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Laguerre Spectral Method for High Order Problems

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Abstract. In this paper, we propose the Laguerre spectral method for high order problems with mixed inhomogeneous boundary conditions. It is also available for approximated solutions growing fast at infinity. The spectral accuracy is proved. Numerical results demonstrate its high effectiveness.

AMS subject classifications: 65L60, 65M70

Key words: Laguerre spectral method, high order problems with mixed inhomogeneous boundary conditions.

1. Introduction

The spectral method possesses high accuracy, and so plays an important role in numerical solutions of differential and integral equations, see [2–4, 7–9, 18] and the references therein. During the past two decades, more and more attentions were paid to problems defined on various unbounded domains. The Laguerre spectral method has been used widely for differential equations defined on the half line and the related unbounded domains, as well as certain exterior problems, see [10–12,14,15,17,19,20] and the references therein. But, there are still two unsettled problems. Firstly, in the existing work, we usually reformed original problems by some variable transformations, and then solved the alternative formulations by using the Laguerre approximation. Thus, those spectral schemes seem available essentially for approximated solutions decaying to zero at infinity. However, in many cases, the solutions do not tend to zero at infinity, such as the kink-like solitons, the heteroclinic solutions in biology, the solutions of Harry-Dym equation and some nonlinear

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dynamical systems. Next, since the existing results on the Laguerre approximation are not optimal, the error estimates of numerical solutions are not very precise. On the other hand, we considered second order problems mostly. Whereas, in some practical cases, such as the stream function form of the Navier-Stokes equations, we have to deal with high order problems. Recently, Guo, Sun and Zhang [13] proposed the generalized Laguerre quasi-orthogonal approximation, which leads to the probability of producing new Laguerre spectral method suitable also for high order problems with solutions growing fast at infinity, and deriving better error estimates of numerical solutions. We refer to the review paper of Guo, Zhang and Sun [16].

In this paper, we investigate the new Laguerre spectral method for high order problems with mixed inhomogeneous boundary conditions. The next section is for preliminaries. In Section 3, we consider two fourth order problems with various boundary conditions. We design the spectral schemes and prove their spectral accuracy. They are also suitable for high order problems with solutions growing fast at infinity. Moreover, we provide the Laguerre spectral method with exact imposition of boundary conditions. In Section 4, we present some numerical results demonstrating the high effectiveness of suggested algorithms. The final section is for concluding remarks.

2. Preliminaries

Let $\Lambda = \{x|0 < x < \infty\}$ and $\chi(x)$ be certain a weight function. For integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $\|\cdot\|_{r,\chi,\Lambda}$. In particular, the inner product and the norm of $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. For simplicity, we denote $d^k v/dx^k$ by $\partial_x^k v$. For integer $r \geq 1$,

$$_0H^r_\chi(\Lambda) = \left\{v \in H^r_\chi(\Lambda) | \partial_x^k v(0) = 0,\ 0 \leq k \leq r-1\right\}.$$

We omit the subscript χ in notations whenever $\chi(x) \equiv 1$.

The scaled generalized Laguerre polynomials of degree $l \geq 0$ were given in [17], as

$$L_l^{(\alpha,\beta)}(x) = \frac{1}{l!} x^{-\alpha} e^{\beta x} \partial_x^l (x^{l+\alpha} e^{-\beta x}), \quad \alpha > -1, \quad \beta > 0, \quad l \geq 0.$$

In this work, we shall use the specific base functions $\mathscr{L}_l^{(-m,\beta)}(x)$ with integer $m \geq 1$, namely (see [13]),

$$\mathscr{L}_l^{(-m,\beta)}(x) = x^m L_{l-m}^{(m,\beta)}(x), \quad l \geq m. \tag{2.1}$$

Let $\omega_{-m,\beta}(x) = x^{-m} e^{-\beta x}$. By (4.8) of [13], we have

$$\int_\Lambda \mathscr{L}_l^{(-m,\beta)}(x) \mathscr{L}_{l'}^{(-m,\beta)}(x) \omega_{-m,\beta}(x) dx = \eta_l^{(-m,\beta)} \delta_{l,l'}, \tag{2.2}$$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\eta_l^{(-m,\beta)} = \frac{1}{l!}\beta^{m-1}\Gamma(l+m+1), \quad l \geq m. \tag{2.3}$$

The set of all $\mathscr{L}_l^{(-m,\beta)}(x)$ is complete in $L^2_{\omega_{-m,\beta}}(\Lambda)$. Hence, for any $v \in L^2_{\omega_{-m,\beta}}(\Lambda)$, we have

$$v(x) = \sum_{l=m}^{\infty} \hat{v}_l^{(-m,\beta)} \mathscr{L}_l^{(-m,\beta)}(x) \tag{2.4}$$

with

$$\hat{v}_l^{(-m,\beta)} = \frac{1}{\eta_l^{(-m,\beta)}} \int_\Lambda v(x)\mathscr{L}_l^{(-m,\beta)}(x)\omega_{-m,\beta}(x)dx. \tag{2.5}$$

For integer $r \geq 0$, we introduce the following space

$$H^r_{\omega_{-m,\beta},A}(\Lambda) = \{v | v \text{ is measurable on } \Lambda \text{ and } \|v\|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} < \infty\}$$

equipped with the semi-norm and the norm as

$$|v|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} = \|\partial_x^r v\|_{\omega_{-m+r,\beta},\Lambda}, \qquad \|v\|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} = \left(\sum_{k=0}^{r} |v|^2_{H^k_{\omega_{-m,\beta},A}(\Lambda)}\right)^{\frac{1}{2}}.$$

Moreover, for $1 \leq m \leq r$,

$$\begin{aligned} {}_0H^r_{\omega_{-m,\beta},A}(\Lambda) &= \{v | v \in H^r_{\omega_{-m,\beta},A}(\Lambda) \text{ and } \partial_x^k v(0) = 0, \text{ for } 0 \leq k \leq r-1\}, \\ B^r_{m,\beta}(\Lambda) &= {}_0H^m_{\omega_{-m,\beta},A}(\Lambda) \cap H^r_{\omega_{-m,\beta},A}(\Lambda). \end{aligned}$$

For integers $1 \leq m \leq N$,

$$Q_N^{(-m,\beta)}(\Lambda) = \text{span}\{\mathscr{L}_m^{(-m,\beta)}(x), \mathscr{L}_{m+1}^{(-m,\beta)}(x), \cdots, \mathscr{L}_N^{(-m,\beta)}(x)\}.$$

The projection ${}_0P^m_{N,-m,\beta,\Lambda} : {}_0H^m_{\omega_{-m,\beta},A}(\Lambda) \to Q_N^{(-m,\beta)}(\Lambda)$ is defined by

$$(\partial_x^m({}_0P^m_{N,-m,\beta,\Lambda}v - v), \partial_x^m\phi)_{\omega_{0,\beta},\Lambda} = 0, \quad \forall \phi \in Q_N^{(-m,\beta)}(\Lambda). \tag{2.6}$$

