The Numerical Simulation of Space-Time Variable Fractional Order Diffusion Equation

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Abstract. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable. Numerical methods and analysis of stability and convergence of numerical scheme for the variable fractional order partial differential equations are quite limited and difficult to derive. This motivates us to develop efficient numerical methods as well as stability and convergence of the implicit numerical methods for the space-time variable fractional order diffusion equation on a finite domain. It is worth mentioning that here we use the Coimbra-definition variable time fractional derivative which is more efficient from the numerical standpoint and is preferable for modeling dynamical systems. An implicit Euler approximation is proposed and then the stability and convergence of the numerical scheme are investigated. Finally, numerical examples are provided to show that the implicit Euler approximation is computationally efficient.

1. Introduction

The fractional diffusion equation (FDE) is a generalization of the classical diffusion equation by replacing the integer-order derivatives by fractional-order derivatives, which is a useful approach for the description of transport dynamics in complex systems governed by anomalous dispersion and non-exponential relaxation [1–3]. Recently, more and more researchers find that many dynamic processes appear to exhibit fractional order behavior that may vary with time or space, which indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex

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dynamical problems. For example, Variable-order has applied to viscoelasticity [4], the processing of geographical data [5], signature verification [6], diffusion [7], etc. While the variable-order definitions were studied in the 1990s, Samko and Ross [8] first discussed some properties and the inversion formula of the variable-order operator $(\frac{d}{dx})^{\alpha(x)} f(x)$ using the Riemann-Liouvile definition and Fourier transforms. Hereafter some mapping properties in Hölder spaces and "measure of deviation" of the direct generalized operators of the Riemann-Liouville fractional integration and differentiation and the Marchaud form to the case of variable order $\alpha(x)$ were considered by them [9, 10]. Kikuchi and Negoro [11] investigated the relationship between Markov processes and evolution equations with respect to pseudo differential operators. In 1998, Lorenzo and Hartley [12] suggested that the concept of variable-order (or order structure) operator is allowed to vary either as a function of the independent variable of integration or differentiation (t) or as a function of some other (perhaps spatial) variable (y). At the same time, a preliminary study was done in several potential variable-order definitions and initial properties were forwarded. Hereafter, in 2002, they [13] developed more deeply the concept of variable and distributed order fractional operators based on the Riemann-Liouville definition and other new operators, then the relationship between the mathematical concepts and physical processes were investigated. Afterward, a few researchers put different definitions of variable fractional order operators to suit desired goals and discussed their applications respectively [4, 7, 14, 15]. In the recent research article, Ramirez et al. [16] compared nine variable-order operator definitions based on a very simple criteria: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. They found that only Marchaud-definition and Coimbra-definition satisfied the above elementary requirement, and pointed Coimbra-definition variable-order operator was more efficient from the numerical standpoint and then preferable for modeling dynamics systems.

The research on variable-order fractional partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin et al. [17] studied the stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [18] discussed the stability and convergence of Euler approximation for the variable fractional order advection-diffusion equation with a nonlinear source term, moveover, they presented some other numerical methods for the equation. Chen et al. [19] considered a variable-order anomalous subdiffusion equation, in the paper, two numerical schemes were proposed, one with first order temporal accuracy and fourth order spatial accuracy, the other with second order temporal accuracy and fourth order spatial accuracy. Here, we point out that in above these papers, variable-order derivative is either space derivative or time derivative. According to the authors knowledge, there are not literatures consider numerical approximation of variable-order problem containing both space variable-order derivative and time variable-order derivative.

In the paper we consider the following space-time variable fractional order diffusion equation (STVFODE):

$$\mathbf{D}_t^{\alpha(x,t)}u(x,t) = \mu \, {}_0^R D_x^{\beta(x,t)}u(x,t) + f(x,t), \tag{1.1a}$$

$$(x,t) \in \Omega = [0,L] \times [0,T], \tag{1.1b}$$

with the following initial and boundary conditions

$$u(x,0) = \varphi(x), \tag{1.2a}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{1.2b}$$

where $\mu(>0)$ is a diffusion coefficient, $0 < \underline{\alpha} \leq \alpha(x,t) \leq \overline{\alpha} \leq 1$, $1 < \underline{\beta} \leq \beta(x,t) \leq \overline{\beta} \leq 2$.

