_1(t,s)y'(s)]ds, \quad t \in [0,T], \qquad (1.1a) \\
y(0) = y_0, \qquad (1.1b)
\end{aligned}
$$

by the Jacobi spectral collocation method. Here, $a, b : [0,T] \to R$ and $K_0, K_1 : D \to R$ (where $D := \{(t,s) : 0 \le s \le t \le T\}$) are given smooth functions (see [5]). As for

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the numerical treatment of VIDE, the reader is referred [2–4, 26, 32, 36] (VIDE with a regular kernel) and [6, 7, 12, 23, 28, 30] (VIDE with a weakly singular kernel), most of these references contain information about other relevant papers.

There are many existing numerical methods for solving VIDE, such as polynomial collocation method [3,7,23,26,27,33], Taylor series method [13], block-by-block method [18, 19], multistep method [21,35] and Runge-Kutta method [2,36]. However, very few works touched the spectral approximation to VIDE. Spectral method has excellent error properties with the so-called "exponential convergence" being the fastest possible. The literature [31] is the first paper proposed a spectral method for Volterra integral equation with a smooth kernel. Subsequently, Y. Chen and T. Tang developed the spectral method for Volterra integral equation with weakly singular kernel in [9, 10].

Usually, the numerical analysis for weakly singular VIDE without neutral term (i.e., $K_1(t,s)=0$) can be based on either of two second-kind Volterra integral equations that are equivalent to the original initial-value problem (1.1a)-(1.1b) (see [5,6]). Its first reformulation has the form

$$y(t)=f(t)+\int_0^t H_1(t,s)y(s)ds, \quad t\in[0,T], \tag{1.2}$$

where

$$f(t)=y_0+\int_0^t b(s)ds, \tag{1.3a}$$

$$H_1(t,s)=a(s)+\int_s^t (v-s)^{-\frac{1}{2}}K_0(v,s)dv. \tag{1.3b}$$

Alternatively, we may consider the equivalent Volterra integral equation for $z(t)=y'(t)$, namely,

$$z(t)=g(t)+\int_0^t H_2(t,s)z(s)ds, \quad t\in[0,T], \tag{1.4}$$

with

$$g(t)=b(t)+\Big(a(t)+\int_0^t (t-s)^{-\frac{1}{2}}K_0(t,s)ds\Big)y_0, \tag{1.5a}$$

$$H_2(t,s)=a(t)+\int_s^t (t-v)^{-\frac{1}{2}}K_0(t,v)dv. \tag{1.5b}$$

But (1.2) and (1.4) are not much suitable for the spectral method since the kernels defined in (1.3b) and (1.5b) have new singularities along the lines $s=0$ and $t=0$, respectively, in addition to those for $s=t$ admitted in [34].

In [15], Y. Jiang considers the Legendre spectral collocation method for VIDE with a smooth kernel, which has smooth solution on the entire interval of integration $[0,T]$ if the given data are sufficiently smooth. The paper [15] only shows the spectral rate of convergence of the approximate solution in L^∞ norm based on the integration of both sides of VIDE. Let us now turn our attention toward the neutral VIDE with a weakly singular kernel $(t-s)^{-\mu}$ $(0<\mu<1)$. It can be shown using the techniques in [17] (see also [5]) that if the given functions have continuous derivatives of order m then there exists a function $Y=Y(t,v)$ possessing continuous derivatives of order $m+1$, such that the solution can be written as $y(t)=Y(t,t^{2-\mu})$ for $t\in[0,T]$, $T>0$. It is known that the weakly singular factor $(t-s)^{-\mu}$ complicates the numerical treatment of VIDE since the solution will be such that $y\in\mathscr{C}^1([0,T])$ but $y''(t)\approx t^{-\mu}$ as $t\to 0$. The case $\mu=1/2$ is encountered in a variety of problems in physics and chemistry (see [1]). In this case, we can use the variable transformations $t=z^2$ and $s=w^2$, so the solution of the new neutral VIDE can be written as $v(z)=y(z^2)=Y(z^2,z^3)$ which is smooth. Thus, the Jacobi spectral collocation method can be applied accordingly.

In the present paper, we will restate the initial condition as an equivalent integral equation instead of integrating both sides of (1.1a). Then, we get the discrete scheme by using Gauss quadrature formula for all the integral terms. We will provide a rigorous error analysis not only for approximate solution but also for approximate derivative of the solution in L^∞ norm and weighted L^2 norm and justify the spectral rate of convergence in both cases. This paper extends spectral method to a wider class of equations than in previous work.

The remainder of the paper is organized as follows. Jacobi collocation discretization for the neutral VIDE (1.1a) is presented in Section 2, and some lemmas useful for establishing the convergence results are given in Section 3. In Section 4 the convergence analysis is outlined, and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with conclusion and future work.

2. Jacobi collocation discretization

Let $\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$, for $\alpha,\beta>-1$ denote a weight function in the usual sense. It is well known, the set of Jacobi polynomial $\{J_N^{\alpha,\beta}\}_{N=0}^\infty$ forms a complete $L^2_{\omega^{\alpha,\beta}}(I)$ orthogonal system, where I stands for the open interval $(-1,1)$ and $L^2_{\omega^{\alpha,\beta}}(I)$ is the space of functions u with $\|u\|_{L^2_{\omega^{\alpha,\beta}}(I)}<+\infty$, equipped with the norm

$$\|u\|_{L^2_{\omega^{\alpha,\beta}}(I)}=\left(\int_{-1}^1|u(x)|^2\omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta}}=\int_{-1}^1 u(x)v(x)\omega^{\alpha,\beta}(x)dx,\quad \forall u,v\in L^2_{\omega^{\alpha,\beta}}(I).$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Jacobi Gauss points, and by $\{v_i\}_{i=0}^N$ the corresponding weights. Thus, the

Jacobi-Gauss quadrature formula is

$$\int_{-1}^{1} f(x)\omega^{\alpha,\beta}(x)dx \approx \sum_{k=0}^{N} f(x_k)v_k. \tag{2.1}$$

In particular, we give the Gauss quadrature formula as follows:

$$\int_{-1}^{1} f(x)(1-x)^{-\frac{1}{2}}dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k, \tag{2.2a}$$

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N} f(\tilde{\theta}_k)\tilde{\omega}_k. \tag{2.2b}$$

In order to obtain the smooth solution, we use the variable transformations $t = z^2$ and $s = w^2$, Eq. (1.1a) becomes

$$\begin{aligned} v'(z) = \hat{a}(z)v(z) + \hat{b}(z) + &\int_{0}^{z}(z-w)^{-\frac{1}{2}}\hat{K}_0(z,w)v(w)dw \\ &+ \int_{0}^{z}(z-w)^{-\frac{1}{2}}\hat{K}_1(z,w)v'(w)dw, \quad z \in [0,\sqrt{T}], \end{aligned} \tag{2.3a}$$

$$v(0) = y_0, \tag{2.3b}$$

where

$$\hat{a}(z) = 2za(z^2), \qquad \hat{b}(z) = 2zb(z^2), \tag{2.4a}$$

$$\hat{K}_0(z,w) = 4(z+w)^{-\frac{1}{2}}zwK_0(z^2,w^2), \qquad \hat{K}_1(z,w) = 2(z+w)^{-\frac{1}{2}}zK_1(z^2,w^2), \tag{2.4b}$$

and $v(z) = y(z^2)$ is the smooth solution of problem (2.3a)-(2.3b). For the sake of applying the theory of orthogonal polynomials, we use the variable transformations $z = \sqrt{T}(1 + x)/2$, $x \in [-1,1]$ and $w = \sqrt{T}(1+\tau)/2$, $\tau \in [-1,x]$ to rewrite (2.3a) as follows

$$\begin{aligned} u'(x) = \tilde{a}(x)u(x) + \tilde{b}(x) + &\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)u(\tau)d\tau \\ &+ \int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)u'(\tau)d\tau, \quad x \in [-1,1], \end{aligned} \tag{2.5}$$