In numerical analysis of the new Laguerre spectral method, we need the Laguerre quasi-orthogonal approximation. To do this, let

$$v_{b,m}(x) = \sum_{j=0}^{m-1} \partial_x^j v(0)\frac{x^j}{j!}. \tag{2.7}$$

For any $v \in H^m_{\omega_{-m,\beta},A}(\Lambda)$, we set $\tilde{v}(x) = v(x) - v_{b,m}(x)$. Since $\tilde{v} \in {}_0H^m_{\omega_{-m,\beta},A}(\Lambda)$, we define the Laguerre quasi-orthogonal projection as

$$P^m_{N,-m,\beta}v(x) = {}_0P^m_{N,-m,\beta}\tilde{v}(x) + v_{b,m}(x) \in \mathscr{P}_N(\Lambda). \tag{2.8}$$

Obviously,

$$\partial_x^k P^m_{N,-m,\beta}v(0) = \partial_x^k v(0), \quad \text{for } 0 \le k \le m-1.$$

According to Theorem 4.4 of [13], we assert that if $v \in H^m_{\omega_{-m,\beta},A}(\Lambda)$, $\partial_x^r v \in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1 \le m \le \min(r,N), 0 \le k \le r \le N+1$ and $m \le k$, then

$$\left\|\partial_x^k(P^m_{N,-m,\beta}v - v)\right\|_{\omega_{-m+k,\beta}} \le c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega_{-m+r,\beta}}. \tag{2.9}$$

Hereafter, we denote by c a generic positive constant independent of any function and N.

It is noted that Everitt, Littlejohn and Wellman [6] also considered the orthogonal approximation using the base functions (2.1) with $\beta = 1$, without error estimation.

3. New Laguerre spectral method for high order problems

This section is devoted to the new Laguerre spectral method for high order problems defined on the half line.

3.1. Some preparations

As examples, we shall consider fourth order problems with Dirichlet or mixed Dirichlet-Neumann boundary conditions. Let $\lambda > 0$. For simplicity of statements, we introduce the bilinear form

$$\mathscr{A}_\lambda(u,v) = a_{1,\lambda}(u,v) + a_2(u,v), \quad \forall u,v \in H^2_{\omega_{0,\beta}}(\Lambda), \tag{3.1}$$

with

$$a_{1,\lambda}(u,v) = \int_\Lambda \partial_x^2 u(x)\partial_x^2 v(x)e^{-\beta x}dx + \lambda\int_\Lambda u(x)v(x)e^{-\beta x}dx, \tag{3.2a}$$

$$a_2(u,v) = -2\beta\int_\Lambda \partial_x^2 u(x)\partial_x v(x)e^{-\beta x}dx + \beta^2\int_\Lambda \partial_x^2 u(x)v(x)e^{-\beta x}dx. \tag{3.2b}$$

The space $H^1_{\omega_{0,\beta}}(\Lambda)$ can be regarded as the interpolation between the spaces $H^2_{\omega_{0,\beta}}(\Lambda)$ and $L^2_{\omega_{0,\beta}}(\Lambda)$. Accordingly, there exists a positive constant d_β such that $\|\partial_x v\|^2_{\omega_{0,\beta}} \le d_\beta\big(\|\partial_x^2 v\|^2_{\omega_{0,\beta}} + \|v\|^2_{\omega_{0,\beta}}\big)$.

It is clear that

$$
\begin{aligned}
|\mathscr{A}_\lambda(u,v)| \leq & \frac{3}{2}\|\partial_x^2 u\|^2_{\omega_{0,\beta}} + \frac{1}{2}\|\partial_x^2 v\|^2_{\omega_{0,\beta}} + \beta^2\|\partial_x v\|^2_{\omega_{0,\beta}} \\
& + \frac{\lambda}{2}\|u\|^2_{\omega_{0,\beta}} + \frac{1}{2}(\lambda+\beta^4)\|v\|^2_{\omega_{0,\beta}} \\
\leq & \frac{3}{2}\|\partial_x^2 u\|^2_{\omega_{0,\beta}} + \Big(\frac{1}{2}+\beta^2 d_\beta\Big)\|\partial_x^2 v\|^2_{\omega_{0,\beta}} \\
& + \frac{\lambda}{2}\|u\|^2_{\omega_{0,\beta}} + \frac{1}{2}(\lambda+2\beta^2 d_\beta+\beta^4)\|v\|^2_{\omega_{0,\beta}}, \quad \forall u,v\in H^2_{\omega_{0,\beta}}(\Lambda).
\end{aligned} \tag{3.3}
$$

Next, for any $v\in H^2_{\omega_{0,\beta}}(\Lambda)$, we have

$$
\sqrt{x}\partial_x^k v(x)e^{-\frac{\beta}{2}x}\to 0 \quad \text{as } \ x\to\infty, \quad k=0,1,2.
$$

Thus,

$$
\begin{aligned}
\beta^2\int_\Lambda \partial_x^2 v(x)v(x)e^{-\beta x}dx = & -\beta^2\int_\Lambda (\partial_x v(x))^2 e^{-\beta x}dx + \beta^3\int_\Lambda \partial_x v(x)v(x)e^{-\beta x}dx \\
& -\beta^2\partial_x v(0)v(0).
\end{aligned} \tag{3.4}
$$

With the aid of (3.4), we verify that

$$
\begin{aligned}
-2\beta\int_\Lambda \partial_x^2 v(x)\partial_x v(x)e^{-\beta x}dx = & -\beta^2\int_\Lambda (\partial_x v(x))^2 e^{-\beta x}dx + \beta(\partial_x v(0))^2 \\
= & \beta^2\int_\Lambda \partial_x^2 v(x)v(x)e^{-\beta x}dx - \beta^3\int_\Lambda \partial_x v(x)v(x)e^{-\beta x}dx \\
& + \beta(\partial_x v(0))^2 + \beta^2\partial_x v(0)v(0) \\
= & \beta^2\int_\Lambda \partial_x^2 v(x)v(x)e^{-\beta x}dx - \frac{1}{2}\beta^4\int_\Lambda v^2(x)e^{-\beta x}dx \\
& + \beta(\partial_x v(0))^2 + \frac{1}{2}\beta^3(v(0))^2 + \beta^2\partial_x v(0)v(0).
\end{aligned} \tag{3.5}
$$

Inserting (3.5) into (3.2b), we obtain

$$
\begin{aligned}
a_2(v,v) = & 2\beta^2\int_\Lambda \partial_x^2 v(x)v(x)e^{-\beta x}dx - \frac{1}{2}\beta^4\int_\Lambda v^2(x)e^{-\beta x}dx \\
& + \beta(\partial_x v(0))^2 + \frac{1}{2}\beta^3(v(0))^2 + \beta^2\partial_x v(0)v(0) \\
\geq & -\frac{1}{2}\int_\Lambda (\partial_x^2 v(x))^2 e^{-\beta x}dx - \frac{5}{2}\beta^4\int_\Lambda v^2(x)e^{-\beta x}dx \\
& + \frac{1}{4}\beta(\partial_x v(0))^2 + \frac{1}{6}\beta^3(v(0))^2.
\end{aligned} \tag{3.6}
$$

