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Superconvergence and L^{∞}-Error Estimates of the Lowest Order Mixed Methods for Distributed Optimal Control Problems Governed by Semilinear Elliptic Equations

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Abstract. In this paper, we investigate the superconvergence property and the L^{∞}-error estimates of mixed finite element methods for a semilinear elliptic control problem. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. We derive some superconvergence results for the control variable. Moreover, we derive L^{∞}-error estimates both for the control variable and the state variables. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear elliptic equations, distributed optimal control problems, superconvergence, L^{∞}-error estimates, mixed finite element methods.

1. Introduction

It is well known that the finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1,6,11–13,19–23]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [9,16].

In many control problems, the objective functional contains the gradient of the state variable. Thus, the accuracy of the gradient is important in the numerical approximation of the state equations. In the finite element community, mixed finite element methods are optimal for discretization of the state equations in such cases, since both the scalar variable and its flux variable can be approximated in the same accuracy by using mixed finite element methods. For a priori error estimates and superconvergence properties of mixed

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finite elements for optimal control problems, see, for example, [4,5,8]. In [5], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch (see [19]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, Chen et al. derived error estimates and superconvergence of mixed methods for convex optimal control problems in [8]. However, as far as we know there is no superconvergence analysis in mixed finite element methods for optimal control problems governed by semilinear elliptic equations except [7].

The goal of this paper is to derive the superconvergence property and the L^∞-error estimates of mixed finite element approximation for a semilinear elliptic control problem. Firstly, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is $h^{3/2}$ as that obtained in [8]. Then, a global superconvergence result for the control variable can be obtained by using a recovery operator. We also derive the L^∞-error estimates for both the control variable and the state variables. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence and L^∞-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with pointwise constraint:

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\mathbf{grad}y) + \phi(y) = u, \qquad x \in \Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p} + \phi(y) = u, \quad \boldsymbol{p} = -A(x)\mathbf{grad}y, \quad x \in \Omega, \tag{1.3}$$

and the boundary condition

$$y = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad} = \{u \in L^\infty(\Omega):\ u \geq 0,\ \ a.e.\ \text{ in } \Omega\}. \tag{1.5}$$

We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R) \cap H^3(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi' \geq 0$. Moreover, we assume that $y_d \in W^{1,\infty}(\Omega)$ and $\boldsymbol{p}_d \in (H^2(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall(\xi,x) \in \mathbb{R}^2 \times \bar{\Omega}, \ \ c_* > 0.$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)-(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3 and Section 4. In Section 3, we derive the superconvergence properties between the average L^2 projection and the approximation, as well as between the postprocessing solution and the exact control solution. In Section 4, we will study the L^∞-error estimates for optimal control problem. In Section 5, we present a numerical example to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum_{|\alpha|\le m}\|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}$. We set

$$W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v|_{\partial\Omega}=0\}.$$

For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$ and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods of optimal control problems

In this section we shall construct mixed finite element approximation scheme of the control problem (1.1)-(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\mathbf{grad}z+\boldsymbol{p}-\boldsymbol{p}_d))+\phi'(y)z=y-y_d,\quad x\in\Omega,\tag{2.1}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{q}+\phi'(y)z=y-y_d,\quad \boldsymbol{q}=-A(x)(\mathbf{grad}z+\boldsymbol{p}-\boldsymbol{p}_d),\quad x\in\Omega,\tag{2.2}$$

and the boundary condition

$$z=0,\quad x\in\partial\Omega.\tag{2.3}$$

Let

$$\boldsymbol{V}=H(\mathrm{div};\Omega)=\big\{\boldsymbol{v}\in(L^2(\Omega))^2,\mathrm{div}\boldsymbol{v}\in L^2(\Omega)\big\},\quad W=L^2(\Omega).\tag{2.4}$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p},y,u)\in\boldsymbol{V}\times W\times U_{ad}$ such that

$$\min_{u\in U_{ad}}\Big\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\Big\},\tag{2.5a}$$

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\mathrm{div}\boldsymbol{v})=0,\qquad\forall\boldsymbol{v}\in\boldsymbol{V},\tag{2.5b}$$

$$(\mathrm{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w),\qquad\forall w\in W.\tag{2.5c}$$

It follows from [16] that the optimal control problem (2.5a)-(2.5c) has a solution $(\boldsymbol{p},y,u)$, and that a triplet $(\boldsymbol{p},y,u)$ is the solution of (2.5a)-(2.5c) if there is a co-state $(\boldsymbol{q},z)\in \boldsymbol{V}\times W$ such that $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ satisfies the following optimality conditions:

$$
\begin{aligned}
&(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\mathrm{div}\boldsymbol{v})=0, && \forall \boldsymbol{v}\in \boldsymbol{V}, && \text{(2.6a)}\\
&(\mathrm{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w), && \forall w\in W, && \text{(2.6b)}\\
&(A^{-1}\boldsymbol{q},\boldsymbol{v})-(z,\mathrm{div}\boldsymbol{v})=-(\boldsymbol{p}-\boldsymbol{p}_d,\boldsymbol{v}), && \forall \boldsymbol{v}\in \boldsymbol{V}, && \text{(2.6c)}\\
&(\mathrm{div}\boldsymbol{q},w)+(\phi'(y)z,w)=(y-y_d,w), && \forall w\in W, && \text{(2.6d)}\\
&(\nu u+z,\tilde{u}-u)\geq 0, && \forall \tilde{u}\in U_{ad}, && \text{(2.6e)}
\end{aligned}
$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

In [19, 20], the expression of the control variable is given. Here, we adopt the same method to derive the following operator

$$u=\max\{0,-z\}/\nu. \tag{2.7}$$

Let $\mathcal{T}_h$ denotes a regular triangulation of the polygonal domain Ω, h_T denotes the diameter of T and $h=\max h_T$. Let $\boldsymbol{V}_h\times W_h\subset \boldsymbol{V}\times W$ denotes the lowest order Raviart-Thomas mixed finite element space [10, 24], namely,

$$\forall T\in\mathcal{T}_h,\quad \boldsymbol{V}(T)=\boldsymbol{P}_0(T)\oplus \mathrm{span}(\boldsymbol{x}P_0(T)),\quad W(T)=P_0(T),$$

where $P_m(T)$ denotes polynomials of total degree at most m, $\boldsymbol{P}_0(T)=(P_0(T))^2$, $\boldsymbol{x}=(x_1,x_2)$, which is treated as a vector, and

$$
\begin{aligned}
\boldsymbol{V}_h&:=\{\boldsymbol{v}_h\in\boldsymbol{V}:\forall T\in\mathcal{T}_h,\boldsymbol{v}_h|_T\in\boldsymbol{V}(T)\}, && \text{(2.8a)}\\
W_h&:=\{w_h\in W:\forall T\in\mathcal{T}_h,w_h|_T\in W(T)\}. && \text{(2.8b)}
\end{aligned}
$$