The variable-order derivative operators in Eq. (1.1) are defined as follows:

$$_0^R D_x^{\beta(x,t)}u(x,t) = \left[\frac{1}{\Gamma(m - \beta(x,t))} \frac{d^m}{d\xi^m} \int_0^\xi (\xi - \eta)^{m-\beta(x,t)-1} u(\eta,t)d\eta \right]_{\xi=x}, \tag{1.3}$$

which was called Riemann-Liouville variable fractional order derivative [17,18] and $m - 1 < \beta(x,t) < m$.

$$\mathbf{D}_t^{\alpha(x,t)}u(x,t) = \frac{1}{\Gamma(1 - \alpha(x,t))} \int_{0^+}^t (t - \sigma)^{-\alpha(x,t)} \frac{\partial u(x,\sigma)}{\partial \sigma} d\sigma$$
$$+ \frac{(u(x,0^+) - u(x,0^-))t^{-\alpha(x,t)}}{\Gamma(1 - \alpha(x,t))}, \tag{1.4}$$

which was defined by Coimbra [4] and $0 < \alpha(x,t) < 1$. Here it is worth mentioning that the Coimbra-definition is first used in numerical approximation of the variable-order partial differential equation.

2. Implicit Euler numerical approximation

Let us suppose that the function $f(x) \in C^{(m-1)}[a,b]$ and $f^{(m)}(x) \in L[a,b]$, then for every α $(0 \leq m - 1 < \alpha(x) < m)$, the Riemann-Liouville fractional derivative exists and coincides with the Grünwald-Letnikov fractional derivative [20]. The relationship between the Riemann-Liouville and Grünwald-Letnikov definitions is important for the numerical approximation of fractional differential equations, manipulation with fractional derivatives, and formulation of physically meaningful initial- and boundary-value problems for fractional differential equations. This allows the use of the Riemann-Liouville definition during problem formulation and then the Grünwald-Letnikov definitions for obtaining the numerical solution.

Let $t_k = k\tau$ $(0 \leq t_k \leq T)$, $k = 0,1,2,\cdots,N$, $x_i = ih$, $i = 0,1,2,\cdots,M$, where $\tau = T/N$ and $h = L/M$ are time and space steps respectively. Defining u_i^k as the numerical approximation to $u(x_i,t_k)$. Also we denote that $\alpha_i^k = \alpha(x_i,t_k)$, $\beta_i^k = \beta(x_i,t_k)$ and $f_i^k = f(x_i,t_k)$.

Lemma 2.1. [18,21] *For $0 \leq n-1 < \zeta(x,t) < n$, if $u(x,t) \in L_1(\Omega)$, and ${}_0^R D_x^{\zeta(x,t)} u(x,t) \in \mathscr{C}(\Omega)$, then it can be obtained the "shifted" Grünwald approximation*

$$
{}_0^R D_x^{\zeta_i^k} u(x_i, t_k) = h^{-\zeta_i^k} \sum_{j=0}^{i+1} g_{\zeta_i^k}^j u(x_{i+1-j}, t_k) + \mathbb{O}(h),
\tag{2.1}
$$

where $g_{\zeta_i^k}^j$ is the Grünwald weights defined by

$$
g_{\zeta_i^k}^j = \frac{\Gamma(j - \zeta(x_i, t_k))}{\Gamma(-\zeta(x_i, t_k))\Gamma(j+1)}, \quad j = 0, 1, 2, \cdots .
\tag{2.2}
$$

In this paper the solution $u(x,t)$ is considered when $t \geq 0$, so we suppose $u(x, 0^+) = u(x, 0^-)$. Furthermore, let $u \in C^{(2)}(\Omega)$, similar to the proof of literature [22] the operator $\mathbf{D}_t^{\alpha(x,t)} u(x,t)$ can be discretized as

$$
\begin{aligned}
&\mathbf{D}_t^{\alpha(x_i, t_{k+1})} u(x_i, t_{k+1}) \\
={}&\frac{1}{\Gamma(1 - \alpha(x_i, t_{k+1}))} \int_0^{(k+1)\tau} (t_{k+1} - \eta)^{-\alpha(x_i, t_{k+1})} \frac{\partial u(x_i, \eta)}{\partial \eta} d\eta \\
={}&\frac{1}{\Gamma(1 - \alpha(x_i, t_{k+1}))} \sum_{j=0}^{k} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{1}{(t_{k+1} - \eta)^{\alpha(x_i, t_{k+1})}} d\eta \\
&+ \mathbb{O}(\tau^{2 - \alpha(x_i, t_{k+1})}) \\
={}&\frac{1}{\Gamma(2 - \alpha(x_i, t_{k+1}))} \sum_{j=0}^{k} \frac{u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})}{\tau^{\alpha(x_i, t_{k+1})}} \\
&\cdot [(j+1)^{1 - \alpha(x_i, t_{k+1})} - j^{1 - \alpha(x_i, t_{k+1})}] + \mathbb{O}(\tau^{2 - \alpha(x_i, t_{k+1})})
\end{aligned}
\tag{2.3}
$$

at the mesh point (x_i, t_{k+1}). Denote

$$
\mathbf{D}_t^{\alpha_i^{k+1}} u(x_i, t_{k+1}) = \frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2 - \alpha_i^{k+1})} \sum_{j=0}^{k} (u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})) b_{j,i}^{k+1},
\tag{2.4}
$$

where $\alpha_i^k = \alpha(x_i, t_k)$ and $b_{j,i}^{k+1} = (j+1)^{1 - \alpha_i^{k+1}} - j^{1 - \alpha_i^{k+1}}$.