with the initial condition

$$u(-1) = u_{-1} = y_0. \tag{2.6}$$

Here,

$$\tilde{a}(x) = \frac{\sqrt{T}}{2}\hat{a}\Big(\frac{\sqrt{T}}{2}(1+x)\Big), \qquad \tilde{b}(x) = \frac{\sqrt{T}}{2}\hat{b}\Big(\frac{\sqrt{T}}{2}(1+x)\Big), \tag{2.7a}$$

$$\tilde{K}_0(x,\tau) = \Big(\frac{\sqrt{T}}{2}\Big)^{\frac{3}{2}}\hat{K}_0\Big(\frac{\sqrt{T}}{2}(1+x), \frac{\sqrt{T}}{2}(1+\tau)\Big), \tag{2.7b}$$

$$\tilde{K}_1(x,\tau) = \Big(\frac{\sqrt{T}}{2}\Big)^{\frac{1}{2}}\hat{K}_1\Big(\frac{\sqrt{T}}{2}(1+x), \frac{\sqrt{T}}{2}(1+\tau)\Big), \tag{2.7c}$$

and $u(x) = v\big(\sqrt{T}(1+x)/2\big)$ is the smooth solution of problem (2.5)-(2.6). In order that the Jacobi collocation method is carried out naturally, we restate (2.6) as

$$u(x) = u_{-1} + \int_{-1}^{x} u'(\tau)d\tau. \tag{2.8}$$

Firstly, Eqs. (2.5) and (2.8) hold at the collocation points $\{x_i\}_{i=0}^N$ on $[-1,1]$, i.e.,

$$\begin{aligned} u'(x_i) = \tilde{a}(x_i)u(x_i) + \tilde{b}(x_i) + \int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}\tilde{K}_0(x_i,\tau)u(\tau)d\tau \\ + \int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}\tilde{K}_1(x_i,\tau)u'(\tau)d\tau, \end{aligned} \tag{2.9a}$$

$$u(x_i) = u_{-1} + \int_{-1}^{x_i} u'(\tau)d\tau. \tag{2.9b}$$

In order to obtain high order accuracy for the problem (2.9a)-(2.9b), the main difficulty is to compute the integral terms. In particular, for small values of x_i, there is little information available for $u(\tau)$ and $u'(\tau)$. To overcome this difficulty, we transfer the integral interval $[-1,x_i]$ to a fixed interval $[-1,1]$

$$\begin{aligned} u'(x_i) = \tilde{a}(x_i)u(x_i) + \tilde{b}(x_i) + \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\int_{-1}^{1}(1-\theta)^{-\frac{1}{2}}\tilde{K}_0(x_i,\tau(x_i,\theta))u(\tau(x_i,\theta))d\theta \\ + \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\int_{-1}^{1}(1-\theta)^{-\frac{1}{2}}\tilde{K}_1(x_i,\tau(x_i,\theta))u'(\tau(x_i,\theta))d\theta, \end{aligned} \tag{2.10a}$$

$$u(x_i) = u_{-1} + \frac{1+x_i}{2}\int_{-1}^{1} u'(\tau(x_i,\theta))d\theta, \tag{2.10b}$$

by using the following variable change

$$\tau = \tau(x_i,\theta) = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}, \quad \theta\in[-1,1]. \tag{2.11}$$

Next, let u_i, u_i' be the approximation of the function value $u(x_i)$, $u'(x_i)$, respectively, and use Gauss quadrature formulas (2.2a) and (2.2b), (2.10a)-(2.10b) becomes

$$\begin{aligned} u_i' = \tilde{a}(x_i)u_i + \tilde{b}(x_i) + \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\sum_{k=0}^{N}\tilde{K}_0(x_i,\tau(x_i,\theta_k))u(\tau(x_i,\theta_k))\omega_k \\ + \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\sum_{k=0}^{N}\tilde{K}_1(x_i,\tau(x_i,\theta_k))u'(\tau(x_i,\theta_k))\omega_k, \end{aligned} \tag{2.12a}$$

$$u_i = u_{-1} + \frac{1+x_i}{2}\sum_{k=0}^{N}u'(\tau(x_i,\tilde{\theta}_k))\tilde{\omega}_k. \tag{2.12b}$$

Denote

$$\tilde{u}_N(x)=\sum_{j=0}^{N}u_jF_j(x) \quad \text{and} \quad \tilde{u}'_N(x)=\sum_{j=0}^{N}u'_jF_j(x)$$

(although $\tilde{u}'_N(x)$ differs from the exact derivative of $\tilde{u}_N(x)$, we still use this notation), where $F_j(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{x_j\}_{j=0}^N$. Now, we use $\tilde{u}_N(x)$ to approximate the solution $u(x)$ and use $\tilde{u}'_N(x)$ to approximate the first derivative $u'(x)$ of the solution $u(x)$, namely, $u(x)\approx\tilde{u}_N(x)$, $u'(x)\approx\tilde{u}'_N(x)$. Then, the Jacobi spectral collocation method is to seek $\tilde{u}_N(x)$, $\tilde{u}'_N(x)$ such that $\{u_i\}_{i=0}^N$, $\{u'_i\}_{i=0}^N$ satisfy the following collocation equations:

$$\begin{aligned} u'_i=&\tilde{a}(x_i)u_i+\tilde{b}(x_i)+\Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\sum_{j=0}^{N}u_j\left(\sum_{k=0}^{N}\tilde{K}_0(x_i,\tau(x_i,\theta_k))F_j(\tau(x_i,\theta_k))\omega_k\right)\\ &+\Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}}\sum_{j=0}^{N}u'_j\left(\sum_{k=0}^{N}\tilde{K}_1(x_i,\tau(x_i,\theta_k))F_j(\tau(x_i,\theta_k))\omega_k\right), \end{aligned} \tag{2.13a}$$

$$u_i=u_{-1}+\frac{1+x_i}{2}\sum_{j=0}^{N}u'_j\left(\sum_{k=0}^{N}F_j(\tau(x_i,\tilde{\theta}_k))\tilde{\omega}_k\right). \tag{2.13b}$$

Let the error function of the solution be written as $e_u(x):=u(x)-\tilde{u}_N(x)$ and the error function of the first derivative of the solution has the form $e_{u'}(x):=u'(x)-\tilde{u}'_N(x)$. Since the exact solution of the initial-value problem (1.1a)-(1.1b) can be written as $y(t)=u(x)$, $t=\frac{T}{4}(1+x)^2$, $t\in[0,T]$, $x\in[-1,1]$, we can define its approximate solution $\tilde{y}_N(t)=\tilde{u}_N(x)$ and the approximate derivative of the solution $\tilde{y}'_N(t)=\frac{2}{T(1+x)}\tilde{u}'_N(x)$. Then the corresponding error functions satisfy

$$\varepsilon_y(t):=y(t)-\tilde{y}_N(t)=e_u(x)=e_u\Big(2\sqrt{\frac{t}{T}}-1\Big), \tag{2.14a}$$

$$\varepsilon_{y'}(t):=y'(t)-\tilde{y}'_N(t)=\frac{2}{T(1+x)}e_{u'}(x)=\frac{1}{\sqrt{Tt}}e_{u'}\Big(2\sqrt{\frac{t}{T}}-1\Big). \tag{2.14b}$$

By using variable transformation $t=T(1+x)^2/4$, we obtain the following equalities

$$\int_0^T(y(t)-\tilde{y}_N(t))^2\Big(2-2\frac{\sqrt{t}}{\sqrt{T}}\Big)^{\alpha}\Big(2\frac{\sqrt{t}}{\sqrt{T}}\Big)^{\beta}\frac{1}{\sqrt{Tt}}dt=\|u-\tilde{u}_N\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{2.15a}$$

$$\|y(t)-\tilde{y}_N(t)\|_{L^\infty(0,T)}=\|u(x)-\tilde{u}_N(x)\|_{L^\infty(I)}, \tag{2.15b}$$

$$\int_0^T(y'(t)-\tilde{y}'_N(t))^2\Big(2-2\frac{\sqrt{t}}{\sqrt{T}}\Big)^{\alpha}\Big(2\frac{\sqrt{t}}{\sqrt{T}}\Big)^{\beta}\sqrt{Tt}dt=\|u'-\tilde{u}'_N\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{2.15c}$$

$$\|\sqrt{Tt}y'(t)-\sqrt{Tt}y'_N(t)\|_{L^\infty(0,T)}=\|u'(x)-\tilde{u}'_N(x)\|_{L^\infty(I)}. \tag{2.15d}$$