Thereby, for any $v \in H^2_{\omega_{0,\beta}}(\Lambda)$,

$$
\begin{aligned}
&\frac{1}{2}\|\partial_x^2 v\|^2_{\omega_{0,\beta}} + \Big(\lambda - \frac{5}{2}\beta^4\Big)\|v\|^2_{\omega_{0,\beta}} + \frac{1}{4}\beta(\partial_x v(0))^2 + \frac{1}{6}\beta^3(v(0))^2 \\
\le& \mathscr{A}_\lambda(v,v) \le \frac{3}{2}\|\partial_x^2 v\|^2_{\omega_{0,\beta}} + \Big(\lambda + \frac{3}{2}\beta^4\Big)\|v\|^2_{\omega_{0,\beta}} + \frac{5}{4}\beta(\partial_x v(0))^2 + \frac{3}{2}\beta^3(v(0))^2. \qquad (3.7)
\end{aligned}
$$

On the other hand, if $u, v \in H^2_{\omega_{0,\beta}}(\Lambda)$ and $\partial_x^3 u(x)v(x)e^{-\beta x} \to 0$ as $x \to \infty$, then

$$
\begin{aligned}
&\int_\Lambda \partial_x^4 u(x)v(x)e^{-\beta x}dx \\
=& -\int_\Lambda \partial_x^3 u(x)\partial_x v(x)e^{-\beta x}dx + \beta\int_\Lambda \partial_x^3 u(x)v(x)e^{-\beta x}dx - \partial_x^3 u(0)v(0) \\
=& \int_\Lambda \partial_x^2 u(x)\partial_x^2 v(x)e^{-\beta x}dx - 2\beta\int_\Lambda \partial_x^2 u(x)\partial_x v(x)e^{-\beta x}dx \\
&+ \beta^2\int_\Lambda \partial_x^2 u(x)v(x)e^{-\beta x}dx - \partial_x^3 u(0)v(0) + \partial_x^2 u(0)\partial_x v(0) - \beta\partial_x^2 u(0)v(0). \qquad (3.8)
\end{aligned}
$$

Furthermore, let $W(\Lambda)$, $\overline{W}(\Lambda) \subseteqq H^2_{\omega_{0,\beta}}(\Lambda)$. We set $W_N(\Lambda) = W(\Lambda) \cap \mathscr{P}_N(\Lambda)$ and $\overline{W}_N(\Lambda) = \overline{W}(\Lambda) \cap \mathscr{P}_N(\Lambda)$. We define the operator ${}_*P^2_{N,\lambda} : W(\Lambda) \to W_N(\Lambda)$, by

$$
\mathscr{A}_\lambda({}_*P^2_{N,\lambda}v - v, \phi) = 0, \quad \forall \phi \in \overline{W}_N(\Lambda). \qquad (3.9)
$$

The above operator possesses a nice property stated below.

Proposition 3.1. *Let $v \in W(\Lambda)$, $w \in W_N(\Lambda)$ and $\lambda > 5\beta^4/2$. If ${}_*P^2_{N,\lambda}v - w \in \overline{W}_N(\Lambda)$, then*

$$
\mathscr{A}_\lambda(v - {}_*P^2_{N,\lambda}v, v - {}_*P^2_{N,\lambda}v) \le \mathscr{A}_\lambda(v - w, v - w). \qquad (3.10)
$$

Proof. A direct calculation shows

$$
\begin{aligned}
\mathscr{A}_\lambda(v - w, v - w) =& \mathscr{A}_\lambda(v - {}_*P^2_{N,\lambda}v, v - {}_*P^2_{N,\lambda}v) + \mathscr{A}_\lambda({}_*P^2_{N,\lambda}v - w, {}_*P^2_{N,\lambda}v - w) \\
&+ 2\mathscr{A}_\lambda(v - {}_*P^2_{N,\lambda}v, {}_*P^2_{N,\lambda}v - w).
\end{aligned}
$$

Thanks to (3.9), we have

$$
\mathscr{A}_\lambda(v - {}_*P^2_{N,\lambda}v, {}_*P^2_{N,\lambda}v - w) = 0.
$$

Due to $\lambda > 5\beta^4/2$, (3.7) implies

$$
\mathscr{A}_\lambda({}_*P^2_{N,\lambda}v - w, {}_*P^2_{N,\lambda}v - w) \ge 0.
$$

Then, the desired result (3.10) follows from the previous statements. $\square$

3.2. Dirichlet problem of high order equation

We first consider the following simple model problem,

$$\begin{cases} \partial_x^4 U(x) + \lambda U(x) = f(x), & x \in \Lambda, \\ \partial_x U(0) = b, & U(0) = a, \end{cases} \tag{3.11}$$

where $f \in L^2_{\omega_{0,\beta}}(\Lambda)$, a, b and $\lambda > 0$ are given constants. The solution $U(x)$ might tend to infinity as x goes to ∞. We assume that $x^{\frac{1}{2}} e^{-\frac{\beta}{2}x} \partial_x^k U(x) \to 0$, as $x \to \infty$, $0 \le k \le 2$, and $x^{-\frac{1}{2}} e^{-\frac{\beta}{2}x} \partial_x^3 U(x) \to 0$, as $x \to \infty$. Let

$$\begin{aligned} &\tilde{H}^2_{\omega_{0,\beta}}(\Lambda) = H^2_{\omega_{0,\beta}}(\Lambda) \cap \{v | x^{-\frac{1}{2}} e^{-\frac{\beta}{2}x} \partial_x^3 v(x) \to 0, \text{ as } x \to \infty\}, \\ &V(\Lambda) = \{v | v \in \tilde{H}^2_{\omega_{0,\beta}}(\Lambda) \text{ and } \partial_x v(0) = b,\ v(0) = a\}, \quad \overline{V}(\Lambda) = {}_0H^2_{\omega_{0,\beta}}(\Lambda). \end{aligned}$$

Multiplying (3.11) by $v(x)e^{-\beta x} \in \overline{V}(\Lambda)$ and integrating the resulting equation by parts, we use (3.8) to derive a weak formulation of (3.11). It is to look for $U \in V(\Lambda)$ such that

$$\mathscr{A}_\lambda(U, v) = (f, v)_{\omega_{0,\beta}}, \quad \forall v \in \overline{V}(\Lambda). \tag{3.12}$$

If $\lambda > 5\beta^4/2$, then by (3.3), (3.7) and the Lax-Milgram lemma, the problem (3.12) admits a unique solution.

Remark 3.1. The condition of $\lambda > 5\beta^4/2$ plays an important role for the existence and uniqueness of solution of (3.12), as well as for ensuring the convergence of the related spectral method. However, it is only a sufficient condition. On the other hand, for any fixed β, we may take a positive parameter $\gamma < \beta^{-1}\sqrt[4]{2\lambda/5}$, and make the following variable transformation,

$$y = \gamma x, \quad V(y) = U\left(\frac{y}{\gamma}\right), \quad F(y) = \gamma^{-4} f\left(\frac{y}{\gamma}\right).$$

Then, the original problem (3.11) is changed to

$$\begin{cases} \partial_y^4 V(y) + \lambda^* V(y) = F(y), & \lambda^* = \dfrac{\lambda}{\gamma^4}, \quad y \in \Lambda, \\ \partial_y V(0) = \dfrac{b}{\gamma}, & V(0) = a. \end{cases}$$

In this case, $\lambda^* > 5\beta^4/2$.

