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Legendre-Gauss Spectral Collocation Method for Second Order Nonlinear Delay Differential Equations

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Abstract. In this paper, we present and analyze a single interval Legendre-Gauss spectral collocation method for solving the second order nonlinear delay differential equations with variable delays. We also propose a novel algorithm for the single interval scheme and apply it to the multiple interval scheme for more efficient implementation. Numerical examples are provided to illustrate the high accuracy of the proposed methods.

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Key words: Legendre-Gauss spectral collocation method, second order nonlinear delay differential equations, error analysis.

1. Introduction

Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance in engineering, chemical process, economics and biological systems. Over the past few decades, rapid progress has been made in numerical methods for various DDEs, see, for example, [2, 3, 6, 33] for an overview. Many numerical schemes mainly based on the Taylor's expansions or quadrature formulas introduced for initial value problems of ordinary differential equations (ODEs) have also been frequently used for numerical solutions of DDEs (cf. [7, 16, 17, 20]).

As we know, spectral methods are widely used in numerical solutions of partial differential equations (cf. [4, 5, 8, 10–12, 24, 25]), which have become powerful tools for solving many kinds of differential equations arising in various fields of engineering and science. Among many types of spectral methods that are more applicable and

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frequently used are collocation methods. In recent years, spectral collocation methods have become increasingly popular in numerical solutions of initial value problems of ODEs and DDEs due to their high-order accuracy and easy implementation. For instance, Guo et al. developed several Legendre-Gauss-type spectral collocation methods for initial value problems of ODEs (cf. [13–15,26]); Kanyamee and Zhang [19] investigated the Legendre/Chebyshev-Gauss-Lobatto spectral collocation method for solving Hamiltonian dynamical systems; Ito et al. [18] proposed a Legendre-tau method for linear DDEs with one constant delay; Ali et al. [1] developed a Legendre collocation method for linear DDEs with vanishing proportional delays; Wei and Chen [28, 29] studied Legendre collocation methods for linear Volterra integro-differential equations and linear Volterra integro-differential equations with pantograph delay. Actually, due to the nature of the DDEs and the behavior of the solutions, it is a difficult task to design efficient codes for the numerical solutions of DDEs, particularly, for the nonlinear DDEs. Very recently, we note that Wang et al. [27, 30] presented Legendre-Gauss-type spectral collocation methods for solving first order nonlinear DDEs. However, to the best of our knowledge, there are few discussions on the numerical methods for second order nonlinear DDEs.

The aim of this paper is to develop a Legendre-Gauss spectral collocation method for solving the second order nonlinear DDE with variable delay:

$$\begin{cases} U''(t) = f(U(t), U'(t), V(t), W(t), t), & 0 < t \le T, \\ U(t) = \varphi(t), \quad U'(t) = \varphi'(t), & t \le 0, \end{cases} \tag{1.1}$$

where $V(t) = U(t-\theta(t)), W(t) = U'(t-\theta(t))$, f, φ are given functions and the delay variable $\theta(t) \ge 0$.

We first propose a single interval Legendre-Gauss spectral collocation scheme for problem (1.1) motivated by [15, 27], and design a novel algorithm by full utilizing properties of the Legendre polynomials. Roughly speaking, we expand the numerical solution by a truncated shifted Legendre polynomial series, and collocate the numerical scheme at the Legendre-Gauss points to determine the expansion coefficients. For more efficient implementation, we also introduce a multiple interval Legendre-Gauss spectral collocation scheme. These approaches we present here have several striking features:

- The single interval Legendre-Gauss collocation scheme can be implemented easily and efficiently for nonlinear problems due to the proposed novel algorithm (see Subsection 2.2).
- The multiple interval Legendre-Gauss collocation scheme enable us to solve the resultant system efficiently and economically. Specifically, if T is large, we can divide the solution interval $(0, T)$ into subintervals and solve the subsystems successively. Moreover, the resultant system for the expansion coefficients of the numerical solution with a modest number of unknowns can be solved quickly.
- In actual computation, we only need to store the expansion coefficients of the

numerical solution in the related “delay” subintervals to recover the solution in the current subinterval, which saves a lot of work.

Some numerical examples are given to confirm the theoretical results. It is shown that the proposed single/multiple interval Legendre-Gauss collocation scheme and the algorithm are numerically stable and possess the desired spectral accuracy.

The rest of this paper is arranged as follows. In Section 2, we introduce the single interval Legendre-Gauss spectral collocation method and design an algorithm for it. The error analysis is also given for the single interval scheme. In Section 3, we propose the multiple interval scheme for more efficient implementation. In Section 4, we present some numerical experiments to demonstrate the theoretical results. We end with some concluding remarks in Section 5.

2. Single interval Legendre-Gauss collocation method

In this section, we shall introduce and analyze a single interval Legendre-Gauss collocation method for the DDE (1.1), based on the Legendre-Gauss interpolation.

2.1. Preliminaries

Let $L_l(x), x \in [-1,1]$ be the standard Legendre polynomial of degree l. It can be defined as the normalized eigenfunction of the singular Strum-Liouville problem

$$((1-x^2)L_l'(x))' + l(l+1)L_l(x) = 0, \quad x \in [-1,1], \quad l \geq 0. \tag{2.1}$$

The shifted Legendre polynomial $L_{T,l}(t)$ is defined by (cf. [13])

$$L_{T,l}(t) = L_l\Big(\frac{2t}{T} - 1\Big), \quad t \in [0,T], \quad l = 0,1,2\cdots.$$

Particularly,

$$L_{T,0}(t) = 1, \quad L_{T,1}(t) = \frac{2t}{T} - 1, \quad L_{T,2}(t) = \frac{6t^2}{T^2} - \frac{6t}{T} + 1, \tag{2.2a}$$

$$L_{T,3}(t) = \frac{5}{2}\Big(\frac{2t}{T} - 1\Big)^3 - \frac{3t}{T} + \frac{3}{2}. \tag{2.2b}$$

Using the properties of the standard Legendre polynomials we can get (cf. [13])

$$(l+1)L_{T,l+1}(t) - (2l+1)\Big(\frac{2t}{T} - 1\Big)L_{T,l}(t) + lL_{T,l-1}(t) = 0, \quad l \geq 1, \tag{2.3}$$

and

$$L'_{T,l+1}(t) - L'_{T,l-1}(t) = \frac{2(2l+1)}{T}L_{T,l}(t), \quad l \geq 1. \tag{2.4}$$

Obviously, the set of $L_{T,l}(t)$ forms a complete $L^2(0,T)$-orthogonal system, namely,

$$\int_0^T L_{T,l}(t)L_{T,m}(t)dt = \frac{T}{2l+1}\delta_{l,m}, \tag{2.5}$$

where $\delta_{l,m}$ is the Kronecker symbol. Hence for any $v \in L^2(0,T)$, there holds

$$v(t) = \sum_{l=0}^{\infty} \hat{v}_l L_{T,l}(t), \qquad \widehat{v}_l = \frac{2l+1}{T} \int_0^T v(t) L_{T,l}(t) dt. \tag{2.6}$$

Moreover, by (2.1) we have

$$\int_0^T L'_{T,l}(t) L'_{T,m}(t) t(T-t) dt = \frac{l(l+1)T}{2l+1} \delta_{l,m}, \quad l,m \geq 1. \tag{2.7}$$

Using some classical properties of the Jacobi polynomials, we can deduce that

$$\int_0^T L''_{T,l}(t) L''_{T,m}(t) t^2 (T-t)^2 dt = \frac{(l+2)(l+1)l(l-1)T}{2l+1} \delta_{l,m}, \quad l,m \geq 2. \tag{2.8}$$

Let $t_j^N, 0 \leq j \leq N$ be the nodes of the standard Legendre-Gauss interpolation on the standard interval $(-1,1)$, and $\omega_j^N, 0 \leq j \leq N$ be the corresponding Christoffel numbers. Accordingly, the nodes of the shifted Legendre-Gauss interpolation on the interval $(0,T)$ are the zeros of $L_{T,N+1}(t)$, denoted by $t_{T,j}^N, 0 \leq j \leq N$. It can be easily verify that $t_{T,j}^N = \frac{T}{2}(t_j^N + 1)$ and $\omega_{T,j}^N = \frac{T}{2}\omega_j^N$ for $0 \leq j \leq N$.

Denote by $\mathcal{P}_N(0,T)$ the set of polynomials of degree not exceeding N. By the property of the standard Legendre-Gauss quadrature, we obtain for any $\phi \in \mathcal{P}_{2N+1}(0,T)$ (cf. [13]),

$$\int_0^T \phi(t) dt = \frac{T}{2} \int_{-1}^1 \phi\Big(\frac{T}{2}(t+1)\Big) dt = \frac{T}{2} \sum_{j=0}^N \omega_j^N \phi\Big(\frac{T}{2}(t_j^N + 1)\Big) = \sum_{j=0}^N \omega_{T,j}^N \phi(t_{T,j}^N). \tag{2.9}$$

Next, let $(u,v)_T$ and $\|v\|_T$ be the inner product and norm of the space $L^2(0,T)$, respectively, i.e.,

$$(u,v)_T = \int_0^T u(t) v(t) dt, \qquad \|v\|_T = (v,v)_T^{\frac{1}{2}}.$$

The discrete inner product and the discrete norm are defined as

$$(u,v)_{T,N} = \sum_{j=0}^N \omega_{T,j}^N u(t_{T,j}^N) v(t_{T,j}^N), \qquad \|v\|_{T,N} = (v,v)_{T,N}^{\frac{1}{2}}.$$

For any $\phi\psi \in \mathcal{P}_{2N+1}(0,T)$ and $\varphi \in \mathcal{P}_N(0,T)$, we have by (2.9) that (cf. [13])

$$(\phi,\psi)_T = (\phi,\psi)_{T,N}, \qquad \|\varphi\|_T = \|\varphi\|_{T,N}. \tag{2.10}$$

The shifted Legendre-Gauss interpolation operator $I_{T,N}v(t) : C(0,T) \to \mathcal{P}_N(0,T)$ is defined as

$$I_{T,N} v(t_{T,j}^N) = v(t_{T,j}^N), \qquad 0 \leq j \leq N.$$

By (2.10), there holds for any $\phi \in \mathcal{P}_{N+1}(0,T)$ that

$$(I_{T,N}v,\phi)_T = (I_{T,N}v,\phi)_{T,N} = (v,\phi)_{T,N}. \tag{2.11}$$

Note that the shifted Legendre-Gauss interpolation $I_{T,N}v(t)$ can be expanded as

$$I_{T,N}v(t) = \sum_{l=0}^{N} \widetilde{v}_l L_{T,l}(t). \tag{2.12}$$

We can infer from (2.6) and (2.11) that

$$\widetilde{v}_l = \frac{2l+1}{T}(I_{T,N}v, L_{T,l})_T = \frac{2l+1}{T}(v, L_{T,l})_{T,N}. \tag{2.13}$$

For any $\psi \in \mathcal{P}_{N+1}(0,T)$, let

$$\psi(t) = \sum_{l=0}^{N+1} \widehat{\psi}_l L_{T,l}(t) \qquad \text{and} \qquad I_{T,N}\psi(t) = \sum_{l=0}^{N} \widetilde{\psi}_l L_{T,l}(t). \tag{2.14}$$

Then, it can be verified that (cf. [13])

$$\widetilde{\psi}_l = \widehat{\psi}_l, \qquad 0 \le l \le N. \tag{2.15}$$

Moreover, for any $\phi \in \mathcal{P}_{N+1}(0,T)$ there holds (cf. (2.10) of [13])

$$\|\phi\|_{T,N} \le \|\phi\|_T. \tag{2.16}$$

The discrete norm of the higher order polynomial can be bounded by the continuous norm, as stated below.