Finally, we give the equations of the matrix form from (2.13a)-(2.13b). Writing $U_N = (u_0, u_1, \cdots, u_N)^T$ and $U'_N = (u'_0, u'_1, \cdots, u'_N)^T$, yield

$$(I-D)U'_N - (A+C)U_N = b_N, \tag{2.16a}$$

$$-BU'_N + U_N = U_{-1}, \tag{2.16b}$$

where

$$C = \mathrm{diag}(\tilde{a}(x_0), \tilde{a}(x_1), \cdots, \tilde{a}(x_N)),$$

$$b_N = (\tilde{b}(x_0), \tilde{b}(x_1), \cdots, \tilde{b}(x_N))^T, \qquad U_{-1} = u_{-1} \times (1, 1, \cdots, 1)^T.$$

The entries of the matrices are given by

$$A_{ij} = \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}} \sum_{k=0}^{N} \tilde{K}_0(x_i, \tau(x_i, \theta_k)) F_j(\tau(x_i, \theta_k))\omega_k,$$

$$B_{ij} = \frac{1+x_i}{2} \sum_{k=0}^{N} F_j(\tau(x_i, \tilde{\theta}_k))\tilde{\omega}_k,$$

$$D_{ij} = \Big(\frac{1+x_i}{2}\Big)^{\frac{1}{2}} \sum_{k=0}^{N} \tilde{K}_1(x_i, \tau(x_i, \theta_k)) F_j(\tau(x_i, \theta_k))\omega_k.$$

We can get the values of $\{u_i\}_{i=0}^N$ and $\{u'_i\}_{i=0}^N$ by solving (2.16a)-(2.16b) and obtain the expressions of $\tilde{u}_N(x)$ and $\tilde{u}'_N(x)$ accordingly.

3. Some preliminaries and useful lemmas

In this section, we will provide some elementary preliminaries and lemmas, which are important for the derivation of the main convergence results in the subsequent section.

Let $\mathscr{P}_N$ be the space of all polynomials of degree not exceeding N. For any $v \in \mathscr{C}([-1,1])$ we can define the Lagrange interpolation polynomial $I_N^{\alpha,\beta} v(x) \in \mathscr{P}_N$, satisfying

$$I_N^{\alpha,\beta} v(x_i) = v(x_i), \quad 0 \le i \le N,$$

see [8, 29]. It can be written as an expression of the form

$$I_N^{\alpha,\beta} v(x) = \sum_{i=0}^{N} v(x_i) F_i(x).$$

Let $H^m_{\omega^{\alpha,\beta}}(I)$ denote the Sobolev space of all functions $u(x)$ on I such that $u(x)$ and all its weak derivatives up to order m are in $L^2_{\omega^{\alpha,\beta}}(I)$, with the norm and the semi-norm as

$$\|u\|_{H^m_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{k=0}^{m} \|u^{(k)}(x)\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}\right)^{\frac{1}{2}}, \tag{3.1a}$$

$$|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|u^{(k)}(x)\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}\right)^{\frac{1}{2}}, \tag{3.1b}$$

where $u^{(k)}(x)=(\partial^k/\partial x^k)u(x)$.

The following result can be found in [8].

Lemma 3.1. *Assume that a $(N+1)$-point Gauss quadrature formula relative to the Jacobi weight is used to integrate the product $v\phi$, where $v\in H^m_{\omega^{\alpha,\beta}}(I)$ for some $m\geq 1$ and $\phi\in\mathscr{P}_N$. Then there exists a constant C independent of N such that*

$$|(v,\phi)_{\omega^{\alpha,\beta}}-(v,\phi)_N|\leq CN^{-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}\|\phi\|_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{3.2}$$

where

$$(v,\phi)_N=\sum_{i=0}^{N}v(x_i)\phi(x_i)v_i. \tag{3.3}$$

We have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials, see, e.g., [20].

Lemma 3.2. *Let $\{F_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Jacobi Gauss points $\{x_i\}_{i=0}^N$ and $\gamma=\max(\alpha,\beta)$. Then*

$$\|I_N^{\alpha,\beta}\|_\infty:=\max_{x\in[-1,1]}\sum_{j=0}^{N}|F_j(x)|=\begin{cases}\mathscr{O}(\log N), & -1<\alpha,\beta\leq-\frac{1}{2},\\ \mathscr{O}\big(N^{\gamma+\frac{1}{2}}\big), & \text{otherwise.}\end{cases} \tag{3.4}$$

Lemma 3.3. *Assume that $v\in H^m_{\omega^{\alpha,\beta}}(I)$ and denote $I_N^{\alpha,\beta}v$ its interpolation polynomial associated with the Jacobi Gauss points $\{x_i\}_{i=0}^N$. Then the following estimates hold*

$$(1)\quad \|v-I_N^{\alpha,\beta}v\|_{L^2_{\omega^{\alpha,\beta}}(I)}\leq CN^{-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)},$$

$$(2)\quad \|v-I_N^{\alpha,\beta}v\|_{L^\infty(I)}\leq\begin{cases}CN^{\frac{1}{2}-m}\log N|v|_{H^{m;N}_{\omega^c}(I)}, & -1<\alpha,\beta\leq-\frac{1}{2},\\ CN^{\gamma+1-m}|v|_{H^{m;N}_{\omega^c}(I)}, & \text{otherwise,}\end{cases}$$

where $\omega^c=\omega^{-1/2,-1/2}$.

Proof. The inequality (1) can be found in [8]. We only prove (2).

Let $I_N^c v\in\mathscr{P}_N$ denote the interpolant of v at Chebyshev Gauss points. From (5.5.28) in [8], the interpolation error estimate in the maximum norm is given by

$$\|v-I_N^c v\|_{L^\infty(I)}\leq CN^{\frac{1}{2}-m}|v|_{H^{m;N}_{\omega^c}(I)}. \tag{3.6}$$

Note that

$$I_N^{\alpha,\beta}p(x)=p(x),\quad i.e.,\quad (I_N^{\alpha,\beta}-I)p(x)=0,\quad \forall p(x)\in\mathscr{P}_N. \tag{3.7}$$

By using (3.7), Lemma 3.2 and (3.6), we obtain that

$$
\begin{aligned}
\|v-I_N^{\alpha,\beta}v\|_{L^\infty(I)} =&\|v-I_N^c v+I_N^{\alpha,\beta}(I_N^c v)-I_N^{\alpha,\beta}v\|_{L^\infty(I)}\\
\leq&\|v-I_N^c v\|_{L^\infty(I)}+\|I_N^{\alpha,\beta}(I_N^c v-v)\|_{L^\infty(I)}\\
\leq&(1+\|I_N^{\alpha,\beta}\|_\infty)\|v-I_N^c v\|_{L^\infty(I)}\\
\leq&\begin{cases} CN^{\frac{1}{2}-m}\log N|v|_{H^{m;N}_{\omega^c}(I)}, & -1<\alpha,\beta\leq -\frac{1}{2},\\ CN^{\gamma+1-m}|v|_{H^{m;N}_{\omega^c}(I)}, & \text{otherwise}.\end{cases}
\end{aligned}
\tag{3.8}
$$

The proof is complete. □

The following generalization of Gronwall lemma for singular kernels, whose proof can be found in [14] (Lemma 7.1.1), will be essential for establishing our main results.

Lemma 3.4. *Suppose $M,L\geq 0$. If a nonnegative integrable function $E(x)$ satisfies*

$$
E(x)\leq M\int_{-1}^{x}E(\tau)d\tau+L\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}E(\tau)d\tau+J(x),\quad -1<x\leq 1, \tag{3.9}
$$

where $J(x)$ is an integrable function, then

$$
\|E\|_{L^\infty(I)}\leq C\|J\|_{L^\infty(I)}, \tag{3.10a}
$$

$$
\|E\|_{L^p_{\omega^{\alpha,\beta}}(I)}\leq C\|J\|_{L^p_{\omega^{\alpha,\beta}}(I)},\quad p\geq 1. \tag{3.10b}
$$

We shall make use of a result of [24,25] in the following lemma.