In order to obtain a stable implicit discrete scheme of Eq. (1.1), we use the operator $\mathbf{D}^{\alpha_i^{k+1}} u(x_i, t_{k+1})$ to approximate the time variable fractional order operator $\mathbf{D}^{\alpha(x,t)} u(x,t)$ and the "shifted" Grünwald approximation scheme to approximate the space variable fractional order operator ${}_0^R D_x^{\beta(x,t)} u(x,t)$, respectively. Therefore, Eq. (1.1) can be discretized as follows:

$$
\begin{aligned}
&\frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2 - \alpha_i^{k+1})} \sum_{j=0}^{k} (u(x_i, t_{k+1-j}) - u(x_i, t_{k-j}) b_{j,i}^{k+1} \\
={}&\mu h^{-\beta_i^{k+1}} \sum_{j=0}^{i+1} g_{\beta_i^{k+1}}^j u(x_{i+1-j}, t_{k+1}) + f_i^{k+1},
\end{aligned}
\tag{2.5}
$$

where $\beta_i^k = \beta(x_i, t_k)$ and $f_i^k = f(x_i, t_k)$ $(k = 0, 1, \cdots, N-1;\ i = 0, 1, \cdots, M)$. Denote $r_i^{k+1} = \tau^{\alpha_i^{k+1}} \Gamma(2 - \alpha_i^{k+1})$ and $u_i^k = u(x_i, t_k)$, the above formula can be rewritten as

$$u_i^{k+1} = u_i^k - \sum_{j=1}^{k} (u_i^{k+1-j} - u_i^{k-j}) b_{j,i}^{k+1}$$

$$+ \mu r_i^{k+1} h^{-\beta_i^{k+1}} \sum_{j=0}^{i+1} g_{\beta_i^{k+1}}^j u_{i+1-j}^{k+1} + r_i^{k+1} f_i^{k+1}, \qquad (2.6)$$

i.e.,

$$u_i^1 = u_i^0 + \mu r_i^1 h^{-\beta_i^1} g_{\beta_i^1}^1 u_i^1 + \mu r_i^1 h^{-\beta_i^1} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^1}^j u_{i+1-j}^1 + r_i^1 f_i^1, \qquad (2.7a)$$

$$u_i^{k+1} = (1 - b_{1,i}^{k+1}) u_i^k + \sum_{j=1}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1}) u_i^{k-j} + b_{k,i}^{k+1} u_i^0$$

$$+ \mu r_i^{k+1} h^{-\beta_i^{k+1}} g_{\beta_i^{k+1}}^1 u_i^{k+1} + \mu r_i^{k+1} h^{-\beta_i^{k+1}} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^{k+1}}^j u_{i+1-j}^{k+1} + r_i^{k+1} f_i^{k+1}, \quad (2.7b)$$

where $i = 1, 2, \cdots, M-1$; $k = 1, 2, \cdots, N-1$. Furthermore, the following variable fractional-order implicit difference approximation (VFOIDA) to the STVFODE (1.1) is obtained

$$(1 - \mu r_i^1 h^{-\beta_i^1} g_{\beta_i^1}^1) u_i^1 - \mu r_i^1 h^{-\beta_i^1} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^1}^j u_{i+1-j}^1 = u_i^0 + r_i^1 f_i^1, \qquad (2.8a)$$

$$(1 - \mu r_i^{k+1} h^{-\beta_i^{k+1}} g_{\beta_i^{k+1}}^1) u_i^{k+1} - \mu r_i^{k+1} h^{-\beta_i^{k+1}} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^{k+1}}^j u_{i+1-j}^{k+1}$$

$$= \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1}) u_i^{k-j} + b_{k,i}^{k+1} u_i^0 + r_i^{k+1} f_i^{k+1}, \qquad (k \geq 1), \qquad (2.8b)$$

where $i = 1, 2, \cdots, M-1$; $k = 1, 2, \cdots, N-1$.

The boundary and initial conditions are discretized as follows:

$$u_i^0 = \varphi(ih) = \varphi_i, \quad u_0^k = 0, \quad u_M^k = 0, \qquad (2.9)$$

where $i = 0, 1, 2, \cdots, M$; $k = 0, 1, 2, \cdots, N$.