Lemma 2.1. *For any $\psi \in \mathcal{P}_{N+2}(0,T)$, there holds*

$$\|\psi\|_{T,N} \le \gamma_N \|\psi\|_T, \tag{2.17}$$

where

$$\gamma_N = \sqrt{2 + \frac{2N+5}{2(N+2)^2(2N+1)}}.$$

Proof. Let (u,v) and $\|v\|$ be the inner product and norm of the space $L^2(-1,1)$, and let $(u,v)_N$ and $\|v\|_N$ be the discrete inner product and norm of the space $L^2(-1,1)$. We set $\varphi(x) = \psi(\frac{T}{2}(x+1))$ with $x \in [-1,1]$, then $\varphi \in \mathcal{P}_{N+2}(-1,1)$. We first show that

$$\|\varphi\|_N \le \gamma_N \|\varphi\|. \tag{2.18}$$

Let $I_N\varphi \in \mathcal{P}_N(-1,1)$ be the standard Legendre-Gauss interpolation on the interval $(-1,1)$, we can write

$$I_N\varphi(x) = \sum_{l=0}^{N} \widetilde{\varphi}_l L_l(x), \qquad \varphi(x) = \sum_{l=0}^{N+2} \widehat{\varphi}_l L_l(x). \tag{2.19}$$

Obviously,

$$\widetilde{\varphi}_l = \frac{2l+1}{2}(I_N\varphi, L_l) = \frac{2l+1}{2}(\varphi, L_l)_N = \widehat{\varphi}_l, \quad 0 \le l \le N-1. \tag{2.20}$$

Noting the fact $L_{N+1}(t_j^N) = 0,\ 0 \le j \le N$, and the recurrence relation

$$L_{n+2}(x) = \frac{2n+3}{n+2}xL_{n+1}(x) - \frac{n+1}{n+2}L_n(x), n \ge 0,$$

we have

$$\begin{aligned}
\widetilde{\varphi}_N &= \frac{2N+1}{2}(I_N\varphi, L_N) = \frac{2N+1}{2}(I_N\varphi, L_N)_N = \frac{2N+1}{2}(\varphi, L_N)_N \\
&= \frac{2N+1}{2}\Big(\sum_{l=0}^{N-1}\widehat{\varphi}_l L_l + \widehat{\varphi}_N L_N + \widehat{\varphi}_{N+1}L_{N+1} + \widehat{\varphi}_{N+2}L_{N+2}, L_N\Big)_N \\
&= \frac{2N+1}{2}(\widehat{\varphi}_N L_N + \widehat{\varphi}_{N+2}L_{N+2}, L_N)_N \\
&= \frac{2N+1}{2}\Big(\widehat{\varphi}_N L_N - \frac{N+1}{N+2}\widehat{\varphi}_{N+2}L_N, L_N\Big)_N \\
&= \frac{2N+1}{2}\Big(\widehat{\varphi}_N - \frac{N+1}{N+2}\widehat{\varphi}_{N+2}\Big)(L_N, L_N)_N \\
&= \widehat{\varphi}_N - \frac{N+1}{N+2}\widehat{\varphi}_{N+2}.
\end{aligned} \tag{2.21}$$

Hence, using (2.20) and (2.21) we thus obtain for any $\epsilon > 0$,

$$\begin{aligned}
&\|I_N\varphi\|^2 = \sum_{l=0}^{N}\frac{2}{2l+1}\widetilde{\varphi}_l^2 = \sum_{l=0}^{N-1}\frac{2}{2l+1}\widehat{\varphi}_l^2 + \frac{2}{2N+1}\Big(\widehat{\varphi}_N - \frac{N+1}{N+2}\widehat{\varphi}_{N+2}\Big)^2 \\
&= \sum_{l=0}^{N}\frac{2}{2l+1}\widehat{\varphi}_l^2 - \frac{4(N+1)}{(2N+1)(N+2)}\widehat{\varphi}_N\widehat{\varphi}_{N+2} + \frac{2(N+1)^2}{(2N+1)(N+2)^2}\widehat{\varphi}_{N+2}^2 \\
&\le \sum_{l=0}^{N}\frac{2}{2l+1}\widehat{\varphi}_l^2 + \frac{2(N+1)}{(2N+1)(N+2)}(\epsilon^{-1}\widehat{\varphi}_N^2 + \epsilon\widehat{\varphi}_{N+2}^2) + \frac{2(N+1)^2}{(2N+1)(N+2)^2}\widehat{\varphi}_{N+2}^2 \\
&= \sum_{l=0}^{N-1}\frac{2}{2l+1}\widehat{\varphi}_l^2 + \Big(1 + \frac{(N+1)}{\epsilon(N+2)}\Big)\frac{2}{2N+1}\widehat{\varphi}_N^2 \\
&\quad + \frac{(N+1)(2N+5)}{(N+2)(2N+1)}\Big(\epsilon + \frac{N+1}{N+2}\Big)\frac{2}{2N+5}\widehat{\varphi}_{N+2}^2 \\
&\le \max\Big\{1 + \frac{N+1}{\epsilon(N+2)}, \frac{(N+1)(2N+5)}{(N+2)(2N+1)}\Big(\epsilon + \frac{N+1}{N+2}\Big)\Big\}\|\varphi\|^2.
\end{aligned} \tag{2.22}$$

Take

$$1 + \frac{N+1}{\epsilon(N+2)} = \frac{(N+1)(2N+5)}{(N+2)(2N+1)}\Big(\epsilon + \frac{N+1}{N+2}\Big),$$

and solve ϵ from this equation, then by a direct calculation we have

$$\begin{aligned}1+\frac{N+1}{\epsilon(N+2)} &= 1+\frac{\sqrt{1+4(N+1)^2(N+2)^2(2N+1)(2N+5)}+1}{2(N+2)^2(2N+1)} \\ &\le 1+\frac{2(N+1)(N+2)(2N+3)+1}{2(N+2)^2(2N+1)} \\ &= 2+\frac{2N+5}{2(N+2)^2(2N+1)}. \end{aligned} \tag{2.23}$$

From (2.22) and (2.23) we deduce that

$$\|I_N\varphi\| \le \sqrt{1+\frac{N+1}{\epsilon(N+2)}}\|\varphi\| \le \sqrt{2+\frac{2N+5}{2(N+2)^2(2N+1)}}\|\varphi\|. \tag{2.24}$$

Since

$$\|\varphi\|_N = \|I_N\varphi\|_N = \|I_N\varphi\|,$$

which together with (2.24) yields (2.18). Finally, a simple variable transformation of (2.18) leads to (2.17). □

Let $H^r(0,T)$ be the usual Sobolev space, and denote by $\|\cdot\|_{r,T}$ and $|\cdot|_{r,T}$ its norm and semi-norm, respectively. For simplicity of statement, we sometimes use the notations $\partial_t U$ and $\partial_t^2 U$ instead of U' and U'', respectively. In view of (5.4.33) and (5.4.34) of [8], the following estimates are valid.

Lemma 2.2. *For any $u \in H^r(0,T)$ with integer $1 \le r \le N+1$, there hold*

$$\|I_{T,N}u-u\|_T \le cT^rN^{-r}|u|_{r,T}, \tag{2.25}$$

$$\big\|\partial_t(I_{T,N}u-u)\big\|_T \le cT^{r-1}N^{\frac{3}{2}-r}|u|_{r,T}, \tag{2.26}$$

$$\big\|\partial_t^2(I_{T,N}u-u)\big\|_T \le cT^{r-2}N^{\frac{7}{2}-r}|u|_{r,T}. \tag{2.27}$$

2.2. The single interval collocation scheme

In this subsection, we shall construct a single interval Legendre-Gauss spectral collocation scheme for the delay differential equation (1.1). For this purpose, we denote by $\Lambda_N := \big\{t_{T,k}^N : \ 0 \le k \le N\big\} \subset (0,T)$ the grid set. The single interval collocation scheme is to find $u^N(t) \in \mathcal{P}_{N+2}(0,T)$, such that

$$\begin{cases} \partial_t^2 u^N(t) = f(u^N(t),\partial_t u^N(t),v^N(t),w^N(t),t), \quad \forall\, t\in\Lambda_N, \\ ]u^N(0) = U(0) = \varphi(0), \quad \partial_t u^N(0) = \partial_t U(0) = \partial_t\varphi(0), \end{cases} \tag{2.28}$$

with the delay term

$$v^N(t) = \begin{cases} u^N(t-\theta(t)), & \forall\, t\in\Lambda_N^1, \\ \varphi(t-\theta(t)), & \forall\, t\in\Lambda_N^0, \end{cases} \tag{2.29}$$

and

$$w^N(t)=\begin{cases}\partial_t u^N(t-\theta(t)), & \forall\, t\in\Lambda_N^1,\\ \partial_t\varphi(t-\theta(t)), & \forall\, t\in\Lambda_N^0,\end{cases} \tag{2.30}$$

where

$$\Lambda_N^0=\big\{t\in\Lambda_N:\ t\le\theta(t)\big\},\quad \Lambda_N^1=\big\{t\in\Lambda_N:\ t>\theta(t)\big\}.$$

For convenience, we set $\tilde v^N(t)=u^N(t-\theta(t))$ and $\tilde w^N(t)=\partial_t u^N(t-\theta(t))$, then the collocation scheme (2.28)-(2.30) can be rewritten as: find $u^N(t)\in\mathcal{P}_{N+2}(0,T)$ such that $u^N(0)=\varphi(0),\partial_t u^N(0)=\partial_t\varphi(0)$ and

$$\partial_t^2u^N(t)=\begin{cases}f\big(u^N(t),\partial_t u^N(t),\tilde v^N(t),\tilde w^N(t),t\big), & \forall\, t\in\Lambda_N^1,\\ f\big(u^N(t),\partial_t u^N(t),\varphi(t-\theta(t)),\partial_t\varphi(t-\theta(t)),t\big), & \forall\, t\in\Lambda_N^0.\end{cases} \tag{2.31}$$

We have from (2.28) that

$$\partial_t^2u^N(t)=I_{T,N}f(u^N(t),\partial_t u^N(t),v^N(t),w^N(t),t). \tag{2.32}$$

Next, let

$$u^N(t)=\sum_{k=0}^{N+2}\widehat u_k L_{T,k}(t). \tag{2.33}$$

Clearly

$$\partial_t u^N(t)=\sum_{k=1}^{N+2}\widehat u_k\partial_t L_{T,k}(t). \tag{2.34}$$

Let

$$I_{T,N}f(u^N(t),\partial_t u^N(t),v^N(t),w^N(t),t)=\sum_{k=0}^{N}\widehat f_k L_{T,k}(t). \tag{2.35}$$

Using (2.13) and (2.10) we find for $0\le k\le N$,

$$\begin{aligned}\widehat f_k&=\frac{2k+1}{T}(I_{T,N}f(u^N,\partial_t u^N,v^N,w^N,\cdot),L_{T,k})_T\\ &=\frac{2k+1}{T}(f(u^N,\partial_t u^N,v^N,w^N,\cdot),L_{T,k})_{T,N}\\ &=\frac{2k+1}{T}\sum_{j=0}^{N}\omega_{T,j}^N f(u^N(t_{T,j}^N),\partial_t u^N(t_{T,j}^N),v^N(t_{T,j}^N),w^N(t_{T,j}^N),t_{T,j}^N)L_{T,k}(t_{T,j}^N).\end{aligned} \tag{2.36}$$