Lemma 3.5. *For nonnegative integer r and $\kappa\in(0,1)$, there exists a constant $C_{r,\kappa}>0$ such that for any function $v\in\mathscr{C}^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathscr{T}_N v\in\mathscr{P}_N$ such that*

$$
\|v-\mathscr{T}_N v\|_{L^\infty(I)}\leq C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa}, \tag{3.11}
$$

where $\|\cdot\|_{r,\kappa}$ is the standard norm in $\mathscr{C}^{r,\kappa}([-1,1])$. Actually, as stated in [24,25], $\mathscr{T}_N$ is a linear operator from $\mathscr{C}^{r,\kappa}([-1,1])$ into $\mathscr{P}_N$.

The proof of the following lemma can be found in [10]. A similar result can be found in Theorem 3.4 of [11].

Lemma 3.6. *Let $\kappa\in(0,1)$ and $\mathscr{M}$ be defined by*

$$
(\mathscr{M}v)(x)=\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}(x,\tau)v(\tau)d\tau, \tag{3.12}
$$

where $\tilde{K}(x,\tau)=\tilde{K}_0(x,\tau)$ or $\tilde{K}(x,\tau)=\tilde{K}_1(x,\tau)$. Then, for any function $v\in\mathscr{C}([-1,1])$, there exists a positive constant C such that

$$
\frac{|\mathscr{M}v(x')-\mathscr{M}v(x'')|}{|x'-x''|^\kappa}\leq C\max_{x\in[-1,1]}|v(x)|, \tag{3.13}
$$

under the assumption that $0<\kappa<1/2$*, for any* $x',x''\in[-1,1]$ *and* $x'\neq x''$*. This implies that*

$$\|\mathscr{M}v\|_{0,\kappa}\le C\max_{x\in[-1,1]}|v(x)|,\quad 0<\kappa<\frac{1}{2}. \tag{3.14}$$

4. Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximation. We will carry our convergence analysis in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ space.

4.1. Error estimate in L^∞ norm

Before we state the main results, the following regularity result of the kernel functions $\tilde{K}_0$ and $\tilde{K}_1$ need to be proved.

Lemma 4.1. *Let* $\{x_i\}_{i=0}^N$ *be the set of* $(N+1)$ *Jacobi Gauss points. Then, we have that*

$$\partial_\theta^m\tilde{K}_0(x_i,\tau(x_i,\theta))\in L^2_{\omega^{-1/2,0}}(I),\quad \partial_\theta^m\tilde{K}_1(x_i,\tau(x_i,\theta))\in L^2_{\omega^{-1/2,0}}(I).$$

Thus, it is reasonable to denote

$$\begin{aligned}K^*=&\max_{0\le i\le N}|\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}\|u\|_{L^2_{\omega^{-1/2,0}}(I)}\\&+\max_{0\le i\le N}|\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}\|u'\|_{L^2_{\omega^{-1/2,0}}(I)}.\end{aligned} \tag{4.1}$$

Here, $\tau(x_i,\theta)$ *is given by* (2.11)*,* $\tilde{K}_0(x,\tau)$ *and* $\tilde{K}_1(x,\tau)$ *are defined by* (2.7)*.*

Proof. It directly follows from (2.4) and (2.7) that

$$\begin{aligned}&\tilde{K}_0(x_i,\tau(x_i,\theta))\\=&\Big(\frac{T}{2}\Big)^{\frac{3}{2}}(1+x_i)^{\frac{3}{2}}(3+\theta)^{-\frac{1}{2}}(1+\theta)K_0\Big(\frac{T}{4}(1+x_i)^2,\frac{T}{16}(1+x_i)^2(1+\theta)^2\Big),\\&\tilde{K}_1(x_i,\tau(x_i,\theta))\\=&(2T)^{\frac{1}{2}}(1+x_i)^{\frac{1}{2}}(3+\theta)^{-\frac{1}{2}}K_1\Big(\frac{T}{4}(1+x_i)^2,\frac{T}{16}(1+x_i)^2(1+\theta)^2\Big).\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}\partial_\theta^m\tilde{K}_0(x_i,\tau(x_i,\theta))=P(x_i,\theta)(3+\theta)^{-\frac{1}{2}-m},\\\partial_\theta^m\tilde{K}_1(x_i,\tau(x_i,\theta))=Q(x_i,\theta)(3+\theta)^{-\frac{1}{2}-m},\end{aligned}$$

where $P(x_i,\theta)$ and $Q(x_i,\theta)$ are continuous functions with respect to θ on the interval $[-1,1]$. Hence, we only need to prove $(3+\theta)^{-1/2-m}\in L^2_{\omega^{-1/2,0}}(I)$, use the transformation $\theta=1-v^2$ to derive that

$$\left\|(3+\theta)^{-\frac{1}{2}-m}\right\|^2_{L^2_{\omega^{-1/2,0}}(I)}=\int_{-1}^{1}(3+\theta)^{-1-2m}(1-\theta)^{-\frac{1}{2}}d\theta=2\int_0^{\sqrt{2}}\frac{1}{(4-v^2)^{1+2m}}dv.$$

For the given integer $m\geq 1$, applying the recurrence relation

$$\int\frac{1}{(4-v^2)^{1+2m}}dv=\frac{1}{16m}\left(\frac{x}{(4-v^2)^{2m}}+(4m-1)\int\frac{1}{(4-v^2)^{2m}}dv\right),$$

we can obtain

$$\left\|(3+\theta)^{-\frac{1}{2}-m}\right\|_{L^2_{\omega^{-1/2,0}}(I)}<+\infty.$$

The lemma is proved. □

Theorem 4.1. *Let $u(x)$ be the exact solution of the neutral Volterra integro-differential equation* (2.5) *with* (2.6)*, which is smooth. Assume that $\tilde{u}_N(x)$ is the approximate solution and $\tilde{u}'_N(x)$ is the approximate derivative of the solution, i.e., $u(x)\approx\tilde{u}_N(x)$, $u'(x)\approx\tilde{u}'_N(x)$. If $\gamma=\max(\alpha,\beta)<0$, then the errors $u(x)-\tilde{u}_N(x)$ and $u'(x)-\tilde{u}'_N(x)$ satisfy for $m\geq 1$:*

$$\|u-\tilde{u}_N\|_{L^\infty(I)}\leq\begin{cases}CN^{-m}\log N\left(K^*+N^{\frac{1}{2}}U\right), & -1<\alpha,\beta\leq-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m}\left(K^*+N^{\frac{1}{2}}U\right), & \text{otherwise},\end{cases}\tag{4.2a}$$

$$\|u'-\tilde{u}'_N\|_{L^\infty(I)}\leq\begin{cases}CN^{-m}\log N\left(K^*+N^{\frac{1}{2}}U\right), & -1<\alpha,\beta\leq-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m}\left(K^*+N^{\frac{1}{2}}U\right), & \text{otherwise},\end{cases}\tag{4.2b}$$

provided that N is sufficiently large, where K^* is defined by (4.1)*,*

$$U=|u|_{H^{m;N}_{\omega^c}(I)}+|u'|_{H^{m;N}_{\omega^c}(I)}\tag{4.3}$$

and C is a constant independent of N.