The above equation can be rewritten in the form of matrix as follows:

$$\begin{cases} A^1 U^1 = U^0, \\ A^{k+1} U^{k+1} = \sum_{j=0}^{k-1} (b_j^{k+1} - b_{j+1}^{k+1}) U^{k-j} + b_k^{k+1} U^0 + r^{k+1} f^{k+1}, \qquad k \geq 1, \\ U^0 = \varphi, \end{cases} \qquad (2.10)$$

where $U^k = (u_1^k, u_2^k, \cdots, u_{M-1}^k)^T$, $b_j^k = (b_{j,1}^k, b_{j,2}^k, \cdots, b_{j,M-1}^k)^T$, $f^k = (f_1^k, f_2^k, \cdots, f_{M-1}^k)^T$, $r^k = (r_1^k, r_2^k, \cdots, r_{M-1}^k)^T$, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{M-1})^T$, and $A^k = (a_{ij}^k)$:

$$
a_{ij}^k = \begin{cases}
0, & j > i+1, \\
1 - \mu r_i^k h^{-\beta_i^k} g_{\beta_i^k}^1, & j = i, \\
-\mu r_i^k h^{-\beta_i^k} g_{\beta_i^k}^{i+1-j}, & 1 \le j \le i-1, \\
-\mu r_i^k h^{-\beta_i^k} g_{\beta_i^k}^0, & j = i+1,
\end{cases} \tag{2.11}
$$

for $i = 1, 2, \cdots, M-1$, $j = 1, 2, \cdots, M-1$, $k = 1, 2, \cdots, N$.

Lemma 2.2. *For* $i = 1, 2, \cdots, M$, $k = 1, 2, \cdots, N$ *and* $0 < \underline{\alpha} \le \alpha(x,t) \le \overline{\alpha} < 1$, *the coefficients* $b_{j,i}^k$ *satisfy*

$$
1 = b_{0,i}^k > b_{1,i}^k > b_{2,i}^k > \cdots > 0.
$$

Proof. Let $\varphi(x) = (x+1)^{1-\alpha(x_i,t_k)} - x^{1-\alpha(x_i,t_k)}$ $(0 < \underline{\alpha} \le \alpha(x_i,t_k) \le \overline{\alpha} < 1, \ x > 0)$, then

$$
\varphi'(x) = (1 - \alpha(x_i, t_k))[(x+1)^{-\alpha(x_i,t_k)} - x^{-\alpha(x_i,t_k)}] < 0.
$$

The result is valid. □

Lemma 2.3. *For* $i = 1, 2, \cdots, M$, $k = 1, 2, \cdots, N$ *and* $1 < \underline{\beta} \le \beta(x,t) \le \overline{\beta} \le 2$, *the coefficients* $g_{\beta_i^k}^j$ *satisfy*

$$
\begin{aligned}
&1) \quad g_{\beta_i^k}^0 = 1, \quad g_{\beta_i^k}^1 < 0; \quad g_{\beta_i^k}^j > 0 \quad (j \ge 2); \\
&2) \quad \sum_{j=0}^{\infty} g_{\beta_i^k}^j = 0, \quad \sum_{j=0}^{l} g_{\beta_i^k}^j < 0 \quad (l \ge 1).
\end{aligned}
$$

Proof. 1) According to the formula (2.2), it is easy to obtain $g_{\beta_i^k}^0 = 1$ and $g_{\beta_i^k}^1 = -\beta_i^k < 0$. Due to

$$
\begin{aligned}
g_{\beta_i^k}^{j+1} &= \frac{\Gamma(j+1-\beta(x_i,t_k))}{\Gamma(-\beta(x_i,t_k))\Gamma(j+2)} \\
&= \frac{j - \beta(x_i,t_k)}{j+1} \frac{\Gamma(j-\beta(x_i,t_k))}{\Gamma(-\beta(x_i,t_k))\Gamma(j+1)} = \frac{j - \beta(x_i,t_k)}{j+1} g_{\beta_i^k}^j,
\end{aligned}
$$

and $1 < \underline{\beta} \le \beta(x,t) \le \overline{\beta} \le 2$, we have $g_{\beta_i^k}^j > 0$, $j = 2, 3, \cdots$.

2) Since $(1-x)^{\beta_i^k} = \sum_{j=0}^{\infty} g_{\beta_i^k}^j x^j$, taking $x = 1$, we get $\sum_{j=0}^{\infty} g_{\beta_i^k}^j = 0$, furthermore, $\sum_{j=0}^{l} g_{\beta_i^k}^j < 0$ for $l = 1, 2, \cdots$.

Lemma 2.4. *Let $f(x) = (a+1)^{1-x} - a^{1-x}$ ($a \geq 1$, $0 < x < 1$), then $f(x)$ is monotonically decreasing.*

Proof. $f'(x) = (1-x)\ln a - (1-x)\ln(a+1) < 0$, then result is valid. $\qquad\square$

Definition 2.1. [23, 24] *For any arbitrary initial rounding error E^0, there exists positive number K, independent of h and τ, such that*

$$\|E^k\| \leq K\|E^0\|$$

or

$$\|E^k\| \leq K,$$

the difference approximation (2.8) *is then stable.*

We suppose that $\tilde{u}_i^k$ is the approximate solution of Eq. (2.8) and denote that $\varepsilon_i^k = \tilde{u}_i^k - u_i^k$, where $i = 0, 1, \cdots, M$ and $k = 0, 1, \cdots, N$.