And the approximated space of control is given by

$$U_h:=\{\tilde{u}_h\in U_{ad}:\forall T\in\mathcal{T}_h,\ \tilde{u}_h|_T=\text{constant}\}. \tag{2.9}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [10] $P_h:\ W\to W_h$, which satisfies: for any $\phi\in W$

$$
\begin{aligned}
&(P_h\phi-\phi,w_h)=0, && \forall w_h\in W_h, && \text{(2.10a)}\\
&\|\phi-P_h\phi\|_{0,\rho}\leq Ch\|\phi\|_{1,\rho},\quad 2\leq\rho\leq\infty, && \forall\phi\in W^{1,\rho}(\Omega). && \text{(2.10b)}
\end{aligned}
$$

Next, recall the Fortin projection (see [3] and [10]) $\Pi_h:\boldsymbol{V}\to\boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q}\in\boldsymbol{V}$

$$
\begin{aligned}
&(\mathrm{div}(\Pi_h\boldsymbol{q}-\boldsymbol{q}),w_h)=0, && \forall w_h\in W_h, && \text{(2.11a)}\\
&\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|_{0,\rho}\leq Ch\|\boldsymbol{q}\|_{1,\rho},\quad 2\leq\rho\leq\infty, && \forall\boldsymbol{q}\in(W^{1,\rho}(\Omega))^2, && \text{(2.11b)}\\
&\|\mathrm{div}(\boldsymbol{q}-\Pi_h\boldsymbol{q})\|\leq Ch\|\mathrm{div}\boldsymbol{q}\|_1, && \forall\mathrm{div}\boldsymbol{q}\in H^1(\Omega). && \text{(2.11c)}
\end{aligned}
$$

We have the commuting diagram property

$$\mathrm{div}\circ\Pi_h = P_h\circ\mathrm{div}: \boldsymbol{V}\to W_h \quad \text{and} \quad \mathrm{div}(I-\Pi_h)\boldsymbol{V}\perp W_h, \tag{2.12}$$

where and after, I denote identity operator.

Then the mixed finite element discretization of (2.5a)-(2.5c) is as follows: find $(\boldsymbol{p}_h, y_h, u_h)\in \boldsymbol{V}_h\times W_h\times U_h$ such that

$$\min_{u_h\in U_h}\Big\{\frac{1}{2}\|\boldsymbol{p}_h-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y_h-y_d\|^2+\frac{\nu}{2}\|u_h\|^2\Big\}, \tag{2.13a}$$

$$(A^{-1}\boldsymbol{p}_h,\boldsymbol{v}_h)-(y_h,\mathrm{div}\boldsymbol{v}_h)=0, \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.13b}$$

$$(\mathrm{div}\boldsymbol{p}_h,w_h)+(\phi(y_h),w_h)=(u_h,w_h), \qquad \forall w_h\in W_h. \tag{2.13c}$$

The optimal control problem (2.13a)-(2.13c) again has a solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.13a)-(2.13c) if there is a co-state $(\boldsymbol{q}_h, z_h)\in\boldsymbol{V}_h\times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h,\boldsymbol{v}_h)-(y_h,\mathrm{div}\boldsymbol{v}_h)=0, \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.14a}$$

$$(\mathrm{div}\boldsymbol{p}_h,w_h)+(\phi(y_h),w_h)=(u_h,w_h), \qquad \forall w_h\in W_h, \tag{2.14b}$$

$$(A^{-1}\boldsymbol{q}_h,\boldsymbol{v}_h)-(z_h,\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}_h-\boldsymbol{p}_d,\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.14c}$$

$$(\mathrm{div}\boldsymbol{q}_h,w_h)+(\phi'(y_h)z_h,w_h)=(y_h-y_d,w_h), \qquad \forall w_h\in W_h, \tag{2.14d}$$

$$(\nu u_h+z_h,\tilde{u}_h-u_h)\ge 0, \qquad \forall \tilde{u}_h\in U_h. \tag{2.14e}$$

Similar to (2.7), the control inequality (2.14e) can be expressed as

$$u_h=\max\{0,-z_h\}/\nu. \tag{2.15}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u}\in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))\in(\boldsymbol{V}\times W)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}),\boldsymbol{v})-(y(\tilde{u}),\mathrm{div}\boldsymbol{v})=0, \qquad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{2.16a}$$

$$(\mathrm{div}\boldsymbol{p}(\tilde{u}),w)+(\phi(y(\tilde{u})),w)=(\tilde{u},w), \qquad \forall w\in W, \tag{2.16b}$$

$$(A^{-1}\boldsymbol{q}(\tilde{u}),\boldsymbol{v})-(z(\tilde{u}),\mathrm{div}\boldsymbol{v})=-(\boldsymbol{p}(\tilde{u})-\boldsymbol{p}_d,\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{2.16c}$$

$$(\mathrm{div}\boldsymbol{q}(\tilde{u}),w)+(\phi'(y(\tilde{u}))z(\tilde{u}),w)=(y(\tilde{u})-y_d,w), \qquad \forall w\in W. \tag{2.16d}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))\in(\boldsymbol{V}_h\times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}),\boldsymbol{v}_h)-(y_h(\tilde{u}),\mathrm{div}\boldsymbol{v}_h)=0, \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.17a}$$

$$(\mathrm{div}\boldsymbol{p}_h(\tilde{u}),w_h)+(\phi(y_h(\tilde{u})),w_h)=(\tilde{u},w_h), \qquad \forall w_h\in W_h, \tag{2.17b}$$

$$(A^{-1}\boldsymbol{q}_h(\tilde{u}),\boldsymbol{v}_h)-(z_h(\tilde{u}),\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}_h(\tilde{u})-\boldsymbol{p}_d,\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.17c}$$

$$(\mathrm{div}\boldsymbol{q}_h(\tilde{u}),w_h)+(\phi'(y_h(\tilde{u}))z_h(\tilde{u}),w_h)=(y_h(\tilde{u})-y_d,w_h), \qquad \forall w_h\in W_h. \tag{2.17d}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$
$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

3. Superconvergence analysis

In this section, we will give a detailed superconvergence analysis.