Proof. First, we use the weighted inner product to rewrite (2.10a) as

$$\begin{aligned}u'(x_i)=\tilde{a}(x_i)u(x_i)+\tilde{b}(x_i)&+\left(\frac{1+x_i}{2}\right)^{\frac{1}{2}}(\tilde{K}_0(x_i,\tau(x_i,\cdot)),u(\tau(x_i,\cdot)))_{\omega^{-1/2,0}}\\&+\left(\frac{1+x_i}{2}\right)^{\frac{1}{2}}\left(\tilde{K}_1(x_i,\tau(x_i,\cdot)),u'(\tau(x_i,\cdot))\right)_{\omega^{-1/2,0}}.\end{aligned}\tag{4.4}$$

By using the discrete inner product, we set

$$\big(R(x_i,\tau(x_i,\cdot)),\phi(\tau(x_i,\cdot))\big)_N=\sum_{k=0}^{N}R(x_i,\tau(x_i,\theta_k))\phi(\tau(x_i,\theta_k))\omega_k.$$

Then, the numerical scheme (2.13a)-(2.13b) can be written as

$$\begin{aligned}u_i'=&\tilde a(x_i)u_i+\tilde b(x_i)+\Big(\frac{1+x_i}{2}\Big)^{\frac12}\big(\tilde K_0(x_i,\tau(x_i,\cdot)),\tilde u_N(\tau(x_i,\cdot))\big)_N\\&+\Big(\frac{1+x_i}{2}\Big)^{\frac12}\big(\tilde K_1(x_i,\tau(x_i,\cdot)),\tilde u_N'(\tau(x_i,\cdot))\big)_N,\end{aligned}\tag{4.5a}$$

$$u_i=u_{-1}+\int_{-1}^{x_i}\tilde u_N'(\tau)d\tau,\tag{4.5b}$$

by using the following equality

$$\begin{aligned}\int_{-1}^{x_i}\tilde u_N'(\tau)d\tau=&\int_{-1}^{x_i}\sum_{j=0}^{N}u_j'F_j(\tau)d\tau=\frac{1+x_i}{2}\int_{-1}^{1}\sum_{j=0}^{N}u_j'F_j(\tau(x_i,\theta))d\theta\\=&\frac{1+x_i}{2}\sum_{j=0}^{N}u_j'\Big(\sum_{k=0}^{N}F_j(\tau(x_i,\tilde\theta_k))\tilde\omega_k\Big).\end{aligned}\tag{4.6}$$

We now subtract (4.5a) from (2.10a) and subtract (4.5b) from (2.9b) to get the error equations:

$$\begin{aligned}u'(x_i)-u_i'=&\tilde a(x_i)(u(x_i)-u_i)+\Big(\frac{1+x_i}{2}\Big)^{\frac12}\big(\tilde K_0(x_i,\tau(x_i,\cdot)),e_u(\tau(x_i,\cdot))\big)_{\omega^{-1/2,0}}\\&+\Big(\frac{1+x_i}{2}\Big)^{\frac12}\big(\tilde K_1(x_i,\tau(x_i,\cdot)),e_{u'}(\tau(x_i,\cdot))\big)_{\omega^{-1/2,0}}+I_1(x_i)+I_2(x_i)\\=&\tilde a(x_i)(u(x_i)-u_i)+\int_{-1}^{x_i}(x_i-\tau)^{-\frac12}\tilde K_0(x_i,\tau)e_u(\tau)d\tau\\&+\int_{-1}^{x_i}(x_i-\tau)^{-\frac12}\tilde K_1(x_i,\tau)e_{u'}(\tau)d\tau+I_1(x_i)+I_2(x_i),\end{aligned}\tag{4.7a}$$

$$u(x_i)-u_i=\int_{-1}^{x_i}e_{u'}(\tau)d\tau,\tag{4.7b}$$

where $e_u(x)=u(x)-\tilde u_N(x)$, $e_{u'}(x)=u'(x)-\tilde u_N'(x)$ are the error functions and

$$\begin{aligned}I_1(x_i)=&\Big(\frac{1+x_i}{2}\Big)^{\frac12}\Big(\big(\tilde K_0(x_i,\tau(x_i,\cdot)),\tilde u_N(\tau(x_i,\cdot))\big)_{\omega^{-1/2,0}}\\&-\big(\tilde K_0(x_i,\tau(x_i,\cdot)),\tilde u_N(\tau(x_i,\cdot))\big)_N\Big),\\I_2(x_i)=&\Big(\frac{1+x_i}{2}\Big)^{\frac12}\Big(\big(\tilde K_1(x_i,\tau(x_i,\cdot)),\tilde u_N'(\tau(x_i,\cdot))\big)_{\omega^{-1/2,0}}\\&-\big(\tilde K_1(x_i,\tau(x_i,\cdot)),\tilde u_N'(\tau(x_i,\cdot))\big)_N\Big).\end{aligned}$$

Using the integration error estimate in Lemma 3.1, we have

$$\begin{aligned}|I_1(x_i)| &\le CN^{-m}|\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}\|\tilde{u}_N(\tau(x_i,\cdot))\|_{L^2_{\omega^{-1/2,0}}(I)}\\ &\le CN^{-m}\max_{0\le i\le N}|\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}(\|u\|_{L^2_{\omega^{-1/2,0}}(I)}+\|e_u\|_{L^\infty(I)}), \qquad (4.8a)\\ |I_2(x_i)| &\le CN^{-m}|\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}\|\tilde{u}'_N(\tau(x_i,\cdot))\|_{L^2_{\omega^{-1/2,0}}(I)}\\ &\le CN^{-m}\max_{0\le i\le N}|\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}(\|u'\|_{L^2_{\omega^{-1/2,0}}(I)}+\|e_{u'}\|_{L^\infty(I)}). \qquad (4.8b)\end{aligned}$$

Substituting (4.7b) into (4.7a) gives

$$\begin{aligned}u'(x_i)-u'_i =&\tilde{a}(x_i)\int_{-1}^{x_i}e_{u'}(\tau)d\tau+\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}\tilde{K}_0(x_i,\tau)e_u(\tau)d\tau\\ &+\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}\tilde{K}_1(x_i,\tau)e_{u'}(\tau)d\tau+I_1(x_i)+I_2(x_i). \qquad (4.9)\end{aligned}$$

Multiplying $F_i(x)$ on both sides of Eqs. (4.9) and (4.7b) and summing up from $i=0$ to $i=N$ yield

$$\begin{aligned}I_N^{\alpha,\beta}u'(x)-\tilde{u}'_N(x)=&I_N^{\alpha,\beta}\left(\tilde{a}(x)\int_{-1}^{x}e_{u'}(\tau)d\tau\right)+J_1(x)+J_2(x)\\ &+I_N^{\alpha,\beta}\left(\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)e_u(\tau)d\tau\right)\\ &+I_N^{\alpha,\beta}\left(\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)e_{u'}(\tau)d\tau\right), \qquad (4.10a)\\ I_N^{\alpha,\beta}u(x)-\tilde{u}_N(x)=&I_N^{\alpha,\beta}\left(\int_{-1}^{x}e_{u'}(\tau)d\tau\right), \qquad (4.10b)\end{aligned}$$

where

$$J_1(x)=\sum_{i=0}^{N}I_1(x_i)F_i(x),\quad J_2(x)=\sum_{i=0}^{N}I_2(x_i)F_i(x).$$

Consequently,

$$\begin{aligned}e_{u'}(x)=&\tilde{a}(x)\int_{-1}^{x}e_{u'}(\tau)d\tau+\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)e_u(\tau)d\tau\\ &+\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)e_{u'}(\tau)d\tau+\sum_{i=1}^{6}J_i(x), \qquad (4.11a)\\ e_u(x)=&\int_{-1}^{x}e_{u'}(\tau)d\tau+J_7(x)+J_8(x), \qquad (4.11b)\end{aligned}$$

where

$$J_3(x)=u'(x)-I_N^{\alpha,\beta}u'(x),\quad J_4(x)=I_N^{\alpha,\beta}\left(\tilde{a}(x)\int_{-1}^{x}e_{u'}(\tau)d\tau\right)-\tilde{a}(x)\int_{-1}^{x}e_{u'}(\tau)d\tau,$$
$$J_5(x)=I_N^{\alpha,\beta}\left(\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)e_u(\tau)d\tau\right)-\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)e_u(\tau)d\tau,$$
$$J_6(x)=I_N^{\alpha,\beta}\left(\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)e_{u'}(\tau)d\tau\right)-\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)e_{u'}(\tau)d\tau,$$
$$J_7(x)=u(x)-I_N^{\alpha,\beta}u(x),\quad J_8(x)=I_N^{\alpha,\beta}\left(\int_{-1}^{x}e_{u'}(\tau)d\tau\right)-\int_{-1}^{x}e_{u'}(\tau)d\tau.$$