We now define the finite-dimensional spaces:

$$V_N(\Lambda) = V(\Lambda) \cap \mathscr{P}_N(\Lambda), \qquad \overline{V}_N(\Lambda) = \overline{V}(\Lambda) \cap \mathscr{P}_N(\Lambda).$$

The spectral method for solving (3.12), is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathscr{A}_\lambda(u_N, \phi) = (f, \phi)_{\omega_{0,\beta}}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.13}$$

For checking the existence of solutions of (3.13), it suffices to prove the uniqueness of solution. Assume that $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (3.13), and $\tilde{u}_N(x)=u_N^{(1)}(x)-u_N^{(2)}(x)\in\overline{V}_N(\Lambda)$. Then

$$\mathscr{A}_\lambda(\tilde{u}_N,\phi)=0,\quad \forall\phi\in\overline{V}_N(\Lambda).$$

Putting $\phi=\tilde{u}_N\in\overline{V}_N(\Lambda)$ in the above equation and using (3.7), we obtain

$$\frac{1}{2}\|\partial_x^2\tilde{u}_N\|^2_{\omega_{0,\beta}}+\Big(\lambda-\frac{5}{2}\beta^4\Big)\|\tilde{u}_N\|^2_{\omega_{0,\beta}}\le\mathscr{A}_\lambda(\tilde{u}_N,\tilde{u}_N)=0.$$

If $\lambda>5\beta^4/2$, then $\tilde{u}_N(x)\equiv 0$. This means the uniqueness of solution of (3.13).

We now estimate the error of numerical solution. For this purpose, we introduce the auxiliary operator $\overline{P}^2_{N,\lambda}:V(\Lambda)\to V_N(\Lambda)$, defined by

$$\mathscr{A}_\lambda\big(\overline{P}^2_{N,\lambda}v-v,\phi\big)=0,\quad\forall\phi\in\overline{V}_N(\Lambda).\tag{3.14}$$

We have from (3.12) and (3.14) that

$$\mathscr{A}_\lambda\big(\overline{P}^2_{N,\lambda}U,\phi\big)=(f,\phi)_{\omega_{0,\beta}},\quad\forall\phi\in\overline{V}_N(\Lambda).\tag{3.15}$$

Subtracting (3.15) from (3.13), yields

$$\mathscr{A}_\lambda\big(u_N-\overline{P}^2_{N,\lambda}U,\phi\big)=0,\quad\forall\phi\in\overline{V}_N(\Lambda).\tag{3.16}$$

Taking $\phi=u_N-\overline{P}^2_{N,\lambda}U\in\overline{V}_N(\Lambda)$ in (3.16), we obtain

$$\mathscr{A}_\lambda\big(u_N-\overline{P}^2_{N,\lambda}U,u_N-\overline{P}^2_{N,\lambda}U\big)=0.$$

Using (3.7) again, we assert that $u_N=\overline{P}^2_{N,\lambda}U$. Furthermore, for any $v\in{}_0H^1_{\omega_{0,\beta}}(\Lambda)$ (see Lemma 2.2 of [17]),

$$\|v\|^2_{\omega_{0,\beta}}\le\frac{4}{\beta^2}\|\partial_x v\|^2_{\omega_{0,\beta}}.\tag{3.17}$$

Clearly, $\partial_x^k(U(0)-u_N(0))=0$ for $k=0,1$. Thereby,

$$\|\partial_x^k(U-u_N)\|^2_{\omega_{0,\beta}}\le\Big(\frac{2}{\beta}\Big)^{4-2k}\|\partial_x^2(U-\overline{P}^2_{N,\lambda}U)\|^2_{\omega_{0,\beta}},\quad k=0,1,2.\tag{3.18}$$

We next estimate the right side of (3.18). Let $P^2_{N,-2,\beta}U$ be the Laguerre quasi-orthogonal projection defined by (2.8) with $\mu=m=2$, $v=U$, $\partial_x v(0)=b$ and $v(0)=a$. Obviously, $P^2_{N,-2,\beta}U\in V_N(\Lambda)$. Moreover, $U-P^2_{N,-2,\beta}U\in\overline{V}(\Lambda)$ and $\overline{P}^2_{N,\lambda}U-P^2_{N,-2,\beta}U\in\overline{V}_N(\Lambda)$. Therefore, we could use the result of Proposition 3.1, with

$$\begin{aligned}
&W(\Lambda)=V(\Lambda), && \overline{W}(\Lambda)=\overline{V}(\Lambda), && W_N(\Lambda)=V_N(\Lambda), && \overline{W}_N(\Lambda)=\overline{V}_N(\Lambda),\\
&v=U, && w=P^2_{N,-2,\beta}U, && {}_*P^2_{N,\lambda}U=\overline{P}^2_{N,\lambda}U.
\end{aligned}$$

More precisely, we use (3.7), (3.10) and (3.18) to derive that

$$\begin{aligned}
&\frac{1}{2}\|\partial_x^2(U-\overline{P}_{N,\lambda}^2U)\|^2_{\omega_{0,\beta}}+\Big(\lambda-\frac{5}{2}\beta^4\Big)\|U-\overline{P}_{N,\lambda}^2U\|^2_{\omega_{0,\beta}}\\
\le&\mathscr{A}_\lambda(U-\overline{P}_{N,\lambda}^2U,U-\overline{P}_{N,\lambda}^2U)\\
\le&\mathscr{A}_\lambda(U-P_{N,-2,\beta}^2U,U-P_{N,-2,\beta}^2U)\\
\le&\frac{3}{2}\|\partial_x^2(U-P_{N,-2,\beta}^2U)\|^2_{\omega_{0,\beta}}+\Big(\lambda+\frac{3}{2}\beta^4\Big)\|U-P_{N,-2,\beta}^2U\|^2_{\omega_{0,\beta}}\\
\le&\Big(\frac{3}{2}+\frac{16}{\beta^4}\Big(\lambda+\frac{3}{2}\beta^4\Big)\Big)\|\partial_x^2(U-P_{N,-2,\beta}^2U)\|^2_{\omega_{0,\beta}}. \qquad (3.19)
\end{aligned}$$

If $\partial_x^2U\in L^2_{\omega_{0,\beta}}(\Lambda)$ and $\partial_x^rU\in L^2_{\omega_{-2+r,\beta}}(\Lambda)$, then we use (2.9) and (3.19) to find that for $2\le r\le N+1$,

$$\|\partial_x^2(U-\overline{P}_{N,\lambda}^2U)\|^2_{\omega_{0,\beta}}\le c\Big(1+\frac{\lambda}{\beta^4}\Big)(\beta N)^{2-r}\|\partial_x^rU\|^2_{\omega_{-2+r,\beta}}. \qquad (3.20)$$