Collecting (2.32)-(2.35), (2.2) and (2.4), we thus obtain (cf. [31])

$$
\begin{aligned}
\partial_t^2 u^N(t) &= \sum_{k=2}^{N+2} \widehat{u}_k \partial_t^2 L_{T,k}(t) = \sum_{k=0}^{N} \widehat{f}_k L_{T,k}(t) = \widehat{f}_0 + \widehat{f}_1 L_{T,1}(t) + \sum_{k=2}^{N} \widehat{f}_k L_{T,k}(t) \\
&= \widehat{f}_0 + \widehat{f}_1 L_{T,1}(t) + \frac{T}{2}\sum_{k=2}^{N} \frac{\widehat{f}_k}{2k+1}\partial_t L_{T,k+1}(t) - \frac{T}{2}\sum_{k=2}^{N} \frac{\widehat{f}_k}{2k+1}\partial_t L_{T,k-1}(t) \\
&= \widehat{f}_0 + \widehat{f}_1 L_{T,1}(t) + \frac{T\widehat{f}_N}{2(2N+1)}\partial_t L_{T,N+1}(t) + \frac{T\widehat{f}_{N-1}}{2(2N-1)}\partial_t L_{T,N}(t) \\
&\quad + \frac{T}{2}\sum_{k=3}^{N-1} \frac{\widehat{f}_{k-1}}{2k-1}\partial_t L_{T,k}(t) - \frac{T}{2}\sum_{k=1}^{N-1} \frac{\widehat{f}_{k+1}}{2k+3}\partial_t L_{T,k}(t) \\
&= \frac{T\widehat{f}_N}{2(2N+1)}\partial_t L_{T,N+1}(t) + \frac{T\widehat{f}_{N-1}}{2(2N-1)}\partial_t L_{T,N}(t) \\
&\quad + \frac{T}{2}\sum_{k=1}^{N-1} \Big(\frac{\widehat{f}_{k-1}}{2k-1} - \frac{\widehat{f}_{k+1}}{2k+3}\Big)\partial_t L_{T,k}(t) \\
&=: \sum_{k=1}^{N+1} \tilde{f}_k \partial_t L_{T,k}(t),
\end{aligned}
\tag{2.37}
$$

where

$$
\tilde{f}_{N+1} = \frac{T\widehat{f}_N}{2(2N+1)}, \quad \tilde{f}_N = \frac{T\widehat{f}_{N-1}}{2(2N-1)}, \quad \tilde{f}_k = \frac{T}{2}\Big(\frac{\widehat{f}_{k-1}}{2k-1} - \frac{\widehat{f}_{k+1}}{2k+3}\Big)
\tag{2.38}
$$

for $1 \leq k \leq N-1$.

Moreover, using (2.37), (2.2) and (2.4), a direct computation yields (cf. [32])

$$
\begin{aligned}
&\partial_t^2 u^N(t) = \sum_{k=2}^{N+2} \widehat{u}_k \partial_t^2 L_{T,k}(t) = \sum_{k=1}^{N+1} \tilde{f}_k \partial_t L_{T,k}(t) \\
&= \tilde{f}_1 \partial_t L_{T,1}(t) + \tilde{f}_2 \partial_t L_{T,2}(t) + \sum_{k=3}^{N+1} \tilde{f}_k \partial_t L_{T,k}(t) \\
&= \tilde{f}_1 \partial_t L_{T,1}(t) + \tilde{f}_2 \partial_t L_{T,2}(t) + \frac{T}{2}\sum_{k=3}^{N+1} \frac{\tilde{f}_k}{2k+1}\partial_t^2 L_{T,k+1}(t) - \frac{T}{2}\sum_{k=3}^{N+1} \frac{\tilde{f}_k}{2k+1}\partial_t^2 L_{T,k-1}(t) \\
&= \tilde{f}_1 \partial_t L_{T,1}(t) + \tilde{f}_2 \partial_t L_{T,2}(t) + \frac{T\tilde{f}_{N+1}}{2(2N+3)}\partial_t^2 L_{T,N+2}(t) + \frac{T\tilde{f}_N}{2(2N+1)}\partial_t^2 L_{T,N+1}(t) \\
&\quad + \frac{T}{2}\sum_{k=3}^{N-1} \frac{\tilde{f}_k}{2k+1}\partial_t^2 L_{T,k+1}(t) - \frac{T}{2}\sum_{k=3}^{N+1} \frac{\tilde{f}_k}{2k+1}\partial_t^2 L_{T,k-1}(t)
\end{aligned}
$$

$$= \frac{T\tilde{f}_{N+1}}{2(2N+3)}\partial_t^2 L_{T,N+2}(t) + \frac{T\tilde{f}_N}{2(2N+1)}\partial_t^2 L_{T,N+1}(t) + \frac{T}{2}\sum_{k=2}^{N}\Big(\frac{\tilde{f}_{k-1}}{2k-1} - \frac{\tilde{f}_{k+1}}{2k+3}\Big)\partial_t^2 L_{T,k}(t).$$

From (2.8) we know that $\{\partial_t^2 L_{T,k}(t)\}_{k\geq 2}$ are mutually orthogonal polynomials. Hence, comparing the expansion coefficients in terms of $\partial_t^2 L_{T,k}(t)$, the above formula leads to

$$\widehat{u}_{N+2} = \frac{T\tilde{f}_{N+1}}{2(2N+3)}, \quad \widehat{u}_{N+1} = \frac{T\tilde{f}_N}{2(2N+1)}, \quad \widehat{u}_k = \frac{T}{2}\Big(\frac{\tilde{f}_{k-1}}{2k-1} - \frac{\tilde{f}_{k+1}}{2k+3}\Big) \tag{2.39}$$

for $2 \leq k \leq N$. Inserting (2.38) into (2.39) we find that

$$\widehat{u}_{N+2} = \frac{T^2}{4(2N+3)(2N+1)}\widehat{f}_N, \tag{2.40a}$$

$$\widehat{u}_{N+1} = \frac{T^2}{4(2N+1)(2N-1)}\widehat{f}_{N-1}, \quad N \geq 1, \tag{2.40b}$$

$$\widehat{u}_N = \frac{T^2}{4(2N-1)(2N-3)}\widehat{f}_{N-2} - \frac{T^2}{2(2N+3)(2N-1)}\widehat{f}_N, \quad N \geq 2, \tag{2.40c}$$

$$\widehat{u}_{N-1} = \frac{T^2}{4(2N-3)(2N-5)}\widehat{f}_{N-3} - \frac{T^2}{2(2N+1)(2N-3)}\widehat{f}_{N-1}, \quad N \geq 3, \tag{2.40d}$$

$$\widehat{u}_k = \frac{T^2}{4(2k-1)(2k-3)}\widehat{f}_{k-2} - \frac{T^2}{2(2k+3)(2k-1)}\widehat{f}_k + \frac{T^2}{4(2k+5)(2k+3)}\widehat{f}_{k+2}, \quad 2 \leq k \leq N-2, \tag{2.40e}$$

where $\{\widehat{f}_k\}_{k=0}^N$ is given by (2.36). Next, substituting $t = 0$ into (2.34) and noting the fact $\partial_t L_{T,k}(0) = \frac{1}{T}(-1)^{k-1}k(k+1)$, we have by (2.28) that

$$\widehat{u}_1 = \frac{T}{2}\partial_t U(0) + \frac{1}{2}\sum_{k=2}^{N+2}(-1)^k k(k+1)\widehat{u}_k. \tag{2.41}$$

Furthermore, taking $t = 0$ in (2.33), a combination of (2.28), (2.41) and the fact $L_{T,k}(0) = (-1)^k$ yields

$$\widehat{u}_0 = U(0) - \sum_{k=1}^{N+2}(-1)^k\widehat{u}_k = U(0) + \frac{T}{2}\partial_t U(0) + \frac{1}{2}\sum_{k=2}^{N+2}(-1)^k(k-1)(k+2)\widehat{u}_k. \tag{2.42}$$

In actual computation, an iterative process can be used to obtain the expansion coefficients $\{\widehat{u}_k\}_{k=0}^{N+2}$, as stated below. This algorithm is much easier, simpler and faster to implement.

Algorithm 2.1

1: Compute the values of $v^N(t)$ and $w^N(t)$ for $t \in \Lambda_N^0$;
2: Provide an initial guess for the coefficients $\{\widehat{u}_k\}_{k=0}^{N+2}$;
3: Compute the values of $f(u^N(t_{T,j}^N), \partial_t u^N(t_{T,j}^N), v^N(t_{T,j}^N), w^N(t_{T,j}^N), t_{T,j}^N)$, $0 \le j \le N$ by (2.33) and (2.34);
4: Compute the coefficients $\{\widehat{f}_k\}_{k=0}^{N}$ by (2.36);
5: Compute the coefficients $\{\widehat{u}_k\}_{k=0}^{N+2}$ by (2.40)-(2.42);
6: Renew the data of $u^N(t)$ and $\partial_t u^N(t)$ for $t \in \Lambda_N$ by (2.33) and (2.34), and the data of $v^N(t)$ and $w^N(t)$ for $t \in \Lambda_N^1$ by (2.29) and (2.30);
7: Repeat steps 3–6.
8: Compute $u^N(T) = \sum_{k=0}^{N+2} \widehat{u}_k$ by (2.33).

2.3. Error analysis

In this subsection, we shall analyze the convergence of the single interval collocation scheme (2.28). For this purpose, we set

$$E^N(t) = u^N(t) - I_{T,N}U(t).$$

Let

$$G_{T,1}^N(t) = I_{T,N}\partial_t^2 U(t) - \partial_t^2 I_{T,N}U(t). \tag{2.43}$$

Due to (1.1) and definition of the interpolation operator $I_{T,N}$, there holds

$$I_{T,N}\partial_t^2 U(t) = I_{T,N} f(U(t), \partial_t U(t), V(t), W(t), t) = f(U(t), \partial_t U(t), V(t), W(t), t)$$

for all $t \in \Lambda_N$, which together with (2.43) implies that

$$\partial_t^2 I_{T,N}U(t) = f(U(t), \partial_t U(t), V(t), W(t), t) - G_{T,1}^N(t), \quad \forall\, t \in \Lambda_N.$$

Thus, we obtain

$$\begin{aligned} &\partial_t^2 I_{T,N}U(t) \\ &= \begin{cases} f(U(t), \partial_t U(t), V(t), W(t), t) - G_{T,1}^N(t), & t \in \Lambda_N^1, \\ f\big(U(t), \partial_t U(t), \varphi(t-\theta(t)), \partial_t\varphi(t-\theta(t)), t\big) - G_{T,1}^N(t), & t \in \Lambda_N^0. \end{cases} \end{aligned} \tag{2.44}$$