Due to Eqs. (4.11a)-(4.11b) and using the $Dirichlet's$ formula which states

$$\int_{-1}^{x}\int_{-1}^{\tau}\Phi(\tau,s)dsd\tau=\int_{-1}^{x}\int_{s}^{x}\Phi(\tau,s)d\tau ds$$

provided the integral exists, we obtain

$$e_{u'}(x)=\int_{-1}^{x}\left(\tilde{a}(x)+\int_{\tau}^{x}(x-s)^{-\frac{1}{2}}\tilde{K}_0(x,s)ds+(x-\tau)^{-\frac{1}{2}}\tilde{K}_1(x,\tau)\right)e_{u'}(\tau)d\tau$$
$$+H(x). \tag{4.12}$$

Here,

$$H(x)=\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}\tilde{K}_0(x,\tau)(J_7(\tau)+J_8(\tau))d\tau+\sum_{i=1}^{6}J_i(x).$$

Denote $\tilde{D}:=\{(x,\tau):-1\le x\le 1,-1\le\tau\le x\}$, we have

$$\left|\tilde{a}(x)+\int_{\tau}^{x}(x-s)^{-\frac{1}{2}}\tilde{K}_0(x,s)ds\right|$$
$$\le\max_{x\in[-1,1]}|\tilde{a}(x)|+2\sqrt{2}\max_{(x,s)\in\tilde{D}}|\tilde{K}_0(x,s)|\triangleq M.$$

Let $L=\max_{(x,\tau)\in\tilde{D}}|\tilde{K}_1(x,\tau)|$. Eq. (4.12) gives

$$|e_{u'}(x)|\le M\int_{-1}^{x}|e_{u'}(\tau)|d\tau+L\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}|e_{u'}(\tau)|d\tau+|H(x)|. \tag{4.13}$$

It follows from the Gronwall inequality in Lemma 3.4 that

$$\|e_{u'}\|_{L^\infty(I)}\le C\|H\|_{L^\infty(I)}\le C\sum_{i=1}^{8}\|J_i\|_{L^\infty(I)}. \tag{4.14}$$

It follows from (4.11b) that

$$\|e_u\|_{L^\infty(I)} \le 2\|e_{u'}\|_{L^\infty(I)} + \|J_7\|_{L^\infty(I)} + \|J_8\|_{L^\infty(I)}. \tag{4.15}$$

Using Lemma 3.2, the estimates (4.8a) and (4.8b), we have

$$\begin{aligned}
\|J_1\|_{L^\infty(I)} &\le \begin{cases} C\log N \max\limits_{0\le i\le N} |I_1(x_i)|, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}} \max\limits_{0\le i\le N} |I_1(x_i)|, & \text{otherwise}, \end{cases}\\
&\le \begin{cases} C\log N N^{-m} \max\limits_{0\le i\le N} |\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)} \\ \quad\times\Big(\|u\|_{L^2_{\omega^{-1/2,0}}(I)} + 2\|e_{u'}\|_{L^\infty(I)} + \|J_7\|_{L^\infty(I)} + \|J_8\|_{L^\infty(I)}\Big), & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m} \max\limits_{0\le i\le N} |\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)} \\ \quad\times\Big(\|u\|_{L^2_{\omega^{-1/2,0}}(I)} + 2\|e_{u'}\|_{L^\infty(I)} + \|J_7\|_{L^\infty(I)} + \|J_8\|_{L^\infty(I)}\Big), & \text{otherwise}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\|J_2\|_{L^\infty(I)} &\le \begin{cases} C\log N \max\limits_{0\le i\le N} |I_2(x_i)|, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}} \max\limits_{0\le i\le N} |I_2(x_i)|, & \text{otherwise}, \end{cases}\\
&\le \begin{cases} C\log N N^{-m} \max\limits_{0\le i\le N} |\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)} \\ \quad\times\Big(\|u'\|_{L^2_{\omega^{-1/2,0}}(I)} + \|e_{u'}\|_{L^\infty(I)}\Big), & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m} \max\limits_{0\le i\le N} |\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)} \\ \quad\times\Big(\|u'\|_{L^2_{\omega^{-1/2,0}}(I)} + \|e_{u'}\|_{L^\infty(I)}\Big), & \text{otherwise}. \end{cases}
\end{aligned} \tag{4.16a}$$

Due to Lemma 3.3,

$$\|J_3\|_{L^\infty(I)} \le \begin{cases} C\log N N^{\frac{1}{2}-m} |u'|_{H^{m;N}_{\omega^c}(I)}, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+1-m} |u'|_{H^{m;N}_{\omega^c}(I)}, & \text{otherwise}, \end{cases}$$

$$\|J_7\|_{L^\infty(I)} \le \begin{cases} C\log N N^{\frac{1}{2}-m} |u|_{H^{m;N}_{\omega^c}(I)}, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+1-m} |u|_{H^{m;N}_{\omega^c}(I)}, & \text{otherwise}. \end{cases}$$

By virtue of Lemma 3.3 (2) with $m=1$,

$$\|J_4\|_{L^\infty(I)} \le \begin{cases} C\log N N^{-\frac{1}{2}} \|e_{u'}\|_{L^\infty(I)}, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma} \|e_{u'}\|_{L^\infty(I)}, & \text{otherwise}, \end{cases}$$

$$\|J_8\|_{L^\infty(I)} \le \begin{cases} C\log N N^{-\frac{1}{2}} \|e_{u'}\|_{L^\infty(I)}, & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma} \|e_{u'}\|_{L^\infty(I)}, & \text{otherwise}. \end{cases}$$

We now estimate the term $J_5(x)$ and $J_6(x)$. It follows from Lemma 3.5 and Lemma 3.6 that

$$\begin{aligned}\|J_5\|_{L^\infty(I)} =&\left\|(I_N^{\alpha,\beta}-I)\mathscr{M}e_u\right\|_{L^\infty(I)}\\ =&\left\|(I_N^{\alpha,\beta}-I)\big(\mathscr{M}e_u-\mathscr{T}_N\mathscr{M}e_u\big)\right\|_{L^\infty(I)}\\ \le&\big(1+\|I_N^{\alpha,\beta}\|_\infty\big)CN^{-\kappa}\|\mathscr{M}e_u\|_{0,\kappa}\\ \le&\begin{cases} C\log N N^{-\kappa}\|e_u\|_{L^\infty(I)}, & -1<\alpha,\beta\le-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-\kappa}\|e_u\|_{L^\infty(I)}, & \text{otherwise},\end{cases}\end{aligned}$$

where in the last step we have used Lemma 3.6 under the following assumption:

$$\begin{cases} 0<\kappa<\frac{1}{2}, & -1<\alpha,\beta\le-\frac{1}{2},\\ \max\left\{\frac{1}{2}+\gamma,0\right\}<\kappa<\frac{1}{2}, & \text{otherwise}.\end{cases}$$

Likewise,

$$\|J_6\|_{L^\infty(I)}\le\begin{cases} C\log N N^{-\kappa}\|e_{u'}\|_{L^\infty(I)}, & -1<\alpha,\beta\le-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-\kappa}\|e_{u'}\|_{L^\infty(I)}, & \text{otherwise}.\end{cases}$$

We now obtain the estimate for $\|e_{u'}\|_{L^\infty(I)}$ by using (4.14):

$$\|e_{u'}\|_{L^\infty(I)}\le\begin{cases} CN^{-m}\log N\big(K^*+N^{\frac{1}{2}}U\big), & -1<\alpha,\beta\le-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m}\big(K^*+N^{\frac{1}{2}}U\big), & \text{otherwise}.\end{cases}$$

The above estimate, together with (4.15), yield

$$\|e_{u}\|_{L^\infty(I)}\le\begin{cases} CN^{-m}\log N\big(K^*+N^{\frac{1}{2}}U\big), & -1<\alpha,\beta\le-\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-m}\big(K^*+N^{\frac{1}{2}}U\big), & \text{otherwise}.\end{cases}$$

This completes the proof of the theorem. □

4.2. Error estimate in $L^2_{\omega^{\alpha,\beta}}$ norm

To prove the error estimate in $L^2_{\omega^{\alpha,\beta}}$ norm, we need the generalized Hardy inequality with weights (see [16]).