Therefore, for $k = 0$, ε_i^0 satisfies

$$(1 - \mu r_i^1 h^{-\beta_i^1} g_{\beta_i^1}^1)\varepsilon_i^1 - \mu r_i^1 h^{-\beta_i^1} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^1}^j \varepsilon_{i+1-j}^1 = \varepsilon_i^0; \tag{2.12}$$

and for $k = 1, 2, \cdots, N-1$, ε_i^k satisfy

$$(1 - \mu r_i^{k+1} h^{-\beta_i^{k+1}} g_{\beta_i^{k+1}}^1)\varepsilon_i^{k+1} - \mu r_i^{k+1} h^{-\beta_i^{k+1}} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^{k+1}}^j \varepsilon_{i+1-j}^{k+1}$$

$$= \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1})\varepsilon_i^{k-j} + b_{k,i}^{k+1} \varepsilon_i^0, \tag{2.13}$$

where $i = 1, 2, \cdots, M-1$.

Theorem 2.1. *The fractional implicit difference scheme* (2.8)-(2.9) *is unconditionally stable.*

Proof. . Let

$$\|E^k\|_\infty = |\varepsilon_l^k| = \max_{1 \leq i \leq M-1} |\varepsilon_i^k|.$$

According to the Lemma 2.3, when $k = 0$ we have

$$\|E^1\|_\infty = |\varepsilon_l^1| \leq |\varepsilon_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0}^{l+1} g_{\beta_l^1}^j |\varepsilon_l^1|$$

$$= (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1)|\varepsilon_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^1}^j |\varepsilon_l^1|$$

$$\leq (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1)|\varepsilon_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^1}^j |\varepsilon_{l+1-j}^1|$$

$$\leq \left| (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1) \varepsilon_l^1 - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j \neq 1}^{l+1} g_{\beta_l^1}^j \varepsilon_{l+1-j}^1 \right|$$

$$= |\varepsilon_l^0| \leq \|E^0\|_\infty.$$

Assume that $\|E^j\|_\infty \leq \|E^0\|_\infty$, $j = 2, 3, \cdots, k$, then using Lemmas 2.2 and 2.3 it can be obtained that

$$\|E^{k+1}\|_\infty = |\varepsilon_l^{k+1}| \leq |\varepsilon_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0}^{l+1} g_{\beta_l^{k+1}}^j |\varepsilon_l^{k+1}|$$

$$= (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) |\varepsilon_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0, j \neq 1}^{l+1} g_{\beta_l^{k+1}}^j |\varepsilon_l^{k+1}|$$

$$\leq (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) |\varepsilon_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0, j \neq 1}^{l+1} g_{\beta_l^{k+1}}^j |\varepsilon_{l+1-j}^{k+1}|$$

$$\leq \left| (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) \varepsilon_l^{k+1} - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0, j \neq 1}^{l+1} g_{\beta_l^{k+1}}^j \varepsilon_{l+1-j}^{k+1} \right|$$

$$= \left| \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1}) \varepsilon_l^{k-j} + b_{k,l}^{k+1} \varepsilon_l^0 \right|$$

$$\leq \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1}) \|\varepsilon^{k-j}\|_\infty + b_{k,l}^{k+1} \|\varepsilon^0\|_\infty$$

$$\leq \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1}) \|\varepsilon^0\|_\infty + b_{k,l}^{k+1} \|\varepsilon^0\|_\infty$$

$$= \|\varepsilon^0\|_\infty.$$

Hence the conclusion is valid according to Definition 2.1. $\square$

In the following we consider the convergence of the discrete scheme (2.8)-(2.9).

Let $u(x_i, t_k)$ be the exact solution of the space-time variable fractional order diffusion equation (1.1)-(1.2) and u_i^k be the exact solution of the discrete equation (2.8)-(2.9) at mesh point (x_i, t_k) respectively, where $i = 1, 2, \cdots, M - 1$ and $k = 1, 2, \cdots, N$. Define $e_i^k = u(x_i, t_k) - u_i^k$ ($i = 1, 2, \cdots, M - 1$; $k = 1, 2, \cdots, N$) and $\mathbf{e}^k = (e_1^k, e_2^k, \cdots, e_{M-1}^k)^T$. Obviously, $\mathbf{e}^0 = \mathbf{0}$. Substituting $u(x_i, t_k)$ Eq. (1.1) and u_i^k into (2.8), then (1.1) subtracts (2.8) we have

$$(1 - \mu r_i^1 h^{-\beta_i^1} g_{\beta_i^1}^1) e_i^1 - \mu r_i^1 h^{-\beta_i^1} \sum_{j=0, j \neq 1}^{i+1} g_{\beta_i^1}^j e_{i+1-j}^1 = R_i^1, \qquad (2.14a)$$