Now, we are in the position of deriving the estimate for $\|P_h y(u_h) - y_h\|$ and $\|P_h z(u_h) - z_h\|$, we need an a priori regularity estimate for the following auxiliary problems:

$$-\operatorname{div}(A\nabla\xi) + \Phi\xi = F_1, \qquad x \in \Omega, \qquad \xi|_{\partial\Omega} = 0, \tag{3.1a}$$
$$-\operatorname{div}(A\nabla\zeta) + \phi'(y(u_h))\zeta = F_2, \qquad x \in \Omega, \qquad \zeta|_{\partial\Omega} = 0, \tag{3.1b}$$

where

$$\Phi = \begin{cases} \dfrac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \\ \phi'(y_h), & y(u_h) = y_h. \end{cases}$$

Lemma 3.1. (see [17]) *Let ξ and ζ be the solutions for* (3.1a) *and* (3.1b)*, respectively. Assume that Ω is convex. Then we have*

$$\|\xi\|_{H^2(\Omega)} \leq C\|F_1\|_{L^2(\Omega)}, \tag{3.2a}$$
$$\|\zeta\|_{H^2(\Omega)} \leq C\|F_2\|_{L^2(\Omega)}. \tag{3.2b}$$

Then, we will give the following superconvergence results for the intermediate solutions which are very important for our following work.

Lemma 3.2. *Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of* (2.16a)-(2.16d) *and* (2.17a)-(2.17d) *with $\tilde{u} = u_h$ respectively. Assume that*

$$\boldsymbol{p}(u_h),\ \boldsymbol{q}(u_h) \in (H^1(\Omega))^2 \text{ and } y(u_h),\ z(u_h) \in W^{1,\infty}(\Omega),$$

then we have

$$\|P_h y(u_h) - y_h\| \leq Ch^2, \tag{3.3a}$$
$$\|P_h z(u_h) - z_h\| \leq Ch^2. \tag{3.3b}$$

Proof. From Eqs. (2.16a)-(2.16d) and (2.17a)-(2.17d), we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) - (y(u_h) - y_h, \operatorname{div}\boldsymbol{v}_h) = 0, \tag{3.4a}$$
$$(\operatorname{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \tag{3.4b}$$
$$(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (z(u_h) - z_h, \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \boldsymbol{v}_h), \tag{3.4c}$$
$$(\operatorname{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, w_h) = (y(u_h) - y_h, w_h), \tag{3.4d}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

As a result of (2.10a), we can rewrite (3.4a)-(3.4d) as

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{v}_h)-(P_h y(u_h)-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.5a}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),w_h)+(\phi(y(u_h))-\phi(y_h),w_h)=0, \tag{3.5b}$$
$$(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_h z(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)=-(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\boldsymbol{v}_h), \tag{3.5c}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(\phi'(y(u_h))z(u_h)-\phi'(y_h)z_h,w_h)=(P_h y(u_h)-y_h,w_h), \tag{3.5d}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

For sake of simplicity, we now denote

$$\tau=P_h y(u_h)-y_h, \quad e=P_h z(u_h)-z_h. \tag{3.6}$$

Then, we estimate (3.3a) and (3.3b) in Part I and Part II, respectively.

Part I. As we can see,

$$\|\tau\|=\sup_{\psi\in L^2(\Omega),\psi\neq 0}\frac{(\tau,\psi)}{\|\psi\|}, \tag{3.7}$$

we then need to bound (τ,ψ) for $\psi\in L^2(\Omega)$. Let $\xi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (3.1a). We can see from (2.11a) and (3.5a)

$$\begin{aligned}(\tau,F_1)=&(\tau,-\mathrm{div}(A\mathbf{grad}\xi))+(\tau,\Phi\xi)\\=&-(\tau,\mathrm{div}(\Pi_h(A\mathbf{grad}\xi)))+(\tau,\Phi\xi)\\=&-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\Pi_h(A\mathbf{grad}\xi))+(\tau,\Phi\xi).\end{aligned} \tag{3.8}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\xi)+(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\mathbf{grad}\xi)=0. \tag{3.9}$$

Thus, from (3.5b), (3.8) and (3.9), we derive

$$\begin{aligned}(\tau,F_1)=&(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\mathbf{grad}\xi-\Pi_h(A\mathbf{grad}\xi))\\&+(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\xi-P_h\xi)-(\Phi(y(u_h)-P_h y(u_h)),\xi)\\&+(\phi(y(u_h))-\phi(y_h),\xi-P_h\xi).\end{aligned} \tag{3.10}$$

From (2.11b), we have

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\mathbf{grad}\xi-\Pi_h(A\mathbf{grad}\xi))\le Ch\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|\cdot\|\xi\|_2. \tag{3.11}$$

Let $\tilde{u}=u_h$ and $w=\mathrm{div}\boldsymbol{p}(u_h)+\phi(y(u_h))-u_h$ in (2.16b), we can find that

$$\mathrm{div}\boldsymbol{p}(u_h)+\phi(y(u_h))-u_h=0. \tag{3.12}$$

Similarly, by (2.10a) and (2.14b), it is easy to see that

$$\mathrm{div}\boldsymbol{p}_h = u_h - P_h\phi(y_h). \tag{3.13}$$

By (3.12), (3.13) and (2.10b), we have

$$\begin{aligned}
&(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\xi-P_h\xi)+(\phi(y(u_h))-\phi(y_h),\xi-P_h\xi)\\
=&(P_h\phi(y_h)-\phi(y_h),\xi-P_h\xi)\\
\leq&Ch^2\|\phi\|_1\|\xi\|_1.
\end{aligned}\tag{3.14}$$

For the third term on the right side of (3.10), using (2.10a), (2.10b) and the assumption on ϕ, we get

$$\begin{aligned}
&(\Phi(y(u_h)-P_hy(u_h)),\xi)\\
=&(\Phi(y(u_h)-P_hy(u_h)),\xi-P_h\xi)+(y(u_h)-P_hy(u_h),(\Phi-P_h\Phi)P_h\xi)\\
\leq&Ch\|\phi\|_{1,\infty}\|y(u_h)-P_hy(u_h)\|\cdot\|\xi\|_1+Ch\|\phi\|_{2,\infty}\|y(u_h)-P_hy(u_h)\|\cdot\|\xi\|\\
\leq&Ch^2\|\xi\|_1.
\end{aligned}\tag{3.15}$$

By (3.2a), (3.7), (3.11) and (3.14)-(3.15), we derive

$$\|P_hy(u_h)-y_h\|\leq Ch\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|+Ch^2. \tag{3.16}$$