Lemma 4.2. *For all measurable function $f\ge 0$, the following generalized Hardy inequality*

$$\left(\int_a^b |(Tf)(x)|^q u(x)dx\right)^{1/q}\le C\left(\int_a^b |f(x)|^p v(x)dx\right)^{1/p} \tag{4.17}$$

holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{1/q}\left(\int_a^x v^{1-p'}(t)dt\right)^{1/p'}<\infty,\quad p'=\frac{p}{p-1}\tag{4.18}$$

for the case $1<p\le q<\infty$. *Here,* T *is an operator of the form* $(Tf)(x)=\int_a^x k(x,t)f(t)dt$ *with* $k(x,t)$ *a given kernel,* u, v *weight functions, and* $-\infty\le a<b\le\infty$.

From Theorem 1 in [22], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 4.3. *For every bounded function* $v(x)$, *there exists a constant* C *independent of* v *such that*

$$\sup_N\Big\|\sum_{j=0}^N v(x_j)F_j(x)\Big\|_{L^2_{\omega^{\alpha,\beta}}(I)}\le C\max_{x\in[-1,1]}|v(x)|.\tag{4.19}$$

Theorem 4.2. *If the hypotheses given in Theorem* 4.1 *hold, then*

$$\begin{aligned}&\|u-\tilde u_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}\\ \le&\begin{cases}CN^{-m}\big(\tilde K^*+V+N^{-\kappa}\log N(K^*+N^{\frac12}U)\big), & -1<\alpha,\beta\le-\frac12,\\ CN^{-m}\big(\tilde K^*+V+N^{\gamma+\frac12-\kappa}(K^*+N^{\frac12}U)\big), & \text{otherwise},\end{cases}\end{aligned}\tag{4.20a}$$

$$\begin{aligned}&\|u'-\tilde u'_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}\\ \le&\begin{cases}CN^{-m}\big(\tilde K^*+V+N^{-\kappa}\log N(K^*+N^{\frac12}U)\big), & -1<\alpha,\beta\le-\frac12,\\ CN^{-m}\big(\tilde K^*+V+N^{\gamma+\frac12-\kappa}(K^*+N^{\frac12}U)\big), & \text{otherwise},\end{cases}\end{aligned}\tag{4.20b}$$

for any $\kappa\in(0,1/2)$, *provided that* N *is sufficiently large and* C *is a constant independent of* N, *where* K^* *and* U *are defined by* (4.1) *and* (4.3), *respectively,*

$$\begin{aligned}\tilde K^*=&\left(\max_{0\le i\le N}|\tilde K_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}+\max_{0\le i\le N}|\tilde K_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}(I)}\right)\\ &\times\left(\|u\|_{H^2_{\omega^c}(I)}+\|u\|_{H^1_{\omega^{-1/2,0}}(I)}\right),\end{aligned}\tag{4.21a}$$

$$V=|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}+|u'|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}.\tag{4.21b}$$

Proof. Due to (4.13), we apply the Gronwall inequality (Lemma 3.4) and the Hardy inequality (Lemma 4.2) to obtain that

$$\|e_{u'}\|_{L^2_{\omega^{\alpha,\beta}}(I)}\le C\sum_{i=1}^8\|J_i\|_{L^2_{\omega^{\alpha,\beta}}(I)}.\tag{4.22}$$

Now, using Lemma 4.3, we have

$$\|J_1\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le CN^{-m} \max_{0\le i\le N} |\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}} \Big(\|u\|_{L^2_{\omega^{-1/2,0}}(I)} + \|e_u\|_{L^\infty(I)} \Big), \tag{4.23a}$$

$$\|J_2\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le CN^{-m} \max_{0\le i\le N} |\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}} \Big(\|u'\|_{L^2_{\omega^{-1/2,0}}(I)} + \|e_{u'}\|_{L^\infty(I)} \Big). \tag{4.23b}$$

By the convergence result in Theorem 4.1 ($m=1$), we have

$$\|e_u\|_{L^\infty(I)} \le C\Big(\|u\|_{H^2_{\omega^c}(I)} + \|u\|_{H^1_{\omega^{-1/2,0}}(I)} \Big). \tag{4.24}$$

So that

$$\|J_1\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le CN^{-m} \max_{0\le i\le N} |\tilde{K}_0(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}} \Big(\|u\|_{H^2_{\omega^c}(I)} + \|u\|_{H^1_{\omega^{-1/2,0}}(I)} \Big). \tag{4.25}$$

Similarly,

$$\|J_2\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le CN^{-m} \max_{0\le i\le N} |\tilde{K}_1(x_i,\tau(x_i,\cdot))|_{H^{m;N}_{\omega^{-1/2,0}}} \Big(\|u\|_{H^2_{\omega^c}(I)} + \|u\|_{H^1_{\omega^{-1/2,0}}(I)} \Big).$$

Due to Lemma 3.3,

$$\begin{aligned}
\|J_3\|_{L^2_{\omega^{\alpha,\beta}}(I)} &\le CN^{-m}|u'|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)},\\
\|J_7\|_{L^2_{\omega^{\alpha,\beta}}(I)} &\le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}.
\end{aligned}$$

By virtue of Lemma 3.3 (1) with $m=1$,

$$\begin{aligned}
\|J_4\|_{L^2_{\omega^{\alpha,\beta}}(I)} &\le CN^{-1}\left| \tilde{a}(x)\int_{-1}^{x} e_{u'}(\tau)d\tau \right|_{H^{1;N}_{\omega^{\alpha,\beta}}(I)}\\
&\le CN^{-1}\|e_{u'}\|_{L^2_{\omega^{\alpha,\beta}}(I)},\\
\|J_8\|_{L^2_{\omega^{\alpha,\beta}}(I)} &\le CN^{-1}\|e_{u'}\|_{L^2_{\omega^{\alpha,\beta}}(I)}.
\end{aligned}$$

Finally, it follows from Lemma 3.5 and Lemma 4.3 that

$$\begin{aligned}
\|J_5\|_{L^2_{\omega^{\alpha,\beta}}(I)} =& \|(I_N^{\alpha,\beta}-I)\mathscr{M}e_u\|_{L^2_{\omega^{\alpha,\beta}}(I)}\\
=& \|(I_N^{\alpha,\beta}-I)(\mathscr{M}e_u - \mathscr{T}_N\mathscr{M}e_u)\|_{L^2_{\omega^{\alpha,\beta}}(I)}\\
\le& \|I_N^{\alpha,\beta}(\mathscr{M}e_u - \mathscr{T}_N\mathscr{M}e_u)\|_{L^2_{\omega^{\alpha,\beta}}(I)} + \|\mathscr{M}e_u - \mathscr{T}_N\mathscr{M}e_u\|_{L^2_{\omega^{\alpha,\beta}}(I)}\\
\le& C\|\mathscr{M}e_u - \mathscr{T}_N\mathscr{M}e_u\|_{L^\infty(I)}\\
\le& CN^{-\kappa}\|\mathscr{M}e_u\|_{0,\kappa} \le CN^{-\kappa}\|e_u\|_{L^\infty(I)},
\end{aligned}$$

where in the last step we used Lemma 3.6 for any $\kappa\in(0,1/2)$. By the convergence result in Theorem 4.1, we obtain that

$$\|J_5\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le \begin{cases} CN^{-\kappa-m}\log N\big(K^*+N^{\frac{1}{2}}U\big), & -1<\alpha,\beta\le -\frac{1}{2},\\ CN^{\gamma+\frac{1}{2}-\kappa-m}\big(K^*+N^{\frac{1}{2}}U\big), & \text{otherwise,}\end{cases}$$

for N sufficiently large and for any $\kappa \in (0,1/2)$. $\|J_6\|_{L^2_{\omega^{\alpha,\beta}}(I)}$ has the same bound with $\|J_5\|_{L^2_{\omega^{\alpha,\beta}}(I)}$. The desired estimates (4.20a) and (4.20b) are obtained by combining the above estimates. □

5. Numerical experiment

We give a numerical example to confirm our analysis. The problem (5.2a)-(5.2b) is solved using the proposed Jacobi collocation method for $\alpha = \beta = -1/2$. To examine the accuracy of the results, L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors are employed to assess the efficiency of the method. All the calculations are supported by the software Matlab.