Further, we denote

$$G_{T,2}^N(t) = \begin{cases} f\big(u^N(t), \partial_t u^N(t), \tilde{v}^N(t), \tilde{w}^N(t), t\big) \\ \quad - f\big(u^N(t), \partial_t I_{T,N}U(t), \tilde{v}^N(t), \tilde{w}^N(t), t\big), & t \in \Lambda_N^1, \\ f\big(u^N(t), \partial_t u^N(t), \varphi(t-\theta(t)), \partial_t\varphi(t-\theta(t)), t\big) \\ \quad - f\big(u^N(t), \partial_t I_{T,N}U(t), \varphi(t-\theta(t)), \partial_t\varphi(t-\theta(t)), t\big), & t \in \Lambda_N^0, \end{cases} \tag{2.45}$$

$$G_{T,3}^N(t)=\begin{cases} f\big(u^N(t),\partial_t I_{T,N}U(t),\tilde{v}^N(t),\tilde{w}^N(t),t\big) \\ \ -f\big(I_{T,N}U(t),\partial_t I_{T,N}U(t),\tilde{v}^N(t),\tilde{w}^N(t),t\big), & t\in\Lambda_N^1, \\ f\big(u^N(t),\partial_t I_{T,N}U(t),\varphi(t-\theta(t)),\partial_t\varphi(t-\theta(t)),t\big) \\ \ -f\big(I_{T,N}U(t),\partial_t I_{T,N}U(t),\varphi(t-\theta(t)),\partial_t\varphi(t-\theta(t)),t\big), & t\in\Lambda_N^0, \end{cases} \tag{2.46}$$

$$G_{T,4}^N(t)=\begin{cases} f\big(I_{T,N}U(t),\partial_t I_{T,N}U(t),\tilde{v}^N(t),\tilde{w}^N(t),t\big) \\ \ -f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),\tilde{v}^N(t),\tilde{w}^N(t),t\big), & t\in\Lambda_N^1, \\ f\big(I_{T,N}U(t),\partial_t I_{T,N}U(t),\varphi(t-\theta(t)),\partial_t\varphi(t-\theta(t)),t\big) \\ \ -f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),\varphi(t-\theta(t)),\partial_t\varphi(t-\theta(t)),t\big), & t\in\Lambda_N^0, \end{cases} \tag{2.47}$$

$$G_{T,5}^N(t)=\begin{cases} f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),\tilde{v}^N(t),\tilde{w}^N(t),t\big) \\ \ -f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),I_{T,N}V(t),\tilde{w}^N(t),t\big), & t\in\Lambda_N^1, \\ 0, & t\in\Lambda_N^0, \end{cases} \tag{2.48}$$

$$G_{T,6}^N(t)=\begin{cases} f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),I_{T,N}V(t),\tilde{w}^N(t),t\big) \\ \ -f\big(I_{T,N}U(t),I_{T,N}\partial_t U(t),I_{T,N}V(t),I_{T,N}W(t),t\big), & t\in\Lambda_N^1, \\ 0, & t\in\Lambda_N^0. \end{cases} \tag{2.49}$$

Subtracting (2.44) from (2.31) yields

$$\begin{cases} \partial_t^2 E^N(t)=\displaystyle\sum_{j=1}^{6} G_{T,j}^N(t), \qquad t\in\Lambda_N^0\cup\Lambda_N^1=\big\{t_{T,k}^N:\ 0\le k\le N\big\}, \\ E^N(0)=U(0)-I_{T,N}U(0), \quad \partial_t E^N(0)=\partial_t U(0)-\partial_t I_{T,N}U(0). \end{cases} \tag{2.50}$$

We now multiply the first formula of (2.50) by $2\partial_t E^N(t_{T,k}^N)\omega_{T,k}^N$, and sum the resulting equation for $0\le k\le N$ to obtain

$$2(\partial_t E^N,\partial_t^2 E^N)_{T,N}=\sum_{j=1}^{6} A_{T,j}^N, \tag{2.51}$$

where $A_{T,j}^N=2(\partial_t E^N,G_{T,j}^N)_{T,N}$. Since $\partial_t E^N\in\mathcal{P}_{N+1}(0,T)$ and $\partial_t^2 E^N\in\mathcal{P}_N(0,T)$, we use (2.10) to assert that $(\partial_t E^N,\partial_t^2 E^N)_{T,N}=(\partial_t E^N,\partial_t^2 E^N)_T$. Thus, by using integration by parts, (2.51) reads

$$|\partial_t E^N(T)|^2=\sum_{j=1}^{6} A_{T,j}^N+|\partial_t E^N(0)|^2. \tag{2.52}$$

Noting that $G_{T,1}^N\in\mathcal{P}_N(0,T)$, using (2.10) we find for any $\varepsilon>0$,

$$|A_{T,1}^N|=|2(\partial_t E^N,G_{T,1}^N)_{T,N}|=|2(\partial_t E^N,G_{T,1}^N)_T|\le\varepsilon\|\partial_t E^N\|_T^2+\frac{1}{\varepsilon}\|G_{T,1}^N\|_T^2. \tag{2.53}$$

Inserting the above inequality into (2.52) gives

$$|\partial_t E^N(T)|^2\le\sum_{j=2}^{6} A_{T,j}^N+\varepsilon\|\partial_t E^N\|_T^2+\frac{1}{\varepsilon}\|G_{T,1}^N\|_T^2+|\partial_t E^N(0)|^2. \tag{2.54}$$

Since for any $v \in H^1(0,T)$ (see p. 279 of [13]),

$$\max_{t\in[0,T]} |v(t)| \le T^{-\frac{1}{2}}\|v\|_T + T^{\frac{1}{2}}\|\partial_t v\|_T, \tag{2.55}$$

The above inequality together with (2.26) and (2.27) yields for $0 \le t \le T$ that

$$\begin{aligned}|\partial_t I_{T,N}U(t) - \partial_t U(t)| &\le T^{-\frac{1}{2}}\|\partial_t(I_{T,N}U - U)\|_T + T^{\frac{1}{2}}\|\partial_t^2(I_{T,N}U - U)\|_T \\ &\le cT^{r-\frac{3}{2}}N^{\frac{7}{2}-r}|U|_{r,T},\end{aligned} \tag{2.56}$$

which implies that

$$|\partial_t E^N(0)| = |\partial_t U(0) - \partial_t I_{T,N}U(0)| \le cT^{r-\frac{3}{2}}N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.57}$$

Further, with the aid of (2.55), (2.25) and (2.26), we obtain for $0 \le t \le T$ that

$$\begin{aligned}|I_{T,N}U(t) - U(t)| &\le T^{-\frac{1}{2}}\|I_{T,N}U - U\|_T + T^{\frac{1}{2}}\|\partial_t(I_{T,N}U - U)\|_T \\ &\le cT^{r-\frac{1}{2}}N^{\frac{3}{2}-r}|U|_{r,T}.\end{aligned} \tag{2.58}$$

which implies that

$$|E^N(0)| = |U(0) - I_{T,N}U(0)| \le cT^{r-\frac{1}{2}}N^{\frac{3}{2}-r}|U|_{r,T}. \tag{2.59}$$

Lemma 2.3. *If $U \in H^r(0,T)$ with integer $3 \le r \le N+1$, then*

$$\|G_{T,1}^N\|_T \le cT^{r-2}N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.60}$$

Proof. Utilizing (2.25) with $\partial_t^2 U$ and $r-2$ instead of u and r, respectively, we have for any integer $3 \le r \le N+3$,

$$\|I_{T,N}\partial_t^2 U - \partial_t^2 U\|_T \le cT^{r-2}N^{2-r}|U|_{r,T}, \tag{2.61}$$

which together with (2.27) yields for any integer $3 \le r \le N+1$,

$$\|G_{T,1}^N\|_T \le \|I_{T,N}\partial_t^2 U - \partial_t^2 U\|_T + \|\partial_t^2(U - I_{T,N}U)\|_T \le cT^{r-2}N^{\frac{7}{2}-r}|U|_{r,T}.$$

This ends the proof. □

We now consider several typical f and analyze the numerical errors. Hereafter, β denotes a certain positive number less than 1, and $c(\varepsilon)$ denotes a positive constant depends on ε.

Case I. Consider (2.31) with the linear variable delay:

$$\theta(t) = \lambda t, \quad 0 \le \lambda < 1. \tag{2.62}$$

We assume that $f(z_1, z_2, z_3, z_4, t)$ satisfies the following Lipschitz conditions with respect to z_1, z_2, z_3 and z_4, respectively. Namely, there exist real numbers $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$ such that

$$|f(z_1, z_2, z_3, z_4, t) - f(z_1', z_2, z_3, z_4, t)| \leq \gamma_1 |z_1 - z_1'|, \tag{2.63}$$
$$|f(z_1, z_2, z_3, z_4, t) - f(z_1, z_2', z_3, z_4, t)| \leq \gamma_2 |z_2 - z_2'|, \tag{2.64}$$
$$|f(z_1, z_2, z_3, z_4, t) - f(z_1, z_2, z_3', z_4, t)| \leq \gamma_3 |z_3 - z_3'|, \tag{2.65}$$
$$|f(z_1, z_2, z_3, z_4, t) - f(z_1, z_2, z_3, z_4', t)| \leq \gamma_4 |z_4 - z_4'|. \tag{2.66}$$

Clearly, in this case, $\Lambda_N^0 = \emptyset$. From (2.64) and (2.16) we obtain

$$\|G_{T,2}^N\|_{T,N} \leq \gamma_2 \|\partial_t(u^N - I_{T,N}U)\|_{T,N} = \gamma_2 \|\partial_t E^N\|_{T,N} \leq \gamma_2 \|\partial_t E^N\|_T. \tag{2.67}$$

Using (2.63) and (2.17) we have

$$\|G_{T,3}^N\|_{T,N} \leq \gamma_1 \|u^N - I_{T,N}U\|_{T,N} = \gamma_1 \|E^N\|_{T,N} \leq \gamma_1 \gamma_N \|E^N\|_T. \tag{2.68}$$

From (2.64), (2.10), (2.25) and (2.26) we infer that

$$\begin{aligned}\|G_{T,4}^N\|_{T,N} &\leq \gamma_2 \|\partial_t I_{T,N}U - I_{T,N}\partial_t U\|_{T,N} = \gamma_2 \|\partial_t I_{T,N}U - I_{T,N}\partial_t U\|_T \\ &\leq \gamma_2 \big(\|\partial_t(I_{T,N}U - U)\|_T + \|\partial_t U - I_{T,N}\partial_t U\|_T\big) \\ &\leq cT^{r-1}N^{\frac{3}{2}-r}|U|_{r,T}.\end{aligned} \tag{2.69}$$

Moreover, using (2.65), (2.17) and (2.25) we get

$$\begin{aligned}&\|G_{T,5}^N\|_{T,N} \leq \gamma_3 \|\tilde{v}^N - I_{T,N}V\|_{T,N} \leq \gamma_3 \gamma_N \|\tilde{v}^N - I_{T,N}V\|_T \\ &\leq \gamma_3 \gamma_N \big(\|\tilde{v}^N - V\|_T + \|V - I_{T,N}V\|_T\big) \\ &\leq \gamma_3 \gamma_N (1-\lambda)^{-\frac{1}{2}} \|U - u^N\|_T + cT^r N^{-r}|U|_{r,T} \\ &\leq \gamma_3 \gamma_N (1-\lambda)^{-\frac{1}{2}} \|E^N\|_T + \gamma_3 \gamma_N (1-\lambda)^{-\frac{1}{2}} \|U - I_{T,N}U\|_T + cT^r N^{-r}|U|_{r,T} \\ &\leq \gamma_3 \gamma_N (1-\lambda)^{-\frac{1}{2}} \|E^N\|_T + cT^r N^{-r}|U|_{r,T}.\end{aligned} \tag{2.70}$$

Similarly, using (2.66), (2.16), (2.25) and (2.26) we get

$$\begin{aligned}&\|G_{T,6}^N\|_{T,N} \leq \gamma_4 \|\tilde{w}^N - I_{T,N}W\|_{T,N} \leq \gamma_4 \|\tilde{w}^N - I_{T,N}W\|_T \\ &\leq \gamma_4 \big(\|\tilde{w}^N - W\|_T + \|W - I_{T,N}W\|_T\big) \\ &\leq \gamma_4 (1-\lambda)^{-\frac{1}{2}} \|\partial_t(U - u^N)\|_T + \gamma_4 (1-\lambda)^{-\frac{1}{2}} \|\partial_t U - I_{T,N}\partial_t U\|_T \\ &\leq \gamma_4 (1-\lambda)^{-\frac{1}{2}} \|\partial_t E^N\|_T + \gamma_4 (1-\lambda)^{-\frac{1}{2}} \|\partial_t(U - I_{T,N}U)\|_T + cT^{r-1}N^{1-r}|U|_{r,T} \\ &\leq \gamma_4 (1-\lambda)^{-\frac{1}{2}} \|\partial_t E^N\|_T + cT^{r-1}N^{\frac{3}{2}-r}|U|_{r,T}.\end{aligned} \tag{2.71}$$