Choosing $\boldsymbol{v}_h=\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h$ in (3.5a) and $w_h=P_hy(u_h)-y_h$ in (3.5b), respectively. Then adding the two equations to get

$$\begin{aligned}
&(A^{-1}(\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)+(\phi(P_hy(u_h))-\phi(y_h),P_hy(u_h)-y_h)\\
=&-(A^{-1}(\boldsymbol{p}(u_h)-\Pi_h\boldsymbol{p}(u_h)),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)\\
&-(\phi(y(u_h))-\phi(P_hy(u_h)),P_hy(u_h)-y_h).
\end{aligned}\tag{3.17}$$

Note that

$$(\phi(y(u_h))-\phi(P_hy(u_h)),P_hy(u_h)-y_h)\leq Ch\|\phi\|_{1,\infty}\|y(u_h)\|_1\|P_hy(u_h)-y_h\|. \tag{3.18}$$

Using (3.17), (3.18), (2.11b) and the assumptions on A and ϕ, we find that

$$\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|\leq Ch+\|P_hy(u_h)-y_h\|. \tag{3.19}$$

Substituting (3.19) into (3.16), using (2.11b), for sufficiently small h, we have

$$\|P_hy(u_h)-y_h\|\leq Ch^2, \tag{3.20}$$

which yields (3.3a).

Part II. Since

$$\|e\|=\sup_{\psi\in L^2(\Omega),\psi\neq0}\frac{(e,\psi)}{\|\psi\|}, \tag{3.21}$$

we then need to bound (e,ψ) for $\psi \in L^2(\Omega)$. Let $\zeta \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of (3.1b). We can see from (2.11a) and (3.5c)

$$\begin{aligned}(e,F_2) =&(e,-\mathrm{div}(A\mathbf{grad}\zeta))+(e,\phi'(y)\zeta)\\ =&-(e,\mathrm{div}(\Pi_h(A\mathbf{grad}\zeta)))+(e,\phi'(y)\zeta)\\ =&-(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\Pi_h(A\mathbf{grad}\zeta))+(e,\phi'(y)\zeta)\\ &-(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\Pi_h(A\mathbf{grad}\zeta)).\end{aligned} \tag{3.22}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\zeta)+(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),A\mathbf{grad}\zeta)=0. \tag{3.23}$$

Thus, it follows from (2.10a), (3.5d), (3.22) and (3.23), we derive

$$\begin{aligned}(e,F_2) =&(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta))\\ &+(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\zeta-P_h\zeta)-(P_hy(u_h)-y_h,P_h\zeta)\\ &+(\phi'(y(u_h))z(u_h)-\phi'(y_h)z_h,\zeta-P_h\zeta)\\ &+(\phi'(y(u_h))(P_hz(u_h)-z(u_h)),\zeta)+(z_h(\phi'(y_h)-\phi'(y(u_h))),\zeta)\\ &-(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\Pi_h(A\mathbf{grad}\zeta))\\ =:&\sum_{i=1}^{7}I_i.\end{aligned} \tag{3.24}$$

For I_1, by (2.11b), we have

$$I_1\le C\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|\cdot\|A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta)\|\le Ch\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|\cdot\|\zeta\|_2. \tag{3.25}$$

Let $\tilde{u}=u_h$ and $w=\mathrm{div}\boldsymbol{q}(u_h)+\phi'(y(u_h))z(u_h)-y(u_h)+y_d$ in (2.16d), we can find that

$$\mathrm{div}\boldsymbol{q}(u_h)+\phi'(y(u_h))z(u_h)=y(u_h)-y_d. \tag{3.26}$$

Similarly, by (2.10a) and (2.14d), it is easy to see that

$$\mathrm{div}\boldsymbol{q}_h=y_h-P_hy_d-P_h\phi'(y_h)z_h. \tag{3.27}$$

By (2.10b) and (3.26)-(3.27), we have

$$\begin{aligned}I_2 =&(P_h\phi'(y_h)z_h-\phi'(y(u_h))z(u_h),\zeta-P_h\zeta)+(P_hy_d-y_d,\zeta-P_h\zeta)\\ &+(y(u_h)-P_hy(u_h),\zeta-P_h\zeta)+(P_hy(u_h)-y_h,\zeta-P_h\zeta)\\ =&(P_h(\phi'(y(u_h))z(u_h))-\phi'(y(u_h))z(u_h),\zeta-P_h\zeta)\\ &+(P_hy_d-y_d,\zeta-P_h\zeta)+(y(u_h)-P_hy(u_h),\zeta-P_h\zeta)\\ \le&Ch^2(\|\phi\|_2\|z(u_h)\|_{1,\infty}+\|y_d\|_1+\|y(u_h)\|_1)\|\zeta\|_1.\end{aligned} \tag{3.28}$$

From (3.3a), we arrive at

$$I_3\le C\|P_hy(u_h)-y_h\|\cdot\|P_h\zeta\|\le Ch^2\|\zeta\|. \tag{3.29}$$

Note that

$$\phi'(y(u_h))z(u_h)-\phi'(y_h)z_h=z(u_h)(\phi'(y(u_h))-\phi'(y_h))+\phi'(y_h)(z(u_h)-z_h). \tag{3.30}$$

Then, by (2.10b), (3.3a) and the assumption on ϕ, we find that

$$\begin{aligned}I_4\le& C\|z(u_h)\|_{0,\infty}\|\phi\|_{2,\infty}\|y(u_h)-y_h\|\cdot\|\zeta-P_h\zeta\|+C\|\phi\|_{1,\infty}\|z(u_h)-z_h\|\cdot\|\zeta-P_h\zeta\|\\ \le& Ch^2\|z(u_h)\|_{0,\infty}\|\phi\|_{2,\infty}\|\zeta\|_1+Ch\|\phi\|_{1,\infty}\|\zeta\|_1\|P_hz(u_h)-z_h\|.\end{aligned} \tag{3.31}$$

As for I_5, by the assumption on ϕ, (2.10a) and (2.10b), we derive

$$\begin{aligned}I_5=&(\phi'(y(u_h))(P_hz(u_h)-z(u_h)),\zeta-P_h\zeta)\\ &+(P_hz(u_h)-z(u_h),(\phi'(y(u_h))-P_h(\phi'(y(u_h))))P_h\zeta)\\ \le& C\|\phi\|_{1,\infty}\|z(u_h)-P_hz(u_h)\|\cdot\|\zeta-P_h\zeta\|\\ &+Ch\|\phi\|_{2,\infty}\|z(u_h)-P_hz(u_h)\|\cdot\|P_h\zeta\|\\ \le& Ch^2\|\phi\|_{2,\infty}\|z(u_h)\|_{1,\infty}\|\zeta\|_1.\end{aligned} \tag{3.32}$$