$$(1 - \mu r_i^{k+1} h^{-\beta_i^{k+1}} g_{\beta_i^{k+1}}^1) e_i^{k+1} - \mu r_i^{k+1} h^{-\beta_i^{k+1}} \sum_{j=0, j\neq 1}^{i+1} g_{\beta_i^{k+1}}^j e_{i+1-j}^{k+1}$$

$$= \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1}) e_i^{k-j} + R_i^{k+1}, \quad (k \geq 1), \tag{2.14b}$$

where $i = 1, 2, \cdots, M-1$. According to the formula (2.3) and Lemma 2.1 it is easy to obtain that the truncation error R_i^k satisfy

$$|R_i^k| \leq C(\tau^2 + \tau^{\alpha_i^k} h), \tag{2.15}$$

where $k = 1, 2, \cdots, N$.

Theorem 2.2. *The fractional implicit difference scheme (2.8)-(2.9) is convergent, and the solution u_i^k of the discrete scheme (2.8)-(2.9) and the solution $u(x_i, t_k)$ of Eqs. (1.1)-(1.2) satisfy*

$$\|u(\cdot, t_k) - u^k\|_\infty \leq C(\tau^{2-\overline{\alpha}} + \tau^{\underline{\alpha}-\overline{\alpha}} h), \quad k = 1, 2, \cdots, N, \tag{2.16}$$

where C is a constance independent of τ and h.

Proof. For $k = 1$, let $\|\mathbf{e}^1\|_\infty = |e_l^1| = \max_{1 \leq i \leq M-1} |e_i^1|$, it obtains

$$\|\mathbf{e}^1\|_\infty = |e_l^1| \leq |e_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0}^{l+1} g_{\beta_l^1}^j |e_l^1|$$

$$= (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1) |e_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^1}^j |e_l^1|$$

$$\leq (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1) |e_l^1| - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^1}^j |e_{l+1-j}^1|$$

$$\leq \left| (1 - \mu r_l^1 h^{-\beta_l^1} g_{\beta_l^1}^1) e_l^1 - \mu r_l^1 h^{-\beta_l^1} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^1}^j e_{l+1-j}^1 \right|$$

$$= |R_l^1| \leq C(\tau^2 + \tau^{\alpha_l^1} h).$$

Suppose that $\|\mathbf{e}^j\|_\infty \leq C(b_{j-1}^{\overline{\alpha}})^{-1}(\tau^2 + \tau^{\underline{\alpha}} h)$ $(1 < j \leq k)$, denote $b_j^{\overline{\alpha}} = (j+1)^{1-\overline{\alpha}} - j^{1-\overline{\alpha}}$ and $|e_l^{k+1}| = \max_{1 \leq i \leq M-1} |e_i^{k+1}|$, then by Lemmas 2.2 and 2.4 we have

$$|e_l^{k+1}| \leq |e_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0}^{l+1} g_{\beta_l^{k+1}}^j |e_l^{k+1}|$$

$$= (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) |e_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0, j\neq 1}^{l+1} g_{\beta_l^{k+1}}^j |e_l^{k+1}|$$

$$\leq (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) |e_l^{k+1}| - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0,j\neq 1}^{l+1} g_{\beta_l^{k+1}}^j |e_{l+1-j}^{k+1}|$$

$$\leq \left| (1 - \mu r_l^{k+1} h^{-\beta_l^{k+1}} g_{\beta_l^{k+1}}^1) e_l^{k+1} - \mu r_l^{k+1} h^{-\beta_l^{k+1}} \sum_{j=0,j\neq 1}^{l+1} g_{\beta_l^{k+1}}^j e_{l+1-j}^{k+1} \right|$$

$$= \left| \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1}) e_l^{k-j} + R_l^{k+1} \right|$$

$$\leq \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1}) \|\mathbf{e}^{k-j}\|_\infty + |R_l^{k+1}|$$

$$\leq \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1})(b_{k-j-1}^{\overline{\alpha}})^{-1} C(\tau^2 + \tau^{\underline{\alpha}} h) + C(\tau^2 + \tau^{\alpha_l^{k+1}} h)$$

$$\leq \sum_{j=0}^{k-1} (b_{j,l}^{k+1} - b_{j+1,l}^{k+1})(b_k^{\overline{\alpha}})^{-1} C(\tau^2 + \tau^{\underline{\alpha}} h) + C b_{k,l}^{k+1}(b_{k,l}^{k+1})^{-1}(\tau^2 + \tau^{\alpha_l^{k+1}} h)$$

$$\leq C(b_k^{\overline{\alpha}})^{-1}.$$

Because

$$\lim_{k\to\infty} \frac{(b_k^{\overline{\alpha}})^{-1}}{k^{\overline{\alpha}}} = \lim_{k\to\infty} \frac{k^{-\overline{\alpha}}}{(k+1)^{1-\overline{\alpha}} - k^{1-\overline{\alpha}}} = \lim_{k\to\infty} \frac{k^{-1}}{(1+\frac{1}{k})^{1-\overline{\alpha}} - 1}$$

$$= \lim_{k\to\infty} \frac{k^{-1}}{(1-\overline{\alpha})k^{-1}} = \frac{1}{1-\overline{\alpha}}$$

and $k\tau \leq T$ it can be obtained

$$\|e^{k+1}\|_\infty \leq \frac{C}{1-\overline{\alpha}} k^{\overline{\alpha}}(\tau^2 + \tau^{\underline{\alpha}} h) \leq C(\tau^{2-\overline{\alpha}} + \tau^{\underline{\alpha}-\overline{\alpha}} h) \quad (k = 0, 1, \cdots, N).$$