We now turn to the estimation of $A_{T,j}^N$ for $2 \leq j \leq 6$. Thanks to (2.16) and (2.67) we deduce that

$$|A_{T,2}^N| = |2(\partial_t E^N, G_{T,2}^N)_{T,N}| \leq 2\|\partial_t E^N\|_{T,N} \|G_{T,2}^N\|_{T,N} \leq 2\gamma_2 \|\partial_t E^N\|_T^2. \tag{2.72}$$

Combine (2.16) and (2.68) we infer that

$$|A_{T,3}^N| = |2(\partial_t E^N, G_{T,3}^N)_{T,N}| \le 2\|\partial_t E^N\|_{T,N}\|G_{T,3}^N\|_{T,N} \le \varepsilon\|\partial_t E^N\|_T^2 + \frac{1}{\varepsilon}\|G_{T,3}^N\|_{T,N}^2$$
$$\le \varepsilon\|\partial_t E^N\|_T^2 + \frac{\gamma_1^2\gamma_N^2}{\varepsilon}\|E^N\|_T^2. \tag{2.73}$$

Next, we use (2.16) and (2.69) to get

$$|A_{T,4}^N| = |2(\partial_t E^N, G_{T,4}^N)_{T,N}| \le 2\|\partial_t E^N\|_{T,N}\|G_{T,4}^N\|_{T,N}$$
$$\le \varepsilon\|\partial_t E^N\|_T^2 + c(\varepsilon)T^{2r-2}N^{3-2r}|U|_{r,T}^2. \tag{2.74}$$

Moreover, using (2.16) and (2.70) we obtain

$$|A_{T,5}^N| = |2(\partial_t E^N, G_{T,5}^N)_{T,N}| \le 2\|\partial_t E^N\|_{T,N}\|G_{T,5}^N\|_{T,N} \le \varepsilon\|\partial_t E^N\|_T^2 + \frac{1}{\varepsilon}\|G_{T,5}^N\|_{T,N}^2$$
$$\le \varepsilon\|\partial_t E^N\|_T^2 + \frac{2\gamma_3^2\gamma_N^2(1-\lambda)^{-1}}{\varepsilon}\|E^N\|_T^2 + c(\varepsilon)T^{2r}N^{-2r}|U|_{r,T}^2. \tag{2.75}$$

Similarly, using (2.16) and (2.71) we have

$$|A_{T,6}^N| = |2(\partial_t E^N, G_{T,6}^N)_{T,N}| \le 2\|\partial_t E^N\|_{T,N}\|G_{T,6}^N\|_{T,N} \le \varepsilon\|\partial_t E^N\|_T^2 + \frac{1}{\varepsilon}\|G_{T,6}^N\|_{T,N}^2$$
$$\le \varepsilon\|\partial_t E^N\|_T^2 + \frac{2\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\|\partial_t E^N\|_T^2 + c(\varepsilon)T^{2r-2}N^{3-2r}|U|_{r,T}^2. \tag{2.76}$$

Now, we are ready to present one of the main results of this section.

Theorem 2.1. *Assume that the conditions* (2.62)-(2.66) *hold. If* $U \in H^r(0,T)$ *with integer* $3 \le r \le N+1$, *and for certain* $\varepsilon > 0$, *there hold*

$$\Big(2\gamma_2 + 6\varepsilon + \frac{3\gamma_2^2 + 6\gamma_2\gamma_4(1-\lambda)^{-\frac{1}{2}} + 5\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\Big)T \le \beta < 1, \tag{2.77}$$

and

$$\frac{4\gamma_N^2\Big(4\gamma_1^2 + 6\gamma_1\gamma_3(1-\lambda)^{-\frac{1}{2}} + 5\gamma_3^2(1-\lambda)^{-1}\Big)}{\varepsilon\Big(1 - \big(2\gamma_2 + 6\varepsilon + \frac{3\gamma_2^2 + 6\gamma_2\gamma_4(1-\lambda)^{-\frac{1}{2}} + 5\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\big)T\Big)}T^3 \le \beta < 1. \tag{2.78}$$

Then

$$\|U - u^N\|_T \le c_\beta T^{r-\frac{1}{2}}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.79}$$
$$\|\partial_t(U - u^N)\|_T \le c_\beta T^{r-\frac{3}{2}}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.80}$$

and

$$|U(T) - u^N(T)| \le c_\beta T^{r-1}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.81}$$
$$|\partial_t U(T) - \partial_t u^N(T)| \le c_\beta T^{r-2}N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.82}$$

In particular,

$$\max_{t\in[0,T]}|U(t)-u^N(t)|\le c_\beta T^{r-1}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.83}$$

$$\max_{t\in[0,T]}|\partial_t U(t)-\partial_t u^N(t)|\le c_\beta T^{r-2}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.84}$$

where c_β is a positive constant depending only on β.

Proof. We first inserting (2.60) and (2.57) into (2.54) to obtain that

$$\begin{aligned}&|\partial_t E^N(T)|^2\\ \le&\sum_{j=2}^{6}A^N_{T,j}+\varepsilon\|\partial_t E^N\|_T^2+c(\varepsilon)T^{2r-4}N^{7-2r}|U|^2_{r,T}+cT^{2r-3}N^{7-2r}|U|^2_{r,T}.\end{aligned} \tag{2.85}$$

Next, plugging (2.72)-(2.76) into (2.85) we get that

$$\begin{aligned}&|\partial_t E^N(T)|^2\\ \le&\Big(5\varepsilon+2\gamma_2+\frac{2\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\Big)\|\partial_t E^N\|_T^2+\Big(\frac{\gamma_1^2\gamma_N^2+2\gamma_3^2\gamma_N^2(1-\lambda)^{-1}}{\varepsilon}\Big)\|E^N\|_T^2\\ &+c(\varepsilon)T^{2r-4}N^{7-2r}|U|^2_{r,T}+c(\varepsilon)T^{2r-2}N^{3-2r}|U|^2_{r,T}+cT^{2r-3}N^{7-2r}|U|^2_{r,T}\\ &+c(\varepsilon)T^{2r}N^{-2r}|U|^2_{r,T}\\ \le&\Big(5\varepsilon+2\gamma_2+\frac{2\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\Big)\|\partial_t E^N\|_T^2+\Big(\frac{\gamma_1^2\gamma_N^2+2\gamma_3^2\gamma_N^2(1-\lambda)^{-1}}{\varepsilon}\Big)\|E^N\|_T^2\\ &+c(\varepsilon)(N^7+T^2N^3+TN^7+T^4)T^{2r-4}N^{-2r}|U|^2_{r,T}.\end{aligned} \tag{2.86}$$

We now estimate the lower bound of $|\partial_t E^N(T)|^2$. Obviously, for $0\le t\le T$,

$$|\partial_t E^N(t)|^2=|\partial_t E^N(T)|^2-\int_t^T\partial_s(\partial_s E^N(s))^2ds\le|\partial_t E^N(T)|^2+2\|\partial_t E^N\|_T\|\partial_t^2E^N\|_T.$$

Integrating the above inequality with respect to t over the interval $(0,T)$ gives

$$\begin{aligned}\|\partial_t E^N\|_T^2&\le T|\partial_t E^N(T)|^2+2T\|\partial_t E^N\|_T\|\partial_t^2E^N\|_T\\ &\le T|\partial_t E^N(T)|^2+\varepsilon T\|\partial_t E^N\|_T^2+\frac{T}{\varepsilon}\|\partial_t^2E^N\|_T^2.\end{aligned} \tag{2.87}$$

Since $\partial_t^2E^N, G^N_{T,1}\in\mathcal{P}_N(0,T)$, we use (2.10), (2.50), (2.60) and (2.67)-(2.71) to deduce that

$$\begin{aligned}\|\partial_t^2E^N\|_T=\|\partial_t^2E^N\|_{T,N}&\le\sum_{j=1}^{6}\|G^N_{T,j}\|_{T,N}\\ &\le(\gamma_2+\gamma_4(1-\lambda)^{-\frac12})\|\partial_t E^N\|_T+(\gamma_1\gamma_N+\gamma_3\gamma_N(1-\lambda)^{-\frac12})\|E^N\|_T\\ &\quad+cT^rN^{-r}|U|_{r,T}+cT^{r-2}N^{\frac72-r}|U|_{r,T}+cT^{r-1}N^{\frac32-r}|U|_{r,T}\\ &\le(\gamma_2+\gamma_4(1-\lambda)^{-\frac12})\|\partial_t E^N\|_T+(\gamma_1\gamma_N+\gamma_3\gamma_N(1-\lambda)^{-\frac12})\|E^N\|_T\\ &\quad+c(T^2+N^{\frac72}+TN^{\frac32})T^{r-2}N^{-r}|U|_{r,T}.\end{aligned} \tag{2.88}$$

Putting (2.88) into (2.87), a direct calculation shows that

$$\begin{aligned}\|\partial_t E^N\|_T^2 \le T|\partial_t E^N(T)|^2 + \Big(\varepsilon T + \frac{3T}{\varepsilon}(\gamma_2+\gamma_4(1-\lambda)^{-\frac12})^2\Big)\|\partial_t E^N\|_T^2\\ + \frac{3T}{\varepsilon}(\gamma_1\gamma_N+\gamma_3\gamma_N(1-\lambda)^{-\frac12})^2\|E^N\|_T^2\\ + c(\varepsilon)(T^2+N^{\frac72}+TN^{\frac32})^2T^{2r-3}N^{-2r}|U|_{r,T}^2,\end{aligned} \tag{2.89}$$

which implies that

$$\begin{aligned}|\partial_t E^N(T)|^2 \ge \Big(\frac1T-\varepsilon-\frac{3}{\varepsilon}(\gamma_2+\gamma_4(1-\lambda)^{-\frac12})^2\Big)\|\partial_t E^N\|_T^2\\ - \frac{3}{\varepsilon}(\gamma_1\gamma_N+\gamma_3\gamma_N(1-\lambda)^{-\frac12})^2\|E^N\|_T^2\\ - c(\varepsilon)(T^2+N^{\frac72}+TN^{\frac32})^2T^{2r-4}N^{-2r}|U|_{r,T}^2.\end{aligned} \tag{2.90}$$

Therefore, in view of (2.86) and (2.90), there holds

$$\begin{aligned}&\Big(\frac1T-2\gamma_2-6\varepsilon-\frac{3\gamma_2^2+6\gamma_2\gamma_4(1-\lambda)^{-\frac12}+5\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\Big)\|\partial_t E^N\|_T^2\\ \le&\frac{\gamma_N^2}{\varepsilon}\Big(4\gamma_1^2+6\gamma_1\gamma_3(1-\lambda)^{-\frac12}+5\gamma_3^2(1-\lambda)^{-1}\Big)\|E^N\|_T^2\\ &+c(\varepsilon)(N^7+T^2N^3+TN^7+T^4)T^{2r-4}N^{-2r}|U|_{r,T}^2,\end{aligned} \tag{2.91}$$

or equivalently,

$$\begin{aligned}&\Big(1-\big(2\gamma_2+6\varepsilon+\frac{3\gamma_2^2+6\gamma_2\gamma_4(1-\lambda)^{-\frac12}+5\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\big)T\Big)\|\partial_t E^N\|_T^2\\ \le&\frac{\gamma_N^2}{\varepsilon}\Big(4\gamma_1^2+6\gamma_1\gamma_3(1-\lambda)^{-\frac12}+5\gamma_3^2(1-\lambda)^{-1}\Big)T\|E^N\|_T^2\\ &+c(\varepsilon)(N^7+T^2N^3+TN^7+T^4)T^{2r-3}N^{-2r}|U|_{r,T}^2,\end{aligned} \tag{2.92}$$