This, along with (3.18), gives

$$\|\partial_x^k(U-u_N)\|^2_{\omega_{0,\beta}}\le c\Big(\frac{2}{\beta}\Big)^{4-2k}\Big(1+\frac{\lambda}{\beta^4}\Big)(\beta N)^{2-r}\|\partial_x^rU\|^2_{\omega_{-2+r,\beta}},\quad k=0,1,2. \qquad (3.21)$$

Theorem 3.1. *If $U\in\tilde{H}^2_{\omega_{0,\beta}}(\Lambda)$, $\partial_x^rU\in L^2_{\omega_{-2+r,\beta}}(\Lambda)$, $\lambda>5\beta^4/2$, and integers $2\le r\le N+1$, then*

$$\|U-u_N\|^2_{H^2_{\omega_{0,\beta}}(\Lambda)}\le c\Big(1+\frac{1}{\beta^4}\Big)\Big(1+\frac{\lambda}{\beta^4}\Big)(\beta N)^{2-r}\|\partial_x^rU\|^2_{\omega_{-2+r,\beta}}. \qquad (3.22)$$

3.3. Mixed boundary value problem of high order equation

The Laguerre quasi-orthogonal approximation plays an important role in the Laguerre spectral method for mixed inhomogeneous boundary value problems of high order, as well as domain decomposition spectral method. As an example, we consider the following model problem,

$$\begin{cases}\partial_x^4U(x)+\lambda U(x)=f(x), & x\in\Lambda,\\ \partial_x^2U(0)=b, & U(0)=a.\end{cases} \qquad (3.23)$$

We also suppose that $x^{\frac{1}{2}}e^{-\frac{\beta}{2}x}\partial_x^kU(x)\to0$, as $x\to\infty$, $0\le k\le2$, and $x^{-\frac{1}{2}}e^{-\frac{\beta}{2}x}\partial_x^3U(x)\to 0$, as $x\to\infty$. This problem is similar to steady beam equation and steady extended Fisher-Kolmogorov equation (cf. [5]). Let

$$\begin{aligned}
V(\Lambda)&=\{v\,|\,v\in\tilde{H}^2_{\omega_{0,\beta}}(\Lambda)\text{ and }v(0)=a\},\\
\overline{V}(\Lambda)&=\{v\,|\,v\in H^2_{\omega_{0,\beta}}(\Lambda)\text{ and }v(0)=0\}.
\end{aligned}$$

The bilinear form $\mathscr{A}_\lambda(u,v)$ is the same as in (3.1). By using (3.8), we derive the weak formulation of (3.23). It is to look for $U \in V(\Lambda)$ such that

$$\mathscr{A}_\lambda(U,v) + b\partial_x v(0) = (f,v)_{\omega_{0,\beta}}, \quad \forall v \in \overline{V}(\Lambda). \tag{3.24}$$

If $\lambda > 5\beta^4/2$, then by (3.3), (3.7) and the Lax-Milgram lemma, Problem (3.24) admits a unique solution. But, the condition $\lambda > 5\beta^4/2$ is not essential, as mentioned in Remark 3.1.

In order to design the spectral method for (3.24), we define the finite-dimensional spaces

$$V_N(\Lambda) = V(\Lambda) \cap \mathscr{P}_N(\Lambda), \quad \overline{V}_N(\Lambda) = \overline{V}(\Lambda) \cap \mathscr{P}_N(\Lambda).$$

The spectral method for solving (3.24), is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathscr{A}_\lambda(u_N,\phi) + b\partial_x \phi(0) = (f,\phi)_{\omega_{0,\beta}}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.25}$$

For the existence of solutions of (3.25), it suffices to check the uniqueness of solution. Assume that $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (3.25) and $\tilde{u}_N(x) = u_N^{(1)}(x) - u_N^{(2)}(x) \in \overline{V}_N(\Lambda)$. Then

$$\mathscr{A}_\lambda(\tilde{u}_N,\phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.26}$$

Putting $\phi = \tilde{u}_N$ in (3.26) and using (3.7), we obtain

$$\frac{1}{2}\|\partial_x^2 \tilde{u}_N\|^2_{\omega_{0,\beta}} + \left(\lambda - \frac{5}{2}\beta^4\right)\|\tilde{u}_N\|^2_{\omega_{0,\beta}} + \beta(\partial_x \tilde{u}_N(0))^2 \le \mathscr{A}_\lambda(\tilde{u}_N,\tilde{u}_N) = 0.$$

Thus, if $\lambda > 5\beta^4/2$, then $\tilde{u}_N(x) \equiv 0$. This implies the existence and the uniqueness of solution of (3.25).

We now turn to deal with the convergence of (3.25). We introduce the operator $\overline{P}^2_{N,\lambda}$: $V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathscr{A}_\lambda\big(\overline{P}^2_{N,\lambda} v - v, \phi\big) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.27}$$

We have from (3.24) and (3.27) that

$$\mathscr{A}_\lambda\big(\overline{P}^2_{N,\lambda} U, \phi\big) + b\partial_x \phi(0) = (f,\phi)_{\omega_{0,\beta}}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.28}$$

Subtracting (3.28) from (3.25), yields

$$\mathscr{A}_\lambda\big(u_N - \overline{P}^2_{N,\lambda} U, \phi\big) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.29}$$

Taking $\phi = u_N - \overline{P}^2_{N,\lambda} U \in \overline{V}_N(\Lambda)$ in (3.29), we obtain

$$\mathscr{A}_\lambda\big(u_N - \overline{P}^2_{N,\lambda} U, u_N - \overline{P}^2_{N,\lambda} U\big) = 0.$$

Using (3.7) again, we find that $u_N = \overline{P}^2_{N,\lambda}U$.

We now use Proposition 3.1, with

$$\begin{aligned}
&W(\Lambda) = V(\Lambda), \qquad \overline{W}(\Lambda) = \overline{V}(\Lambda), \qquad W_N(\Lambda) = V_N(\Lambda), \qquad \overline{W}_N(\Lambda) = \overline{V}_N(\Lambda),\\
&v = U, \qquad w = P^2_{N,-2,\beta}U, \qquad {}_*P^2_{N,\lambda}U = \overline{P}^2_{N,\lambda}U.
\end{aligned}$$

Therefore, we use (3.7) and (3.10) to deduce that

$$\begin{aligned}
&\frac{1}{2}\|\partial_x^2(U-\overline{P}^2_{N,\lambda}U)\|^2_{\omega_{0,\beta}} + \left(\lambda - \frac{5}{2}\beta^4\right)\|U-\overline{P}^2_{N,\lambda}U\|^2_{\omega_{0,\beta}} + \frac{1}{4}\beta(\partial_x(U-\overline{P}^2_{N,\lambda}U)(0))^2\\
\le&\mathscr{A}_\lambda(U-\overline{P}^2_{N,\lambda}U, U-\overline{P}^2_{N,\lambda}U) \le \mathscr{A}_\lambda(U-P^2_{N,-2,\beta}U, U-P^2_{N,-2,\beta}U)\\
\le&\frac{3}{2}\|\partial_x^2(U-P^2_{N,-2,\beta}U)\|^2_{\omega_{0,\beta}} + \left(\lambda+\frac{3}{2}\beta^4\right)\|U-P^2_{N,-2,\beta}U\|^2_{\omega_{0,\beta}}\\
&+\frac{5}{4}\beta(\partial_x(U-P^2_{N,-2,\beta}U)(0))^2.
\end{aligned}\tag{3.30}$$