So the above theorem is proved. $\square$

3. Numerical examples

Example 3.1. Consider the following problem:

$$\begin{cases} \mathbf{D}_t^{\alpha(x,t)} u(x,t) = {}_0^R D_x^{\beta(x,t)} u(x,t) + f(x,t), & (x,t) \in \Omega = [0,1] \times [0,T], \\ u(x,0) = 10x^2(1-x), & 0 \leq x \leq 1, \\ u(0,t) = 0, \quad u(1,t) = 0, & 0 \leq t \leq T, \end{cases} \quad (3.1)$$

where

$$f(x,t) = \frac{10x^2(1-x)t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))} - 10(t+1)\left[\frac{2x^{2-\beta(x,t)}}{\Gamma(3-\beta(x,t))} - \frac{6x^{3-\beta(x,t)}}{\Gamma(4-\beta(x,t))}\right].$$

Table 1: Numerical solution, exact solution and absolute error at $T = 1.0$.

x	Numerical solution	exact solution	error
0.1	0.17676417	0.18000000	0.00323583
0.2	0.63449050	0.64000000	0.00550950
0.3	1.25295192	1.26000000	0.00704808
0.4	1.91211319	1.92000000	0.00788681
0.5	2.49195513	2.50000000	0.00804487
0.6	2.87245660	2.88000000	0.00754340
0.7	2.93358378	2.94000000	0.00641622
0.8	2.55528102	2.56000000	0.00471898
0.9	1.61746187	1.62000000	0.00253813

Table 2: Maximum error behavior versus grid size reduction at $T = 1.0$ when $\tau = 0.02$.

h	Maximum error	Convergence order
1/25	3.439×10^{-2}	
1/50	1.709×10^{-2}	1.01
1/100	8.049×10^{-3}	1.09
1/200	3.542×10^{-3}	1.18

Table 3: Maximum error behavior versus grid size reduction at $T = 1.0$ when $h = \tau^{1.15}$.

τ	Maximum error	Convergence order
1/8	8.345×10^{-2}	
1/16	3.625×10^{-2}	1.20
1/32	1.570×10^{-2}	1.21
1/64	6.560×10^{-3}	1.26

The exact solution of the problem is $u(x,t) = 10(t+1)x^2(1-x)$.

Taking $\alpha(x,t) = 1 - 0.5e^{-xt}$ and $\beta(x,t) = 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}$.

Table 1 gives the numerical solution, exact solution and absolute error of them at $T = 1.0$, here $h = \tau = 1/100$.

Fig. 1 shows the comparison of the numerical solution and exact solution at $T = 1.0$, here the space step is equal to the time step, i.e. $h = \tau = 1/100$.

Fig. 2 shows the solution behavior of (3.1) at $T = 1.0$, $T = 0.75$, $T = 0.5$ and $T = 0.25$, respectively. Here $h = \tau = 1/100$.

Table 2 shows Maximum error and Convergence order versus space-step h reduction at $T = 1.0$ for fixed time-step $\tau = 0.02$, where the convergence order is calculated by the formula: Convergence order$= \log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}$.

When $T = 1.0$, $0.5 \leq \alpha(x,t) = 1 - 0.5e^{-xt} \leq 0.85$ ($0 \leq x \leq 1$, $0 \leq t \leq 1$), so $1.15 \leq 2 - \alpha(x,t) \leq 1.5$. Table 3 shows Maximum error and Convergence order versus time-step τ reduction at $T = 1.0$ when $h = \tau^{1.15}$, where the convergence order is calculated by the formula: Convergence order$= \log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}$.
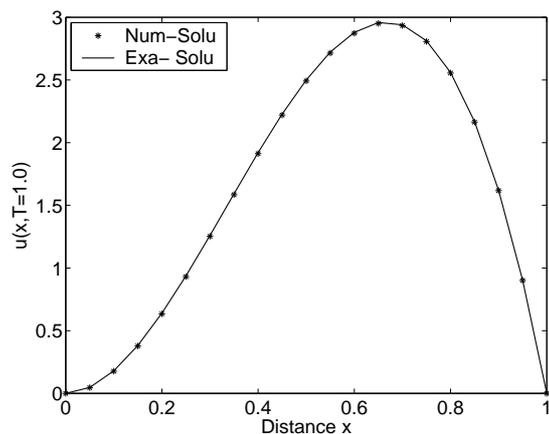
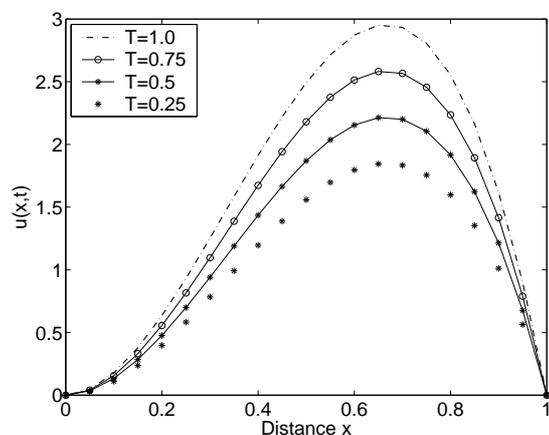Figure 1: The comparison of the numerical solution and exact solution at $T = 1.0$.