On the other hand, it is easy to verify that

$$|E^N(t)|^2 \le |E^N(0)|^2 + 2\|E^N\|_T\|\partial_t E^N\|_T. \tag{2.93}$$

Integrating the above estimate with respect to t over the interval $(0,T)$, we get that

$$\begin{aligned}\|E^N\|_T^2 &\le T|E^N(0)|^2 + 2T\|E^N\|_T\|\partial_t E^N\|_T\\ &\le T|E^N(0)|^2 + \frac12\|E^N\|_t^2 + 2T^2\|\partial_t E^N\|_T^2,\end{aligned} \tag{2.94}$$

which implies

$$\|E^N\|_T^2 \le 2T|E^N(0)|^2 + 4T^2\|\partial_t E^N\|_T^2. \tag{2.95}$$

For convenience, we set

$$d_{T,\varepsilon,\lambda,\gamma_2,\gamma_4}=1-\Big(2\gamma_2+6\varepsilon+\frac{3\gamma_2^2+6\gamma_2\gamma_4(1-\lambda)^{-\frac{1}{2}}+5\gamma_4^2(1-\lambda)^{-1}}{\varepsilon}\Big)T$$

and

$$d_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_N}=1-\frac{4\gamma_N^2}{\varepsilon d_{T,\varepsilon,\lambda,\gamma_2,\gamma_4}}\Big(4\gamma_1^2+6\gamma_1\gamma_3(1-\lambda)^{-\frac{1}{2}}+5\gamma_3^2(1-\lambda)^{-1}\Big)T^3.$$

Inserting (2.92) and (2.59) into (2.95) gives

$$\begin{aligned}&d_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_N}\|E^N\|_T^2\\ \le & cT^{2r}N^{3-2r}|U|_{r,T}^2\\ &+c(\varepsilon)d^{-1}_{T,\varepsilon,\lambda,\gamma_2,\gamma_4}(N^7+T^2N^3+TN^7+T^4)T^{2r-1}N^{-2r}|U|^2_{r,T},\end{aligned}\tag{2.96}$$

which together with (2.77) and (2.78) implies that

$$\begin{aligned}\|E^N\|_T^2&\le c(\varepsilon)d^{-1}_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_N}d^{-1}_{T,\varepsilon,\lambda,\gamma_2,\gamma_4}(N^7+T^2N^3+TN^7+T^4)T^{2r-1}N^{-2r}|U|^2_{r,T}\\ &\le c_\beta T^{2r-1}N^{7-2r}|U|^2_{r,T}.\end{aligned}\tag{2.97}$$

Therefore, a combination of (2.97) and (2.25) yields

$$\begin{aligned}\|U-u^N\|_T&\le\|U-I_{T,N}U\|_T+\|E^N\|_T\le cT^rN^{-r}|U|_{r,T}+c_\beta T^{r-\frac{1}{2}}N^{\frac{7}{2}-r}|U|_{r,T}\\ &\le c_\beta T^{r-\frac{1}{2}}N^{\frac{7}{2}-r}|U|_{r,T},\end{aligned}\tag{2.98}$$

which completes the proof of (2.79).

Next, we have by (2.92) and (2.97) that

$$\|\partial_tE^N\|_T^2\quad\le c_\beta T^{2r-3}N^{7-2r}|U|^2_{r,T}.\tag{2.99}$$

This along with (2.26) yields

$$\|\partial_t(U-u^N)\|_T\le\|\partial_t(U-I_{T,N}U)\|_T+\|\partial_tE^N\|_T\le c_\beta T^{r-\frac{3}{2}}N^{\frac{7}{2}-r}|U|_{r,T},\tag{2.100}$$

which implies (2.80).

Furthermore, using (2.55), (2.97) and (2.99), we thus obtain

$$|E^N(T)|\le T^{-\frac{1}{2}}\|E^N\|_T+T^{\frac{1}{2}}\|\partial_tE^N\|_T\le c_\beta T^{r-1}N^{\frac{7}{2}-r}|U|_{r,T}.\tag{2.101}$$

A combination of (2.58) and (2.101) implies

$$|U(T)-u^N(T)|\le|I_{T,N}U(T)-U(T)|+|E^N(T)|\le c_\beta T^{r-1}N^{\frac{7}{2}-r}|U|_{r,T}.\tag{2.102}$$

Therefore, inserting (2.97) and (2.99) into (2.86) leads to

$$|\partial_tE^N(T)|\le c_\beta T^{r-2}N^{\frac{7}{2}-r}|U|_{r,T},\tag{2.103}$$

which together with (2.56) implies

$$|\partial_t U(T) - \partial_t u^N(T)| \le c_\beta T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}. \tag{2.104}$$

Moreover, using (2.55), (2.79) and (2.80) we deduce that

$$\begin{aligned}\max_{t\in[0,T]} |U(t) - u^N(t)| &\le T^{-\frac{1}{2}} \|U - u^N\|_T + T^{\frac{1}{2}} \|\partial_t (U - u^N)\|_T \\ &\le c_\beta T^{r-1} N^{\frac{7}{2}-r} |U|_{r,T}. \end{aligned} \tag{2.105}$$

Inserting (2.97) and (2.99) into (2.88) yields

$$\|\partial_t^2 E^N\|_T \le c T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}. \tag{2.106}$$

This along with (2.27) implies

$$\|\partial_t^2 (U - u^N)\|_T \le c_\beta T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}. \tag{2.107}$$

Finally, using (2.55), (2.80) and (2.107) we deduce that

$$\begin{aligned}\max_{t\in[0,T]} |\partial_t U(t) - \partial_t u^N(t)| &\le T^{-\frac{1}{2}} \|\partial_t (U - u^N)\|_T + T^{\frac{1}{2}} \|\partial_t^2 (U - u^N)\|_T \\ &\le c_\beta T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}. \end{aligned} \tag{2.108}$$

This completes the proof. □

Remark 2.1. The conditions (2.77) and (2.78) are necessary for the proof, but they should not be essential. Actually, some numerical experiments show that the scheme is still convergent, even if the conditions (2.77) and (2.78) do not hold.

Case II. Assume that the delay function satisfies:

$$t - \theta(t) \le 0, \quad t \in [0, T]. \tag{2.109}$$

Moveover, $f(z_1, z_2, z_3, z_4, t)$ satisfies the Lipschitz conditions (2.63) and (2.64).

Theorem 2.2. *Let $U \in H^r(0,T)$ with integer $3 \le r \le N+1$. If the conditions* (2.109), (2.63) *and* (2.64) *hold, and for certain $\varepsilon > 0$, there hold*

$$\Big(2\gamma_2 + 4\varepsilon + \frac{3\gamma_2^2}{\varepsilon}\Big) T \le \beta < 1, \tag{2.110}$$

and

$$\frac{16\gamma_1^2\gamma_N^2}{\varepsilon\Big(1 - \big(2\gamma_2 + 4\varepsilon + \frac{3\gamma_2^2}{\varepsilon}\big)T\Big)} T^3 \le \beta < 1. \tag{2.111}$$

Then

$$\|U-u^N\|_T \le c_\beta T^{r-\frac{1}{2}} N^{\frac{7}{2}-r} |U|_{r,T}, \tag{2.112}$$

$$\|\partial_t(U-u^N)\|_T \le c_\beta T^{r-\frac{3}{2}} N^{\frac{7}{2}-r} |U|_{r,T}, \tag{2.113}$$

and

$$|U(T)-u^N(T)| \le c_\beta T^{r-1} N^{\frac{7}{2}-r} |U|_{r,T}, \tag{2.114}$$

$$|\partial_t U(T)-\partial_t u^N(T)| \le c_\beta T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}. \tag{2.115}$$

In particular,

$$\max_{t\in[0,T]} |U(t)-u^N(t)| \le c_\beta T^{r-1} N^{\frac{7}{2}-r} |U|_{r,T}, \tag{2.116}$$

$$\max_{t\in[0,T]} |\partial_t U(t)-\partial_t u^N(t)| \le c_\beta T^{r-2} N^{\frac{7}{2}-r} |U|_{r,T}, \tag{2.117}$$

where c_β is a positive constant depending only on β.

Proof. Obviously, in this case, $\Lambda_N^1 = \emptyset$. Then $G_{T,5}^N(t) = G_{T,6}^N(t) = 0$ and $A_{T,5}^N = A_{T,6}^N = 0$. Moreover, the estimates (2.67)-(2.69) still hold. Consequently, the estimates (2.72)-(2.74) are also valid.

We first inserting (2.60) and (2.57) into (2.54) to obtain that

$$\begin{aligned} &|\partial_t E^N(T)|^2 \\ \le&\sum_{j=2}^{4} A_{T,j}^N + \varepsilon\|\partial_t E^N\|_T^2 + c(\varepsilon)T^{2r-4}N^{7-2r}|U|_{r,T}^2 + cT^{2r-3}N^{7-2r}|U|_{r,T}^2. \end{aligned} \tag{2.118}$$

Next, we putting (2.72)-(2.74) into (2.118) yields

$$\begin{aligned} |\partial_t E^N(T)|^2 \le& (3\varepsilon+2\gamma_2)\|\partial_t E^N\|_T^2 + \frac{\gamma_1^2\gamma_N^2}{\varepsilon}\|E^N\|_T^2 \\ &+ c(\varepsilon)(N^7+T^2N^3+TN^7)T^{2r-4}N^{-2r}|U|_{r,T}^2. \end{aligned} \tag{2.119}$$

Since $\partial_t^2 E^N, G_{T,1}^N \in \mathcal{P}_N(0,T)$, we can use (2.10), (2.50), (2.60) and (2.67)-(2.69) to deduce that

$$\begin{aligned} \|\partial_t^2 E^N\|_T =& \|\partial_t^2 E^N\|_{T,N} \le \sum_{j=1}^{4}\|G_{T,j}^N\|_{T,N} \\ \le& \gamma_2\|\partial_t E^N\|_T + \gamma_1\gamma_N\|E^N\|_T + c(N^{\frac{7}{2}}+TN^{\frac{3}{2}})T^{r-2}N^{-r}|U|_{r,T}. \end{aligned} \tag{2.120}$$

Inserting (2.120) into (2.87), a direct calculation shows that

$$\begin{aligned} \|\partial_t E^N\|_T^2 \le& T|\partial_t E^N(T)|^2 + T\Big(\varepsilon+\frac{3\gamma_2^2}{\varepsilon}\Big)\|\partial_t E^N\|_T^2 + \frac{3\gamma_1^2\gamma_N^2 T}{\varepsilon}\|E^N\|_T^2 \\ &+ c(\varepsilon)(N^{\frac{7}{2}}+TN^{\frac{3}{2}})^2T^{2r-3}N^{-2r}|U|_{r,T}^2, \end{aligned} \tag{2.121}$$

which implies that

$$|\partial_t E^N(T)|^2 \geq \Big(\frac{1}{T} - \varepsilon - \frac{3\gamma_2^2}{\varepsilon}\Big)\|\partial_t E^N\|_T^2 - \frac{3\gamma_1^2\gamma_N^2}{\varepsilon}\|E^N\|_T^2 - c(\varepsilon)(N^{\frac{7}{2}} + TN^{\frac{3}{2}})^2 T^{2r-4}N^{-2r}|U|_{r,T}^2. \tag{2.122}$$