For I_6, by (2.10a), (2.10b), (3.3a), the embedding $\|v\|_{0,\infty}\le c\|v\|_2$ and the assumption on ϕ, we obtain

$$\begin{aligned}I_6=&(\phi'(y_h)-\phi'(y(u_h)),(z_h-z(u_h))\zeta)+(\phi'(y_h)-\phi'(P_hy(u_h)),z(u_h)\zeta)\\ &+(\phi''(y(u_h))(P_hy(u_h)-y(u_h)),z(u_h)\zeta)\\ &+\left(\frac{1}{2}\phi'''(y(u_h)+\theta(P_hy(u_h)-y(u_h)))(P_hy(u_h)-y(u_h))^2,z(u_h)\zeta\right)\\ =&(\phi'(y_h)-\phi'(y(u_h)),(z_h-z(u_h))\zeta)+(\phi'(y_h)-\phi'(P_hy(u_h)),z(u_h)\zeta)\\ &+(\phi''(y(u_h))(P_hy(u_h)-y(u_h)),z(u_h)\zeta-P_h(z(u_h)\zeta))\\ &+(P_hy(u_h)-y(u_h),(\phi''(y(u_h))-P_h(\phi''(y(u_h))))P_h(z(u_h)\zeta))\\ &+\frac{1}{2}(\phi'''(y(u_h)+\theta(P_hy(u_h)-y(u_h)))(P_hy(u_h)-y(u_h))^2,z(u_h)\zeta)\\ \le& C\|\phi\|_{2,\infty}\|y(u_h)-y_h\|\cdot\|z(u_h)-z_h\|\cdot\|\zeta\|_{0,\infty}\\ &+C\|\phi\|_{2,\infty}\|P_hy(u_h)-y_h\|\cdot\|z(u_h)\|\cdot\|\zeta\|_{0,\infty}\\ &+Ch\|\phi\|_{2,\infty}\|y(u_h)-P_hy(u_h)\|\cdot\|z(u_h)\|_{1,\infty}\|\zeta\|_1\\ &+Ch\|z(u_h)\|_{0,\infty}\|\phi\|_3\|y(u_h)-P_hy(u_h)\|\cdot\|\zeta\|_{0,\infty}\\ &+C\|\phi\|_3\|y(u_h)-P_hy(u_h)\|_{0,\infty}^2\|z(u_h)\|_{0,\infty}\|\zeta\|\\ \le& Ch^2\|\zeta\|_2+Ch\|P_hz(u_h)-z_h\|\cdot\|\zeta\|_2,\end{aligned} \tag{3.33}$$

where $0\le\theta\le1$.

Finally, for I_7, from (2.11b), (2.11c), (3.3a) and (3.5a), we have

$$\begin{aligned}I_7=&(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta))-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A^2\mathbf{grad}\zeta)\\ =&(\boldsymbol{p}(u_h)-\boldsymbol{p}_h,A\mathbf{grad}\zeta-\Pi_h(A\mathbf{grad}\zeta))-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A^2\mathbf{grad}\zeta-\Pi_h(A^2\mathbf{grad}\zeta))\\ &-(P_hy(u_h)-y_h,\mathrm{div}(\Pi_h(A^2\mathbf{grad}\zeta)))\\ \le& Ch^2\|\zeta\|_2.\end{aligned} \tag{3.34}$$

Substituting the estimates I_1-I_7 in (3.24), by (3.21) and (3.2b), we derive

$$\|P_h z(u_h) - z_h\| \le Ch\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\| + Ch^2. \tag{3.35}$$

Next, using (2.11a), we rewrite (3.5c)-(3.5d) as

$$\begin{aligned}
&(A^{-1}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) \\
&= -(A^{-1}(\boldsymbol{q}(u_h) - \Pi_h \boldsymbol{q}(u_h)), \boldsymbol{v}_h) - (\boldsymbol{p}(u_h) - \Pi_h \boldsymbol{p}(u_h), \boldsymbol{v}_h) \\
&\quad - (\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \qquad (3.36a)\\
&(\mathrm{div}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) + (\phi'(y(u_h))(P_h z(u_h) - z_h), w_h) \\
&= -(\phi'(y(u_h))(z(u_h) - P_h z(u_h)), w_h) + (P_h y(u_h) - y_h, w_h) \\
&\quad + ((\phi'(y(u_h)) - \phi'(y_h))z_h, w_h), \quad \forall w_h \in W_h. \qquad (3.36b)
\end{aligned}$$

Choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h$ in (3.36a) and $w_h = P_h z(u_h) - z_h$ in (3.36b), respectively. Then adding the two equations to get

$$\begin{aligned}
&(A^{-1}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) + (\phi'(y(u_h))(P_h z(u_h) - z_h), P_h z(u_h) - z_h) \\
=&- (A^{-1}(\boldsymbol{q}(u_h) - \Pi_h \boldsymbol{q}(u_h)), \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) - (\boldsymbol{p}(u_h) - \Pi_h \boldsymbol{p}(u_h), \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) \\
&- (\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) + (P_h y(u_h) - y_h, P_h z(u_h) - z_h) \\
&- (\phi'(y(u_h))(z(u_h) - P_h z(u_h)), P_h z(u_h) - z_h) \\
&+ ((\phi'(y(u_h)) - \phi'(y_h))z_h, P_h z(u_h) - z_h). \qquad (3.37)
\end{aligned}$$

Note that

$$\begin{aligned}
&(\phi'(y(u_h))(z(u_h) - P_h z(u_h)), P_h z(u_h) - z_h) \\
\le& Ch\|\phi\|_{1,\infty}\|z(u_h)\|_1\|P_h z(u_h) - z_h\| \qquad (3.38)
\end{aligned}$$

and

$$\begin{aligned}
&((\phi'(y(u_h)) - \phi'(y_h))z_h, P_h z(u_h) - z_h) \\
\le&((\phi'(y(u_h)) - \phi'(P_h y(u_h)))z_h, P_h z(u_h) - z_h) \\
&+ ((\phi'(P_h y(u_h)) - \phi'(y_h))z_h, P_h z(u_h) - z_h) \\
\le& Ch\|\phi\|_{1,\infty}\|y(u_h)\|_{1,\infty}\|z_h\| \cdot \|P_h z(u_h) - z_h\| \\
&+ C\|\phi\|_{1,\infty}\|P_h y(u_h) - y_h\|_{0,\infty}\|z_h\| \cdot \|P_h z(u_h) - z_h\|, \qquad (3.39)
\end{aligned}$$

where

$$\begin{aligned}
\|z_h\| \le&\|z(u_h) - P_h z(u_h)\| + \|P_h z(u_h) - z_h\| + \|z(u_h)\| \\
\le& C\|z(u_h)\|_1 + \|P_h z(u_h) - z_h\|. \qquad (3.40)
\end{aligned}$$