According to the construction of $P^2_{N,-2,\beta}U$, we have $\partial_x^k(U-P^2_{N,-2,\beta}U)(0) = 0$, $k = 0,1$. Thus, thanks to (3.17), the inequality (3.30) reads

$$\begin{aligned}
&\frac{1}{2}\|\partial_x^2(U-\overline{P}^2_{N,\lambda}U)\|^2_{\omega_{0,\beta}} + \left(\lambda-\frac{5}{2}\beta^4\right)\|U-\overline{P}^2_{N,\lambda}U\|^2_{\omega_{0,\beta}}\\
\le& c\left(1+\frac{\lambda}{\beta^4}\right)\|\partial_x^2(U-P^2_{N,-2,\beta}U)\|^2_{\omega_{0,\beta}}.
\end{aligned}\tag{3.31}$$

Since $u_N = \overline{P}^2_{N,\lambda}U$, the following conclusion comes from a combination of (3.31) and (2.9).

Theorem 3.2. *If $U \in \tilde{H}^2_{\omega_{0,\beta}}(\Lambda)$, $\partial_x^r U \in L^2_{\omega_{-2+r,\beta}}(\Lambda)$, $\lambda > 5\beta^4/2$, and integers $2 \le r \le N+1$, then*

$$\|U-u_N\|^2_{H^2_{\omega_{0,\beta}}(\Lambda)} \le c\left(1+\frac{1}{\beta^4}\right)\left(1+\frac{\lambda}{\beta^4}\right)(\beta N)^{2-r}\|\partial_x^r U\|^2_{\omega_{-2+r,\beta}}.\tag{3.32}$$

Remark 3.2. In the early work, we often reformed original problems, and then solved them by using the usual generalized Laguerre approximation. Thus, they are essentially available for solutions decaying to zero as x increases, see, e.g., [10–12, 17]. But our new approach is also efficient for solutions growing up at infinity. This merit is confirmed by the numerical experiments in the next section. In fact, the spectral method proposed in the above is a Petrov-Galerkin spectral method.

We could also design the Laguerre spectral method for problem (3.23) with exact imposition of boundary conditions. In this case, let $V(\Lambda), \overline{V}(\Lambda)$ and $\mathscr{A}_\lambda(u,v)$ be the same as before. We set

$$\begin{aligned}
V_N(\Lambda) &= V(\Lambda) \cap \mathscr{P}_N(\Lambda) \cap \{v \,|\, \partial_x^2 v(0) = b\},\\
\overline{V}_N(\Lambda) &= \overline{V}(\Lambda) \cap \mathscr{P}_N(\Lambda) \cap \{v \,|\, \partial_x^2 v(0) = 0\}.
\end{aligned}$$

The spectral method for solving (3.24), is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathscr{A}_\lambda(u_N, \phi) + b\partial_x \phi(0) = (f, \phi)_{\omega_{0,\beta}}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{3.33}$$

For the convergence analysis, we introduce the operator $\overline{P}^2_{N,\lambda} : V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathscr{A}_\lambda(\overline{P}^2_{N,\lambda} v - v, \phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda).$$

An argument similar to the derivations of (3.27)-(3.29), leads to $u_N = \overline{P}^2_{N,\lambda} U$. Consequently, $\|\partial_x^k (U - u_N)\|^2_{\omega_{0,\beta}} = \|\partial_x^k (U - \overline{P}^2_{N,\lambda} U)\|^2_{\omega_{0,\beta}}$, $k = 0, 1, 2$.

Let $P^3_{N,-3,\beta} U$ be the Laguerre quasi-orthogonal projection given by (2.8) with $\mu = m = 3$, $v = U$, $\partial_x^2 v(0) = b$ and $v(0) = a$. Clearly, $P^3_{N,-3,\beta} U \in V_N(\Lambda)$ and $\overline{P}^2_{N,\lambda} U - P^3_{N,-3,\beta} U \in \overline{V}_N(\Lambda)$. Therefore, we could use Proposition 3.1 with

$$\begin{aligned} &W(\Lambda) = V(\Lambda), && \overline{W}(\Lambda) = \overline{V}(\Lambda), && W_N(\Lambda) = V_N(\Lambda), && \overline{W}_N(\Lambda) = \overline{V}_N(\Lambda), \\ &v = U, && w = P^3_{N,-3,\beta} U, && {}_*P^2_{N,\lambda} U = \overline{P}^2_{N,\lambda} U. \end{aligned}$$

In other words, we follow the same line as in the derivation of (3.19) to reach that

$$\begin{aligned} &\frac{1}{2}\|\partial_x^2 (U - \overline{P}^2_{N,\lambda} U)\|^2_{\omega_{0,\beta}} + \left(\lambda - \frac{5}{2}\beta^4\right)\|U - \overline{P}^2_{N,\lambda} U\|^2_{\omega_{0,\beta}} + \frac{1}{4}\beta\left(\partial_x (U - \overline{P}^2_{N,\lambda} U)(0)\right)^2 \\ \le &\frac{3}{2}\|\partial_x^2 (U - P^3_{N,-3,\beta} U)\|^2_{\omega_{0,\beta}} + \left(\lambda + \frac{3}{2}\beta^4\right)\|U - P^3_{N,-3,\beta} U\|^2_{\omega_{0,\beta}} + \frac{5}{4}\beta(\partial_x (U - P^3_{N,-3,\beta} U)(0))^2. \end{aligned}$$

Since $\partial_x^k (U - P^3_{N,-3,\beta} U)(0) = 0$, $k = 0, 1, 2$, we use (3.17) to obtain

$$\begin{aligned} &\frac{1}{2}\|\partial_x^2 (U - \overline{P}^2_{N,\lambda} U)\|^2_{\omega_{0,\beta}} + \left(\lambda - \frac{5}{2}\beta^4\right)\|U - \overline{P}^2_{N,\lambda} U\|^2_{\omega_{0,\beta}} \\ \le &c\left(1 + \frac{\lambda}{\beta^4}\right)\|\partial_x^2 (U - P^3_{N,-3,\beta} U)\|^2_{\omega_{0,\beta}}. \end{aligned} \tag{3.34}$$

Furthermore, by virtue of (2.9) with $k = 3$, we observe that for $3 \le r \le N + 1$,

$$\|\partial_x^2 (U - P^3_{N,-3,\beta} U)\|^2_{\omega_{0,\beta}} \le \frac{c}{\beta^2}\|\partial_x^3 (U - P^3_{N,-3,\beta} U)\|^2_{\omega_{0,\beta}} \le \frac{c}{\beta^2}(\beta N)^{3-r}\|\partial_x^r U\|^2_{\omega_{-3+r,\beta}}. \tag{3.35}$$

Since $u_N = \overline{P}^2_{N,\lambda} U$, a combination of (3.34), (3.35) and the embedding inequality, leads to that if $\lambda > 5\beta^4/2$ and integers $3 \le r \le N + 1$, then

$$\|U - u_N\|^2_{H^2_{\omega_{0,\beta}}(\Lambda)} \le \frac{c}{\beta^2}\left(1 + \frac{\lambda}{\beta^4}\right)(\beta N)^{3-r}\|\partial_x^r U\|^2_{\omega_{-3+r,\beta}}.$$

Remark 3.3. Auteri, Parolini and Quartapelle [1] considered the Legendre spectral method with essential imposition of Neumann condition for second order problems defined on bounded domains.