Therefore, in view of (2.119) and (2.122), there holds

$$\Big(\frac{1}{T} - 2\gamma_2 - 4\varepsilon - \frac{3\gamma_2^2}{\varepsilon}\Big)\|\partial_t E^N\|_T^2 \leq \frac{4\gamma_1^2\gamma_N^2}{\varepsilon}\|E^N\|_T^2 + c(\varepsilon)(N^7 + T^2N^3 + TN^7)T^{2r-4}N^{-2r}|U|_{r,T}^2, \tag{2.123}$$

or equivalently,

$$\Big(1 - \big(2\gamma_2 + 4\varepsilon + \frac{3\gamma_2^2}{\varepsilon}\big)T\Big)\|\partial_t E^N\|_T^2 \leq \frac{4\gamma_1^2\gamma_N^2 T}{\varepsilon}\|E^N\|_T^2 + c(\varepsilon)(N^7 + T^2N^3 + TN^7)T^{2r-3}N^{-2r}|U|_{r,T}^2. \tag{2.124}$$

For convenience, we set

$$d_{T,\varepsilon,\lambda,\gamma_2} = 1 - \Big(2\gamma_2 + 4\varepsilon + \frac{3\gamma_2^2}{\varepsilon}\Big)T \quad \text{and} \quad d_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_N} = 1 - \frac{16\gamma_1^2\gamma_N^2T^3}{\varepsilon d_{T,\varepsilon,\lambda,\gamma_2}}.$$

Plugging (2.124) and (2.59) into (2.95), we find

$$d_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_N}\|E^N\|_T^2 \leq cT^{2r}N^{3-2r}|U|_{r,T}^2 + c(\varepsilon)d_{T,\varepsilon,\lambda,\gamma_2}^{-1}(N^7 + T^2N^3 + TN^7)T^{2r-1}N^{-2r}|U|_{r,T}^2, \tag{2.125}$$

which together with (2.110) and (2.111) implies that

$$\|E^N\|_T^2 \leq c(\varepsilon)d_{T,\varepsilon,\lambda,\gamma_1,\gamma_2,\gamma_N}^{-1}d_{T,\varepsilon,\lambda,\gamma_2}^{-1}(N^7 + T^2N^3 + TN^7)T^{2r-1}N^{-2r}|U|_{r,T}^2 \leq c_\beta T^{2r-1}N^{7-2r}|U|_{r,T}^2. \tag{2.126}$$

Therefore, a combination of (2.126) and (2.25) yields

$$\|U - u^N\|_T \leq \|U - I_{T,N}U\|_T + \|E^N\|_T \leq cT^rN^{-r}|U|_{r,T} + c_\beta T^{r-\frac{1}{2}}N^{\frac{7}{2}-r}|U|_{r,T} \leq c_\beta T^{r-\frac{1}{2}}N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.127}$$

This ends the proof of (2.112).

Next, we have by (2.124) and (2.126) that

$$\|\partial_t E^N\|_T^2 \leq c_\beta T^{2r-3}N^{7-2r}|U|_{r,T}^2. \tag{2.128}$$

This along with (2.26) yields

$$\|\partial_t(U - u^N)\|_T \leq \|\partial_t(U - I_{T,N}U)\|_T + \|\partial_t E^N\|_T \leq c_\beta T^{r-\frac{3}{2}}N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.129}$$

which implies (2.113).

Furthermore, by virtue of (2.55), (2.126) and (2.128), there holds

$$|E^N(T)| \le T^{-\frac{1}{2}}\|E^N\|_T + T^{\frac{1}{2}}\|\partial_t E^N\|_T \le c_\beta T^{r-1} N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.130}$$

A combination of (2.58) and (2.130) implies

$$|U(T) - u^N(T)| \le |I_{T,N}U(T) - U(T)| + |E^N(T)| \le c_\beta T^{r-1} N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.131}$$

Moreover, inserting (2.126) and (2.128) into (2.119) leads to

$$|\partial_t E^N(T)| \le c_\beta T^{r-2} N^{\frac{7}{2}-r}|U|_{r,T}, \tag{2.132}$$

which together with (2.56) implies

$$|\partial_t U(T) - \partial_t u^N(T)| \le c_\beta T^{r-2} N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.133}$$

Moreover, using (2.55), (2.112) and (2.113) we deduce that

$$\begin{aligned}&\max_{t\in[0,T]}|U(t) - u^N(t)|\\ \le& T^{-\frac{1}{2}}\|U - u^N\|_T + T^{\frac{1}{2}}\|\partial_t(U - u^N)\|_T \le c_\beta T^{r-1} N^{\frac{7}{2}-r}|U|_{r,T}.\end{aligned} \tag{2.134}$$

Putting (2.126) and (2.128) into (2.120) gives

$$\|\partial_t^2 E^N\|_T \le cT^{r-2} N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.135}$$

This along with (2.27) implies

$$\|\partial_t^2(U - u^N)\|_T \le c_\beta T^{r-2} N^{\frac{7}{2}-r}|U|_{r,T}. \tag{2.136}$$

Finally, using (2.55), (2.113) and (2.136) we get (2.117). □

Remark 2.2. From Theorems 2.1 and 2.2, it can be seen that the errors $\|U - u^N\|_T$, $|U(T) - u^N(T)|$, $\|\partial_t(U - u^N)\|_T$ and $|\partial_t U(T) - \partial_t u^N(T)|$ decay rapidly as N and r increase. The convergence rate is $\mathcal{O}(N^{\frac{7}{2}-r})$, which implies that the scheme (2.28) possesses the spectral accuracy.

Remark 2.3. If $\frac{d^rU}{dt^r} \in L^\infty(0,T)$, $3 \le r \le N+1$, we deduce from Theorems 2.1 and 2.2 that

$$\|U - u^N\|_T \le c_\beta T^r N^{\frac{7}{2}-r}\Big\|\frac{d^rU}{dt^r}\Big\|_{L^\infty(0,T)}, \tag{2.137}$$

$$\|\partial_t(U - u^N)\|_T \le c_\beta T^{r-1} N^{\frac{7}{2}-r}\Big\|\frac{d^rU}{dt^r}\Big\|_{L^\infty(0,T)}, \tag{2.138}$$

$$|U(T) - u^N(T)| \le c_\beta T^{r-\frac{1}{2}} N^{\frac{7}{2}-r} \Big\| \frac{d^r U}{dt^r} \Big\|_{L^\infty(0,T)}, \tag{2.139}$$

$$|\partial_t U(T) - \partial_t u^N(T)| \le c_\beta T^{r-\frac{3}{2}} N^{\frac{7}{2}-r} \Big\| \frac{d^r U}{dt^r} \Big\|_{L^\infty(0,T)}. \tag{2.140}$$

Particularly, if $\frac{1}{N} \le T < 1$, we take $r = N+1$ in (2.137)-(2.140) to get that

$$\|U - u^N\|_T = \mathcal{O}(T^{2N-\frac{3}{2}}), \qquad |U(T) - u^N(T)| = \mathcal{O}(T^{2N-2}), \tag{2.141}$$

$$\|\partial_t(U - u^N)\|_T = \mathcal{O}(T^{2N-\frac{5}{2}}), \quad |\partial_t U(T) - \partial_t u^N(T)| = \mathcal{O}(T^{2N-3}). \tag{2.142}$$

3. Multiple interval Legendre-Gauss collocation method

In the previous section, we introduced the single interval Legendre-Gauss collocation method for the DDE (1.1). However, in actual computation, it is not efficient to resolve the scheme (2.31) for large T with very large mode N. It is naturally to divide the interval $(0,T]$ into a finite number of subintervals and solve the equations subsequently on each subinterval economically.

Let M and $N_m,\ 1 \le m \le M$ be any positive integers. We first divide the interval $(0,T]$ into M subintervals $(T_{m-1}, T_m],\ 1 \le m \le M$, such that the set of T_m includes all breaking points, where $T_0 = 0$ and $T_M = T$. Let $\tau_m = T_m - T_{m-1},\ 1 \le m \le M$. We shall use $u_m^{N_m}(t) \in \mathcal{P}_{N_m+2}(0,\tau_m)$ to approximate the solution U in the subinterval $(T_{m-1}, T_m]$.

By replacing T and N by τ_1 and N_1 in (2.31) and all other formulas in Subsection 2.2, we can derive an alternative algorithm, with which we get the numerical solution $u_1^{N_1}(t) \in \mathcal{P}_{N_1+2}(0,\tau_1)$. Then we evaluate the numerical solutions $u_m^{N_m}(t) \in \mathcal{P}_{N_m+2}(0,\tau_m)$, $2 \le m \le M$, step by step. Finally, the global numerical solution of (1.1) is given by

$$u^N(T_{m-1} + t) = u_m^{N_m}(t), \qquad 0 \le t \le \tau_m, \qquad 1 \le m \le M. \tag{3.1}$$

Let $t_{\tau_m,k}^{N_m}$ and $\omega_{\tau_m,k}^{N_m},\ 0 \le k \le N_m$ be the nodes and the corresponding Christoffel numbers of the shifted Legendre-Gauss interpolation on the interval $(0,\tau_m)$. We set

$$\Lambda_{N,m}^0 = \{ t_{\tau_m,k}^{N_m} \mid T_{m-1} + t_{\tau_m,k}^{N_m} - \theta(T_{m-1} + t_{\tau_m,k}^{N_m}) \le 0,\ 0 \le k \le N_m \},$$

and

$$\Lambda_{N,m}^j = \{ t_{\tau_m,k}^{N_m} \mid T_{m-1} + t_{\tau_m,k}^{N_m} - \theta(T_{m-1} + t_{\tau_m,k}^{N_m}) \in (T_{j-1}, T_j],\ 0 \le k \le N_m \}, \quad 1 \le j \le m.$$

The multiple interval Legendre-Gauss collocation scheme for (1.1) is to find $u_m^{N_m}(t) \in \mathcal{P}_{N_m+2}(0,\tau_m)$, such that

$$\begin{cases} \partial_t^2 u_m^{N_m}(t) = f(u_m^{N_m}(t), \partial_t u_m^{N_m}(t), \tilde{v}_m^j(t), \tilde{w}_m^j(t), T_{m-1}+t), & t \in \Lambda_{N,m}^j,\ j > 0, \\ \partial_t^2 u_m^{N_m}(t) = f(u_m^{N_m}(t), \partial_t u_m^{N_m}(t), \varphi(T_{m-1} + t - \theta(T_{m-1}+t)), & \\ \qquad\qquad \partial_t \varphi(T_{m-1} + t - \theta(T_{m-1}+t)), T_{m-1}+t), & t \in \Lambda_{N,m}^0, \\ u_m^{N_m}(0) = u_{m-1}^{N_{m-1}}(\tau_{m-1}), \quad \partial_t u_m^{N_m}(0) = \partial_t u_{m-1}^{N_{m-1}}(\tau_{m-1}), & 2 \le m \le M, \end{cases} \tag{3.2}$$

where

$$\tilde{v}_m^j(t) = u^N(T_{m-1} + t - \theta(T_{m-1} + t)) = u_j^{N_j}(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t))$$

and

$$\tilde{w}_m^j(t) = \partial_t u^N(T_{m-1} + t - \theta(T_{m-1} + t)) = \partial_t u_j^{N_j}(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t)).$$