Using (3.37)-(3.40), (2.11b), (3.3a), (3.19) and the assumptions on A and ϕ, we find that

$$\|\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \le Ch + \|P_h z(u_h) - z_h\|. \tag{3.41}$$

Substituting (3.41) into (3.35), using (2.11b), for sufficiently small h, we have

$$\|P_h z(u_h) - z_h\| \leq Ch^2. \tag{3.42}$$

Thus, we complete the proof. □

By modifying the proof of Lemma 3.3 in [7], we have

Lemma 3.3. *Let* $(\boldsymbol{p}(P_h u), y(P_h u), \boldsymbol{q}(P_h u), z(P_h u))$ *and* $(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u))$ *be the solutions of* (2.16a)-(2.16d) *with* $\tilde{u} = P_h u$ *and* $\tilde{u} = u$*, respectively. Assume that* $u \in H^1(\Omega)$*. Then we have*

$$\|y(u) - y(P_h u)\| + \|\boldsymbol{p}(u) - \boldsymbol{p}(P_h u)\| \leq Ch^2, \tag{3.43a}$$

$$\|z(u) - z(P_h u)\| + \|\boldsymbol{q}(u) - \boldsymbol{q}(P_h u)\| \leq Ch^2. \tag{3.43b}$$

Let $(\boldsymbol{p}(u), y(u))$ be the solutions of (2.5a)-(2.5c) and $J(\cdot) : L^2(\Omega) \to \mathbb{R}$ be a G-differential convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2. \tag{3.44}$$

Then we can find that

$$(J'(u), v) = (\nu u + z, v), \tag{3.45a}$$

$$(J'(u_h), v) = (\nu u_h + z(u_h), v), \tag{3.45b}$$

$$(J'(P_h u), v) = (\nu P_h u + z(P_h u), v). \tag{3.45c}$$

In many applications, $J(\cdot)$ is uniform convex near the solution u. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. Then, there exists a constant $c > 0$, independent of h, such that

$$(J'(P_h u) - J'(u_h), P_h u - u_h) \geq c\|P_h u - u_h\|^2, \tag{3.46}$$

where u and u_h are solutions of (2.6a)-(2.6e) and (2.14a)-(2.14e) respectively, $P_h u$ is the orthogonal projection of u which is defined in (2.10a). We shall assume that the above inequality throughout this paper.

Now, we will discuss the superconvergence for the control variable.

Lemma 3.4. *Let* u *be the solution of* (2.6a)-(2.6e) *and* u_h *be the solution of* (2.14a)-(2.14e)*, respectively. Assume that* $\boldsymbol{p}(u_h)$*,* $\boldsymbol{q}(u_h) \in (H^1(\Omega))^2$ *and* $u, z \in W^{1,\infty}(\Omega)$*. Then, we have*

$$\|P_h u - u_h\| \leq Ch^{\frac{3}{2}}. \tag{3.47}$$

Proof. We choose $\tilde{u} = u_h$ in (2.6e) and $\tilde{u}_h = P_h u$ in (2.14e) to get the following two inequalities:

$$(\nu u + z, u_h - u) \geq 0 \tag{3.48}$$

and

$$(\nu u_h + z_h, P_h u - u_h) \geq 0. \tag{3.49}$$

Note that $u_h - u = u_h - P_h u + P_h u - u$. Adding the two inequalities (3.48) and (3.49), we have

$$(\nu u_h + z_h - \nu u - z, P_h u - u_h) + (\nu u + z, P_h u - u) \geq 0. \tag{3.50}$$

Thus, by (3.46), (3.50) and (2.10a), we find that

$$\begin{aligned}
c\|P_h u - u_h\|^2 \leq &(J'(P_h u) - J'(u_h), P_h u - u_h) \\
= &\nu(P_h u - u_h, P_h u - u_h) + (z(P_h u) - z(u_h), P_h u - u_h) \\
= &\nu(P_h u - u, P_h u - u_h) + \nu(u - u_h, P_h u - u_h) \\
&+ (z(P_h u) - z(u_h), P_h u - u_h) \\
\leq &(z_h - z, P_h u - u_h) + (\nu u + z, P_h u - u) \\
&+ (z(P_h u) - z(u_h), P_h u - u_h) \\
= &(z_h - P_h z(u_h), P_h u - u_h) + (\nu u + z, P_h u - u) \\
&+ (z(P_h u) - z(u), P_h u - u_h).
\end{aligned} \tag{3.51}$$

By Lemma 3.2 and Lemma 3.3, we find that

$$(z_h - P_h z(u_h), P_h u - u_h) \leq Ch^4 + \frac{c}{4}\|P_h u - u_h\|^2 \tag{3.52}$$

and

$$(z(P_h u) - z(u), P_h u - u_h) \leq Ch^4 + \frac{c}{4}\|P_h u - u_h\|^2. \tag{3.53}$$

For the second term at the right side of (3.51), by Theorem 5.1 in [8], we have

$$(\nu u + z, P_h u - u) \leq Ch^3(\|u\|_{1,\infty}^2 + \|z\|_{1,\infty}^2). \tag{3.54}$$

Combining (3.51)-(3.54), we derive (3.47). □

Now, let us recall the recovery operator G_h. Let $G_h v$ be a continuous piecewise linear function (without zero boundary constraint). The value of $G_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes, the details can be refer to the definition of R_h in [15].

Theorem 3.1. *Let u and u_h be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e)*, respectively. Assume that all the conditions in Lemma 3.4 are valid and $u \in W^{1,\infty}(\Omega)$. Then we have*

$$\|u - G_h u_h\| \leq Ch^{\frac{3}{2}}. \tag{3.55}$$

Proof. Let $P_h u$ be defined in (2.10a). Then

$$\|u - G_h u_h\| \le \|u - G_h u\| + \|G_h u - G_h P_h u\| + \|G_h P_h u - G_h u_h\|. \tag{3.56}$$

According to Lemma 4.2 in [15], we have

$$\|u - G_h u\| \le Ch^{\frac{3}{2}}. \tag{3.57}$$

Using the definition of G_h, we find that

$$G_h u = G_h P_h u \tag{3.58}$$

and

$$\|G_h P_h u - G_h u_h\| \le C\|P_h u - u_h\|. \tag{3.59}$$

Combining (3.56)-(3.59) with Lemma 3.4, we complete the proof. □

4. L^∞-error estimates

In this section, we will give the L^∞-error estimates both for the control variable and the state, co-state variables.