4. Numerical results

We present some numerical results to illustrate the high efficiency of spectral schemes proposed in the last section.

4.1. Dirichlet boundary value problem

Let $\xi^{(0,\beta)}_{G,N,j}$ $(0 \le j \le N)$ be the zeros of the polynomial $L^{(0,\beta)}_{N+1}(x)$, which are arranged in ascending order. Meanwhile, $\omega^{(0,\beta)}_{G,N,j}$ $(0 \le j \le N)$ stand for the corresponding Christoffel numbers such that

$$\int_\Lambda \phi(x)\omega_{0,\beta}(x)dx = \sum_{j=0}^{N} \phi\big(\xi^{(0,\beta)}_{G,N,j}\big)\,\omega^{(0,\beta)}_{G,N,j}, \quad \forall \phi \in \mathscr{P}_{2N+1}(\Lambda).$$

The discrete inner product $(u,v)_{N,0,\beta}$ is defined by

$$(u,v)_{N,0,\beta} = \sum_{j=0}^{N} u\big(\xi^{(0,\beta)}_{G,N,j}\big)v\big(\xi^{(0,\beta)}_{G,N,j}\big)\,\omega^{(0,\beta)}_{G,N,j}.$$

We solve problem (3.12) with $\lambda = 210$, by the scheme:

$$\mathscr{A}_\lambda(u_N,\phi) = (f,\phi)_{N,0,\beta}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{4.1}$$

Let

$$\phi_l(x) = \frac{\beta^2}{l(l-1)}\mathscr{L}^{(-2,\beta)}_l(x) = L^{(0,\beta)}_{l-2}(x) - 2L^{(0,\beta)}_{l-1}(x) + L^{(0,\beta)}_l(x), \quad 2 \le l \le N.$$

Clearly, $\phi_l(x) \in \overline{V}_N(\Lambda)$. In actual computation, we expand the numerical solution as

$$u_N(x) = \sum_{l=2}^{N} \hat{u}_{N,l}\phi_l(x) + bx + a \in V_N(\Lambda).$$

Inserting the above expression into (4.1) with $\phi = \phi_k$, we obtain

$$\sum_{l=2}^{N}(a_{k,l} - 2\beta b_{k,l} + \beta^2 d_{k,l} + \lambda g_{k,l})\hat{u}_{N,l} = F_k, \quad 2 \le k \le N, \tag{4.2}$$

where

$$a_{k,l} = (\partial_x^2\phi_l, \partial_x^2\phi_k)_{\omega_{0,\beta}} = \begin{cases} \beta^3, & l = k,\\ 0, & \text{otherwise}, \end{cases} \qquad b_{k,l} = (\partial_x^2\phi_l, \partial_x\phi_k)_{\omega_{0,\beta}} = \begin{cases} \beta^2, & l = k,\\ -\beta^2, & l = k+1,\\ 0, & \text{otherwise}, \end{cases}$$

$$d_{k,l} = (\partial_x^2 \phi_l, \phi_k)_{\omega_{0,\beta}} = \begin{cases} \beta, & l=k,k+2, \\ -2\beta, & l=k+1, \\ 0, & \text{otherwise}, \end{cases} \qquad g_{k,l} = (\phi_l, \phi_k)_{\omega_{0,\beta}} = \begin{cases} \dfrac{6}{\beta}, & l=k, \\ -\dfrac{4}{\beta}, & l=k\pm 1, \\ \dfrac{1}{\beta}, & l=k\pm 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_k = (f,\phi_k)_{N,0,\beta} - b\lambda(x,\phi_k)_{\omega_{0,\beta}} - a\lambda(1,\phi_k)_{\omega_{0,\beta}}.$$

Next, we set

$$\begin{aligned} &A=(a_{k,l})_{2\le k,l\le N}, && B=(b_{k,l})_{2\le k,l\le N}, && D=(d_{k,l})_{2\le k,l\le N}, \\ &G=(g_{k,l})_{2\le k,l\le N}, && \mathbf{u}=(\hat{u}_{N,2},\hat{u}_{N,3},\cdots,\hat{u}_{N,N})^T, && \mathbf{F}=(F_2,F_3,\cdots,F_N)^T. \end{aligned}$$

Then, the system (4.2) becomes

$$(A-2\beta B+\beta^2 D+\lambda G)\mathbf{u}=\mathbf{F}. \tag{4.3}$$

The numerical errors are measured by the discrete norm $E_N = \|U-u_N\|_{N,0,\beta}$.

We first take the test function $U(x)=e^{-x}\sin x$, which oscillates and decays exponentially as x increases. In Fig. 1, we plot the values of $\log_{10} E_N$ with $\beta=1,2,3$, vs. the mode N. Clearly, the errors decay very fast as N increases. This fact agrees the analysis well. Indeed, the parameters $\lambda=210$ and $\beta=1,2,3$ fulfill the condition ensuring the convergence as imposed in Proposition 3.1. It is also shown that a suitable choice of parameter β leads to more accurate numerical results.

According to Remark 3.2, our new method is also available for solutions growing up at infinity. We now take the test function $U(x)=(x^2+1)(\sin x+1)$, which oscillates and grows up as x goes to infinity. In Fig. 2, we plot the values of $\log_{10} E_N$ with different N and β. They coincide with the prediction.

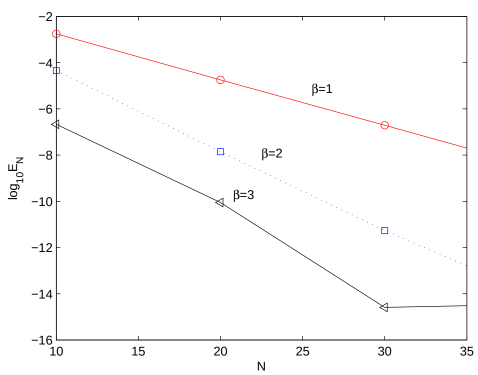

Figure 1: Numerical errors of (4.1), Example 1.