Let $U_m(t) = U(T_{m-1} + t)$ for $0 \le t \le \tau_m$. Due to (1.1) we find that

$$\begin{cases} \partial_t^2 U_m(t) = f(U_m(t), \partial_t U_m(t), V_m^j(t), W_m^j(t), T_{m-1} + t), & t \in \Lambda_{N,m}^j, \ j > 0, \\ \partial_t^2 U_m(t) = f(U_m(t), \partial_t U_m(t), \varphi(T_{m-1} + t - \theta(T_{m-1} + t)), & \\ \qquad\qquad \partial_t \varphi(T_{m-1} + t - \theta(T_{m-1} + t)), T_{m-1} + t), & t \in \Lambda_{N,m}^0, \\ U_m(0) = U_{m-1}(\tau_{m-1}), \quad \partial_t U_m(0) = \partial_t U_{m-1}(\tau_{m-1}), & 2 \le m \le M, \\ U_1(0) = U(0) = \varphi(0), \quad \partial_t U_1(0) = \partial_t U(0) = \partial_t \varphi(0), & \end{cases} \tag{3.3}$$

where

$$V_m^j(t) = U(T_{m-1} + t - \theta(T_{m-1} + t)) = U_j(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t))$$

and

$$W_m^j(t) = \partial_t U(T_{m-1} + t - \theta(T_{m-1} + t)) = \partial_t U_j(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t)).$$

In view of (3.2) and (3.3), we can infer that the local numerical solution $u_m^{N_m}(t)$ is actually an approximation to the local exact solution $U_m(t)$, with the approximate initial data $u_m^{N_m}(0) = u_{m-1}^{N_{m-1}}(\tau_{m-1})$ and $\partial_t u_m^{N_m}(0) = \partial_t u_{m-1}^{N_{m-1}}(\tau_{m-1})$.

4. Numerical results

In this section, we present some numerical examples to illustrate the performance of our methods. Let $u^N(t)$ be the global numerical solution and denote the pointwise error

$$\text{Err}(t) = |U(t) - u^N(t)|. \tag{4.1}$$

For convenience, we shall use uniform time step-size $\tau_m = \tau$ and uniform mode $N_m = N$ in the test of multiple interval collocation scheme. Throughout this paper, we take the initial guess $\widehat{u}_k = 0$, $0 \le k \le N + 2$.

4.1. Linear variable delay (Case I)

Consider a linear DDE with constant coefficients:

$$\begin{cases} U''(t) = \frac{1}{2}U(t) + \frac{1}{3}U'(t) - \frac{1}{2}U(\frac{t}{2}) + \frac{1}{4}U'(\frac{t}{2}) + \frac{5}{6}e^{-t} + \frac{3}{4}e^{-\frac{t}{2}}, \quad 0 < t \le T, \\ U(0) = 1, \quad U'(0) = -1, \end{cases} \tag{4.2}$$

where the exact solution $U(t) = e^{-t}$.

We first test the single interval scheme for problem (4.2) at different T, the pointwise errors are shown in Fig. 1, It can be seen that the numerical errors decay exponentially as N increases. Moreover, we observe that our algorithm is still valid even if the conditions of Theorem 2.1 may not satisfied for large T. We next test the multiple interval scheme at $T = 5$ with different step-size τ. Fig. 2 shows that the numerical errors decay exponentially.

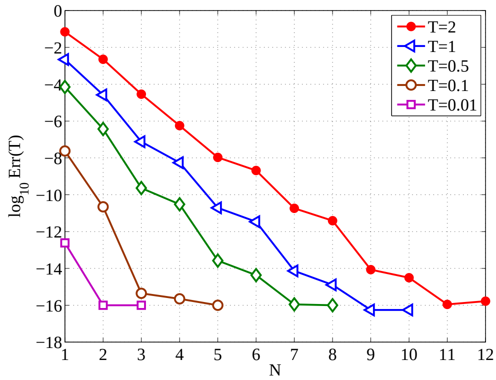

Figure 1: Single interval scheme for problem (4.2) at different T.

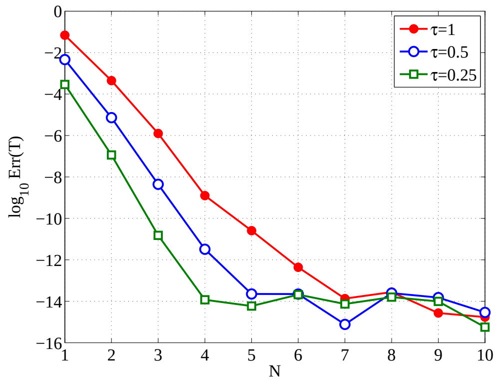

Figure 2: Multiple interval scheme for problem (4.2) at $T = 5$.

4.2. Nonlinear variable delay (Case I)

Consider a nonlinear DDE with variable coefficients:

$$\begin{cases} U''(t) = \sin(t)U(t) + \cos(t)U'(t) \\ \qquad\qquad + \sin(\frac{t}{2})(U(\frac{t}{2}))^2 - (U'(\frac{t}{2}))^3 + g(t), \quad 0 < t \le T, \\ U(0) = 0, \quad U'(0) = 1, \end{cases} \tag{4.3}$$

where $g(t) = -1 - \sin(t) - \sin^3(\frac{t}{2}) + \cos^3(\frac{t}{2})$, and the exact solution $u(t) = \sin(t)$.

The numerical errors obtained by using the single interval scheme at different T are shown in Fig. 3. It can be seen that the numerical errors decay exponentially. Moreover, we observe that our algorithm is still valid even if the conditions of Theorem 2.1 may not satisfied. The numerical errors at $T = 5$ by using the multiple interval scheme with different τ are shown in Fig. 4. It is again shows that the numerical errors decay exponentially.

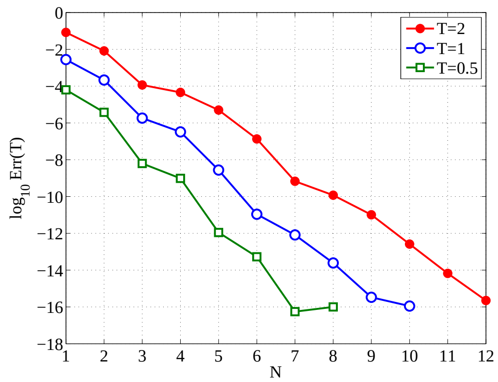

Figure 3: Single interval scheme for problem (4.3) at different T.

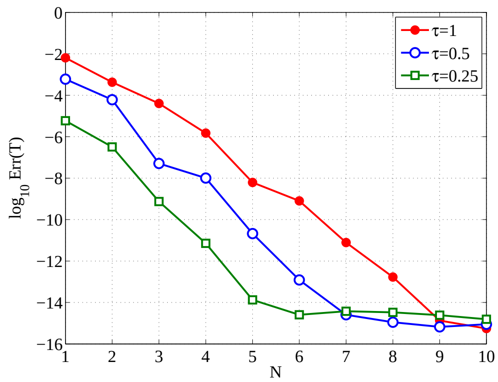

Figure 4: Multiple interval scheme for problem (4.3) at $T = 5$.

4.3. Linear constant delay (Case II)

Consider a linear DDE with constant coefficients (cf. [9]):

$$\begin{cases} U''(t) = -U'(t) - U(t-1) + 1, & 0 < t \le 2, \\ U(t) = 1, \quad U'(0) = -1, & -1 \le t \le 0, \end{cases} \tag{4.4}$$

where the solution is given by

$$U(t) = \begin{cases} e^{-t}, & 0 \le t \le 1, \\ -3 + t + e^{-t} + (1+t)e^{-(t-1)}, & 1 < t \le 2, \end{cases} \tag{4.5}$$

and there is a discontinuous change of $U'''(t)$ at $t = 1$.

In Fig. 5, we plot the numerical errors at $T = 2$ by using the multiple interval method with different τ. It also shows the exponentially converge of the numerical scheme.

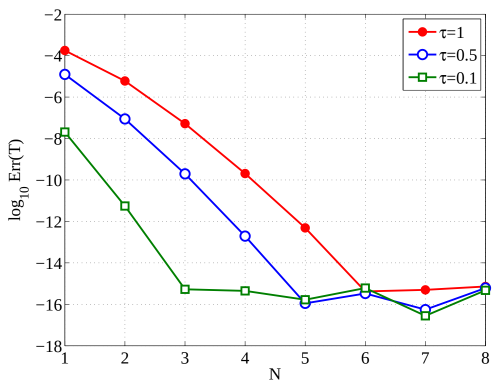

Figure 5: Multiple interval scheme for problem (4.4) at $T = 2$.

4.4. Nonlinear constant delay (Case II)

Consider a nonlinear DDE which describes the propagation of a delayed impulse in an electric circuit (cf. [21,23]):

$$\begin{cases} U''(t) = -a_1 U(t) - a_2 U'(t) - a_3 U'(t-\alpha) + b(U'(t-\alpha))^3, & 0 < t \le T, \\ U(0) = 0.5, \quad U'(t) = 2\pi\cos(20\pi t), & -\alpha \le t \le 0, \end{cases} \tag{4.6}$$

where $a_1 = 100, a_2 = 10, a_3 = 25, b = 0.05$ and $\alpha = 0.1$.

In Fig. 6, we plot the numerical solutions of (4.6) for $t \in [0,5]$ by using the multiple interval method with $\tau = 0.01, N = 10$. It can be observed that after a short transient period the solution describes an almost periodic oscillation with nearly constant amplitude.

We next consider a reference solution at $t = 10$ with $U(10) = -0.5735841564$ (cf. [22]). In Fig. 7, we plot the numerical errors at $T = 10$ with uniform τ and N. It can be seen that the numerical errors decay exponentially.

5. Concluding remarks

In this paper, we have presented and analyzed a single interval Legendre-Gauss spectral collocation method for the second order nonlinear DDEs. We have also de-

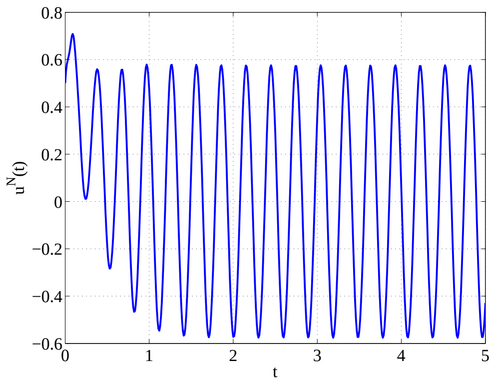

Figure 6: Numerical solution of problem (4.6) with $\tau = 0.01, N = 10$.

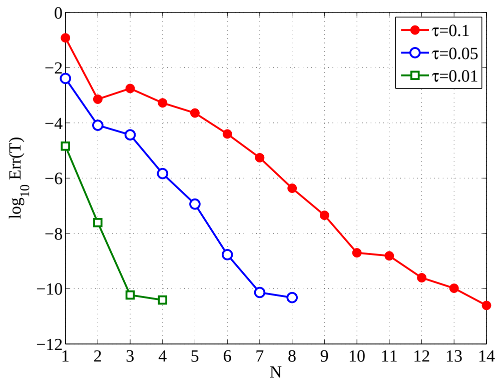

Figure 7: Multiple interval scheme for problem (4.6) at $T = 10$.

signed an efficient algorithm for the single interval scheme. For more efficient implementation, we have further proposed a multiple interval Legendre-Gauss collocation method, which can provide us much flexibility in regard to variable time steps and local approximation orders. The numerical results demonstrated the efficiency of the suggested algorithm and confirmed the well-known exponential convergence property of spectral methods.

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