Now, we recall a result from Bonnans and Casas [2].

Lemma 4.1. *Let $a_0 \ge 0$ be a function in $L^\infty(\Omega)$. Then for every $p \ge 2$ and every function $g \in L^p(\Omega)$, the solution y of*

$$-\mathrm{div}(A\mathbf{grad}y) + a_0 y = g \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0, \tag{4.1}$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant C independent of a_0 such that

$$\|y\|_{W^{2,p}(\Omega)} \le C\|g\|_{L^p(\Omega)}. \tag{4.2}$$

Theorem 4.1. *Let (y,z,u) and (y_h,z_h,u_h) be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively, then we have*

$$\|u - u_h\|_{0,\infty} \le Ch, \tag{4.3a}$$

$$\|y - y_h\|_{0,\infty} + \|z - z_h\|_{0,\infty} \le Ch. \tag{4.3b}$$

Proof. Using (2.10b) and Lemma 3.4, it is easy to see that

$$\begin{aligned}\|u - u_h\| \le& \|u - P_h u\| + \|P_h u - u_h\| \\ \le& Ch\|u\|_1 + \|P_h u - u_h\| \\ \le& Ch.\end{aligned} \tag{4.4}$$

Choosing $\tilde{u}=u$ and $\tilde{u}=u_h$ in (2.16a) and (2.16b), using mean value theorem, we have

$$-\text{div}(A\nabla(y-y(u_h)))+\phi'(\bar{y})(y-y(u_h))=u-u_h, \tag{4.5}$$

for some function $\bar{y}$. Using (4.4), the regularity result (4.2) and the classical Imbedding Theorem, we can see that

$$\begin{aligned}\|y-y(u_h)\|_{0,\infty}\leq& C\|y-y(u_h)\|_2\\ \leq& C\|u-u_h\|\\ \leq& Ch.\end{aligned} \tag{4.6}$$

Thus, by use of Lemma 3.2, (2.10b), (4.6) and the inverse estimate, we find that

$$\begin{aligned}\|y-y_h\|_{0,\infty}\leq&\|y-y(u_h)\|_{0,\infty}+\|y(u_h)-P_hy(u_h)\|_{0,\infty}+\|P_hy(u_h)-y_h\|_{0,\infty}\\ \leq& Ch\|y(u_h)\|_{1,\infty}+Ch^{-1}\|P_hy(u_h)-y_h\|\\ \leq& Ch.\end{aligned} \tag{4.7}$$

Similarly, we have the following error equation

$$\begin{aligned}&-\text{div}(A\nabla(z-z(u_h)))+\phi'(y(u_h))(z-z(u_h))\\ =&-\text{div}(A^2\nabla(y-y(u_h)))+y-y(u_h)-z(\phi'(y)-\phi'(y(u_h))).\end{aligned} \tag{4.8}$$

Using Lemma 4.1 and the classical Imbedding Theorem, we can see that

$$\begin{aligned}\|z-z(u_h)\|_{0,\infty}\leq& C\|z-z(u_h)\|_2\\ \leq& C\|\text{div}(A^2\nabla(y-y(u_h)))-y+y(u_h)+z(\phi'(y)-\phi'(y(u_h)))\|\\ \leq& C\|\text{div}(A^2\nabla(y-y(u_h)))\|+C\|y-y(u_h)\|+C\|z(\phi'(y)-\phi'(y(u_h)))\|\\ \leq& C\|A^2\nabla(y-y(u_h))\|_1+C\|y-y(u_h)\|+C\|z\|_{0,\infty}\|\phi\|_{2,\infty}\|y-y(u_h)\|\\ \leq& C\|A\|^2_{1,\infty}\|y-y(u_h)\|_2+C\|y-y(u_h)\|\\ \leq& C\|y-y(u_h)\|_2.\end{aligned} \tag{4.9}$$

Thus, by use of (2.10b), (4.6), (4.9), Lemma 3.2 and the inverse estimate, we find that

$$\begin{aligned}\|z-z_h\|_{0,\infty}\leq&\|z-z(u_h)\|_{0,\infty}+\|z(u_h)-P_hz(u_h)\|_{0,\infty}+\|P_hz(u_h)-z_h\|_{0,\infty}\\ \leq& Ch.\end{aligned} \tag{4.10}$$

Finally, from (2.7), (2.15) and (4.10), we get

$$\|u-u_h\|_{0,\infty}\leq C\|z-z_h\|_{0,\infty}\leq Ch. \tag{4.11}$$

We complete the proof. □

Theorem 4.2. *Let* $(\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{p}_h,\boldsymbol{q}_h)$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that* $\boldsymbol{p},\boldsymbol{q}\in(W^{1,\infty}(\Omega))^2$, *then we have*

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}\le Ch^{\frac{1}{2}}|\ln h|^{\frac{1}{2}}, \tag{4.12a}$$

$$\|\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h)\|_{0,\infty}+\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|_{0,\infty}\le Ch. \tag{4.12b}$$

Proof. By modifying the proof of Theorem 3.3 in [18], we can derive

$$\|\Pi_h\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}+\|\Pi_h\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}\le Ch^{\frac{1}{2}}|\ln h|^{\frac{1}{2}}. \tag{4.13}$$

Thus, (4.12a) can be proved by (2.11b) and (4.13).

Moreover, from (1.3), (2.10b), (3.13) and (4.3a)-(4.3b), we have

$$\begin{aligned}\|\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h)\|_{0,\infty}=&\|u-\phi(y)-(u_h-P_h\phi(y_h))\|_{0,\infty}\\ \le&\|u-u_h\|_{0,\infty}+\|\phi(y)-P_h\phi(y)\|_{0,\infty}+\|P_h(\phi(y)-\phi(y_h))\|_{0,\infty}\\ \le&\|u-u_h\|_{0,\infty}+Ch\|\phi\|_{1,\infty}+C\|\phi(y)-\phi(y_h)\|_{0,\infty}\\ \le&Ch+C\|\phi\|_{1,\infty}\|y-y_h\|_{0,\infty}\\ \le&Ch.\end{aligned} \tag{4.14}$$