Figure 2: The solution behavior of (3.1) at $T = 1.0$, $T = 0.75$, $T = 0.5$ and $T = 0.25$.

Example 3.2. Consider the following problem:

$$
\begin{cases}
\mathbf{D}_t^{\alpha(x,t)}u(x,t) = 2\,{}_0^R D_x^{\beta(x,t)}u(x,t) + f(x,t), & (x,t) \in \Omega = [0,1] \times [0,T], \\
u(x,0) = 5x(1-x), & 0 \le x \le 1, \\
u(0,t) = 0, \quad u(1,t) = 0, & 0 \le t \le T,
\end{cases}
\tag{3.2}
$$

where $\alpha(x,t) = 0.8 + 0.01\ln(5xt)$, $\beta(x,t) = 1.7 + 0.001\cos(xt)\sin(x)$ and

$$
f(x,t) = \frac{10x(1-x)t^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} - 10(t^2+1)\left[\frac{x^{1-\beta(x,t)}}{\Gamma(2-\beta(x,t))} - \frac{2x^{2-\beta(x,t)}}{\Gamma(3-\beta(x,t))}\right].
$$

The exact solution of the problem is $u(x,t) = 5(t^2+1)x(1-x)$.

The 3-D plot of the exact solution and numerical solution of (3.2) are shown in Fig. 3 and Fig. 4 respectively.
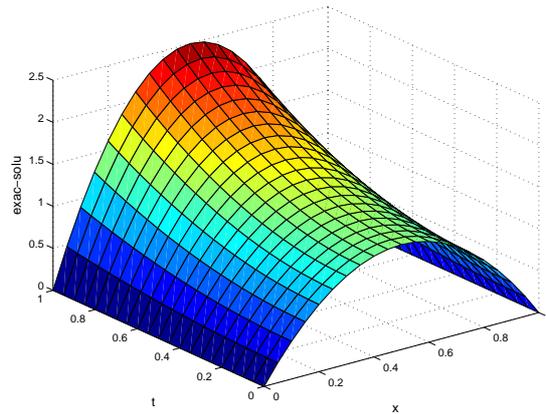
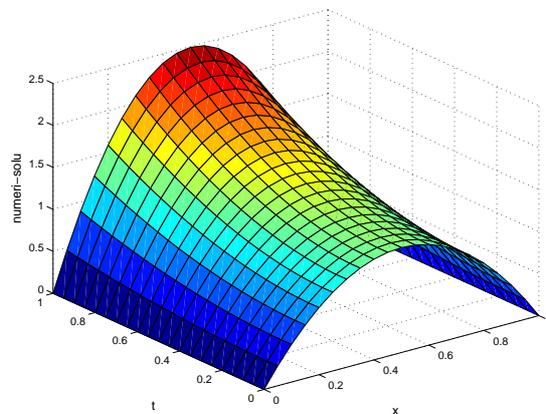Figure 3: The 3-dimension exact solution of (3.2).

Figure 4: The 3-dimension numerical solution of (3.2).

4. Conclusion

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable. Recently, variable fractional derivatives are applied to many fields, such as viscoelasticity, the processing of geographical data, signature verification, diffusion, etc. Numerical approximation of the variable fractional partial differential equation is relatively new and still at an early stage of development. The existing literatures only considered numerical approximation of variable-order problems containing either space variable-order derivative or time variable-order derivative, and the variable-order derivative is either Riemann-Liouville variable fractional derivative or Riesz variable fractional derivative. In the paper, we consider the numerical approximation of the space-time variable fractional diffusion equation, where the time variable-order derivative uses Coimbra-definition variable fractional derivative and the space variable-order derivative uses Riemann-Liouville variable fractional derivative. A implicit numerical scheme is

derived, furthermore, the stability and convergence of the above scheme are discussed. Finally, some numerical examples are presented to show the theoretical conclusions are valid.

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