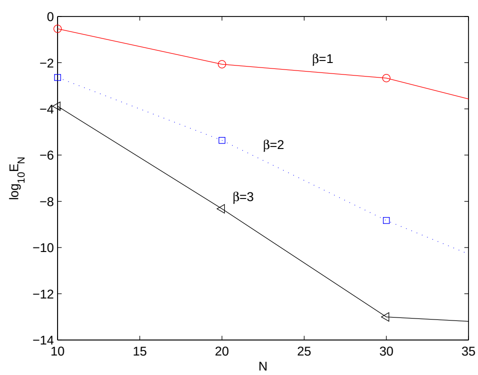

Figure 2: Numerical errors of (4.1), Example 2.

4.2. Mixed boundary value problem

We now solve problem (3.24) with $\lambda = 210$, by the following scheme:

$$\mathscr{A}_\lambda(u_N,\phi) + b\partial_x\phi(0) = (f,\phi)_{N,0,\beta}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{4.4}$$

Let

$$\phi_l(x) = \frac{\beta}{l}\mathscr{L}_l^{(-1,\beta)}(x) = L_{l-1}^{(0,\beta)}(x) - L_l^{(0,\beta)}(x), \quad 1 \le l \le N.$$

Clearly, $\phi_l(x) \in \overline{V}_N(\Lambda)$. In actual computation, we expand the numerical solution as

$$u_N(x) = \sum_{l=1}^{N} \hat{u}_{N,l}\phi_l(x) + a \in V_N(\Lambda).$$

Inserting the above expression into (4.4) with $\phi = \phi_k$, we obtain

$$\sum_{l=1}^{N}(a_{k,l} - 2\beta b_{k,l} + \beta^2 d_{k,l} + \lambda g_{k,l})\hat{u}_{N,l} = F_k, \quad 1 \le k \le N,$$

where

$$\begin{aligned} &a_{k,l} = (\partial_x^2\phi_l, \partial_x^2\phi_k)_{\omega_{0,\beta}}, && b_{k,l} = (\partial_x^2\phi_l, \partial_x\phi_k)_{\omega_{0,\beta}}, \\ &d_{k,l} = (\partial_x^2\phi_l, \phi_k)_{\omega_{0,\beta}}, && g_{k,l} = (\phi_l, \phi_k)_{\omega_{0,\beta}}, \\ &F_k = (f,\phi_k)_{\omega_{0,\beta}} - b\beta - a\lambda(1,\phi_k)_{\omega_{0,\beta}}. \end{aligned}$$

Next, we set

$$\begin{aligned} &A = (a_{k,l})_{1\le k,l\le N}, && B = (b_{k,l})_{1\le k,l\le N}, && D = (d_{k,l})_{1\le k,l\le N}, \\ &G = (g_{k,l})_{1\le k,l\le N}, && \mathbf{u} = (\hat{u}_{N,1}, \hat{u}_{N,2}, \cdots, \hat{u}_{N,N})^T, && \mathbf{F} = (F_1, F_2, \cdots, F_N)^T. \end{aligned}$$

Then, we derive a compact matrix form which is similar to (4.3).

We first take the test function $U(x) = e^{-x}\sin x$. In Fig. 3, we plot the values of $\log_{10} E_N$ with $\beta = 1, 2, 3$, vs. the mode N. Evidently, the errors decay very fast as N increases. Also, a suitable choice of parameter β leads to more accurate numerical results.

We next take the test function $U(x) = (x^2 + 1)(\sin x + 1)$. In Fig. 4, we plot the values of $\log_{10} E_N$ with different N and β. They coincide with the analysis again. In particular, our new method works well for test functions growing up as x increases.

Remark 4.1. We may change inhomogeneous boundary conditions to homogeneous boundary conditions, by using variable transformations. But for nonlinear problems, the resulting differential equations usually lose some properties, such as certain conservations, which play important role in theoretical analysis and actual computations. Thus, it seems better to use our new method directly. On the other hand, in domain decomposition spectral method, certain derivatives of numerical solutions should match properly on the common boundaries of adjacent subdomains. In this case, we also need the Laguerre quasi-orthogonal approximations used in this work.

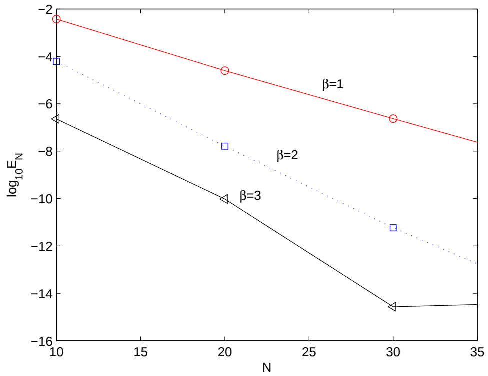

Figure 3: Numerical errors of (4.4), Example 1.

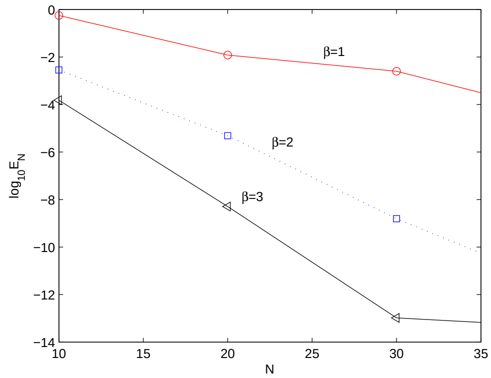

Figure 4: Numerical errors of (4.4), Example 2.

Remark 4.2. We see from the numerical results that for fixed $\lambda > 5\beta^4/2$, a suitable choice of parameter β may raise the numerical accuracy. However, so far, there is no theoretical result on the best choice of β. Generally speaking, it seems reasonable to take certain β so that the asymptotic behaviors of numerical solutions are similar to those of exact solutions. But, for fixed λ, we could not take too big β. To show this, we use the scheme (4.1) with different parameter $\beta = 1, 2, 6$ to solve (3.12) with $\lambda = 50$. Clearly, $\lambda > 5\beta^4/2$ for $\beta = 1, 2$, while this condition does not hold for $\beta = 6$. We take the test function $U(x) = e^{-x}\sin x$ and the mode $N = 30$. The global weighted errors of numerical solutions with $\beta = 1, 2$ are $3.44E-7$ and $1.04E-11$, respectively. Whereas, the global weighted errors of numerical solutions with $\beta = 6$ is $3.81E-5$.

5. Concluding remarks

In this paper, we proposed the new Laguerre spectral method for high order problems defined on the half line, with mixed inhomogeneous boundary conditions. The precise analysis indicated the spectral accuracy of numerical solutions. The numerical results showed their high accuracy and confirm the analysis well. The suggested algorithms also work well, even if the approximated solutions grow up at infinity. In fact, in the existing work, we usually used the variable transformation like $U(x,t) = e^{-\gamma x}W(x,t)$, $\gamma > 0$, and then solved the alternative problem with the unknown function $W(x,t)$ by using the Laguerre approximation. Thus, those spectral schemes seem available essentially for approximated solution $U(x,t)$ decaying to zero at infinity. However, in this work, we directly evaluated $U(x,t)$, which could grow rapidly. Although we only considered two model problems, the main idea and strategy of this work provide an efficient framework for a large class of spectral method for unbounded domains, as well as other high order methods for high order problems.

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