Similarly, using (2.2), (3.27), (2.10b) and (4.3a)-(4.3b), we get

$$\begin{aligned}\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|_{0,\infty}=&\|y-y_d-\phi'(y)z-(y_h-P_hy_d-P_h\phi'(y_h)z_h)\|_{0,\infty}\\ \le&\|y-y_h\|_{0,\infty}+\|y_d-P_hy_d\|_{0,\infty}+\|\phi'(y)z-P_h(\phi'(y)z)\|_{0,\infty}\\ &+\|P_h(\phi'(y)z)-P_h\phi'(y_h)z_h\|_{0,\infty}\\ \le&\|y-y_h\|_{0,\infty}+Ch\|y_d\|_{1,\infty}+Ch\|\phi\|_{2,\infty}\|z\|_{1,\infty}\\ &+C\|(\phi'(y)-\phi'(y_h))z\|_{0,\infty}+C\|\phi'(y_h)(z-z_h)\|_{0,\infty}\\ \le&Ch+C\|\phi\|_{1,\infty}\|z\|_{0,\infty}\|y-y_h\|_{0,\infty}+C\|\phi\|_{1,\infty}\|z-z_h\|_{0,\infty}\\ \le&Ch.\end{aligned} \tag{4.15}$$

We complete the proof. □

5. Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [14]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions, whereas the state $(y,\boldsymbol{p})$ and the co-state $(z,\boldsymbol{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In our examples, we choose the domain $\Omega=[0,1]\times[0,1]$, $\nu=1$ and $A=I$.

Example 5.1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u-u_0\|^2\right\} \tag{5.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p}+y^3=f+u, \quad \boldsymbol{p}=-\mathbf{grad}y, \tag{5.2}$$

where

$$y=\sin(\pi x_1)\sin(\pi x_2), \qquad z=\sin(\pi x_1)\sin(\pi x_2), \tag{5.3a}$$

$$u_0=1.0-0.8\sin\left(\frac{\pi x_1}{2}\right)-0.9\sin(2\pi x_2), \qquad u=\max(u_0-z,0), \tag{5.3b}$$

$$f=2\pi^2y+y^3-u, \qquad y_d=y-2\pi^2y-3y^2z, \tag{5.3c}$$

$$\boldsymbol{p}_d=-\begin{pmatrix}\pi\cos(\pi x_1)\sin(\pi x_2)\\ \pi\sin(\pi x_1)\cos(\pi x_2)\end{pmatrix}. \tag{5.3d}$$

In Table 1, the errors $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$ and $\|u-G_hu_h\|$ obtained on a sequence of uniformly refined meshes are shown. Table 2 displays the errors $\|y-y_h\|_{0,\infty}$, $\|z-z_h\|_{0,\infty}$, $\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$ and $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$. In Fig. 1, the profile of the numerical solution of u on the 64×64 mesh grid is plotted. Moreover, in Figs. 2 and 3, we show the convergence orders by slopes. Theoretical results are clearly recognized from the data.

Table 1: The errors of $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$ and $\|u-G_hu_h\|$.

Resolution	$\|u-u_h\|$	$\|u-u_h\|_{0,\infty}$	$\|P_hu-u_h\|$	$\|u-G_hu_h\|$
16×16	4.95336e-02	2.64503e-01	6.67828e-03	4.16357e-02
32×32	2.49825e-02	1.33711e-01	2.16314e-03	1.67884e-02
64×64	1.25400e-02	6.70237e-02	7.39935e-04	6.13222e-03
128×128	6.28424e-03	3.35326e-02	2.68840e-04	2.16078e-03

Table 2: The errors of $\|y-y_h\|_{0,\infty}$, $\|z-z_h\|_{0,\infty}$, $\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$ and $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$.

Resolution	$\|y-y_h\|_{0,\infty}$	$\|z-z_h\|_{0,\infty}$	$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$	$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$
16×16	9.08039e-02	9.08531e-02	2.69975e-01	2.70048e-01
32×32	4.55233e-02	4.55299e-02	1.90910e-01	1.90960e-01
64×64	2.27762e-02	2.27770e-02	1.35655e-01	1.35674e-01
128×128	1.13906e-02	1.13907e-02	9.59339e-02	9.59396e-02

6. Conclusions and future works

In this paper, we discussed the lowest order Raviart-Thomas mixed finite element methods for the semilinear elliptic optimal control problem (1.1)-(1.4). Our superconvergence

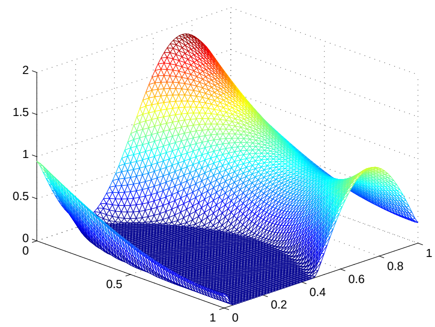

Figure 1: The profile of the numerical solution of u on 64×64 triangle mesh.

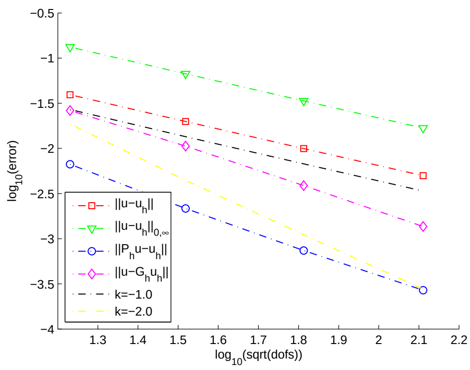

Figure 2: Convergence orders of $u-u_h$, P_hu-u_h and $u-G_hu_h$ in different norms.

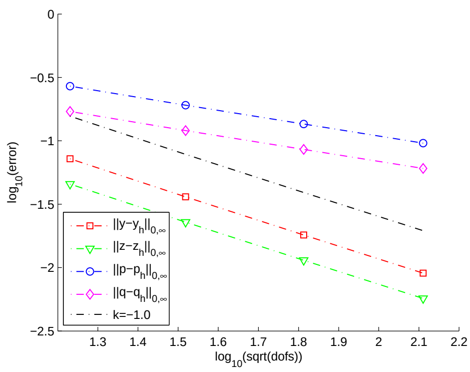

Figure 3: Convergence orders of $y-y_h$, $z-z_h$, $\boldsymbol{p}-\boldsymbol{p}_h$ and $\boldsymbol{q}-\boldsymbol{q}_h$ in L^∞-norm.

analysis and L^∞-error estimates for the semilinear elliptic equations by mixed finite element methods seems to be new, and these results can be extended to general convex problems. In our future work, we will investigate the superconvergence of mixed finite element methods for optimal control problems governed by nonlinear parabolic equations.

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