Example 5.1. For computational simplicity, we consider the following neutral VIDE with $T = 4$

$$\begin{aligned} y'(t) &= \frac{1}{20}y(t) + \frac{3}{2}t^{\frac{1}{2}} - \frac{1}{20}t^{\frac{3}{2}} - \frac{3}{2}\pi \sin\Big(\frac{t}{2}\Big)J_0\Big(\frac{t}{2}\Big) \\ &\quad + \int_0^t (t-s)^{-\frac{1}{2}} \frac{\sin(s)}{s} y'(s)ds, \quad t \in [0,4], \qquad (5.1a) \\ y(0) &= 0, \qquad (5.1b) \end{aligned}$$

where $J_0(z)$ is the Bessel function defined by

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{(k!)^2 4^k}.$$

The corresponding exact solution is given by $y(t) = t^{3/2}$. Applying the variable transformations introduced in Section 2, (5.1a)-(5.1b) becomes

$$\begin{aligned} u'(x) &= \tilde{a}(x)u(x) + \tilde{b}(x) + \int_{-1}^{x} (x-\tau)^{-\frac{1}{2}} \tilde{K}_1(x,\tau)u'(\tau)d\tau, \quad x \in [-1,1], \qquad (5.2a) \\ u(-1) &= 0. \qquad (5.2b) \end{aligned}$$

Here,

$$\begin{aligned} \tilde{a}(x) &= \frac{1}{10}(1+x), \\ \tilde{b}(x) &= 3(1+x)^2 - \frac{1}{10}(1+x)^4 - 3\pi(1+x)\sin\Big(\frac{(1+x)^2}{2}\Big)J_0\Big(\frac{(1+x)^2}{2}\Big), \\ \tilde{K}_1(x,\tau) &= 2(2+x+\tau)^{-\frac{1}{2}}(1+x)\frac{\sin((1+\tau)^2)}{(1+\tau)^2}. \end{aligned}$$

The solution of (5.2a)-(5.2b) is $u(x) = (1+x)^3$. Table 1 shows the errors $\|u - \tilde{u}_N\|_{L^\infty(I)}$ and $\|u - \tilde{u}_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}$ obtained by using the spectral methods described above. Furthermore,

Table 1: The errors $\|u-\tilde{u}_N\|_{L^\infty(I)}$ and $\|u-\tilde{u}_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}$.

N	2	4	6	8
L^∞-error	5.3756e-001	3.6638e-003	2.0663e-007	5.2978e-010
$L^2_{\omega^{\alpha,\beta}}$-error	1.9923e-001	3.6925e-003	1.3176e-007	3.2717e-010
N	10	12	14	16
L^∞-error	6.3566e-012	7.1098e-012	6.1453e-012	6.5174e-012
$L^2_{\omega^{\alpha,\beta}}$-error	6.7069e-012	7.2096e-012	6.2302e-012	6.6075e-012

Table 2: The errors $\|u'-\tilde{u}'_N\|_{L^\infty(I)}$ and $\|u'-\tilde{u}'_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}$.

N	2	4	6	8
L^∞-error	3.7117e-001	4.6845e-003	2.7037e-006	6.4152e-009
$L^2_{\omega^{\alpha,\beta}}$-error	3.9261e-001	5.0468e-003	1.6542e-006	3.8225e-009
N	10	12	14	16
L^∞-error	8.9768e-012	9.4155e-012	8.1188e-012	8.6259e-012
$L^2_{\omega^{\alpha,\beta}}$-error	8.3108e-012	9.8944e-012	8.5593e-012	9.0890e-012

we also compute the errors $\|u'-\tilde{u}'_N\|_{L^\infty(I)}$ and $\|u'-\tilde{u}'_N\|_{L^2_{\omega^{\alpha,\beta}}(I)}$, the results are shown in Table 2. It is observed that the desired exponential rate of convergence is obtained. Fig. 1 presents the approximate and exact solution on left side and presents the approximate and exact derivative of the solution on right side, which are found in excellent agreement. In Fig. 2, the numerical errors $u-\tilde{u}_N$ and $u'-\tilde{u}'_N$ are plotted for $2 \le N \le 16$ in both L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms.

6. Conclusions and future work

In this paper, we elaborated a spectral collocation method based on Jacobi orthogonal polynomials to obtain an approximate solution and an approximate derivative of the solution for weakly singular neutral VIDE. The strategy is derived using some variable transformations to change the equation into an other Volterra integro-differential equation, so that the new equation has the smooth solution and the Jacobi orthogonal polynomial theory can be applied conveniently. The initial condition is restated as an equivalent integral equation, so all the integral terms are approximated by using Gauss quadrature formula. The spectral rate of convergence for the proposed method is established in L^∞ norm and $L^2_{\omega^{\alpha,\beta}}$ norm.

We only investigated the case when $\mu = 1/2$ in the present work, with the availability of this methodology, it will be possible to extend the results of this paper to the weakly singular VIDE with $(t-s)^{-\mu}$, $\mu \neq 1/2$ which will be the subject of our future work.

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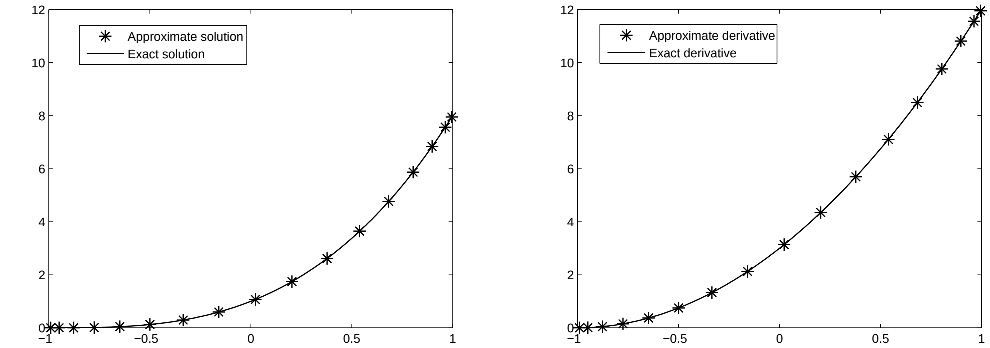

Figure 1: Comparison between approximate solution $\tilde{u}_N$ and exact solution u (left); Comparison between approximate derivative $\tilde{u}'_N$ and exact derivative u' (right).

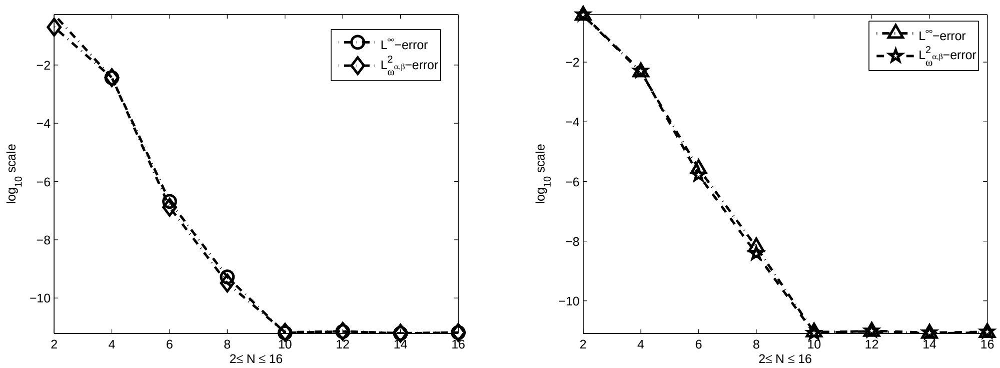

Figure 2: The errors $u-\tilde{u}_N$ (left) and $u'-\tilde{u}'_N$ (right) versus the number of collocation points in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms.

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