Numer. Math. Theor. Meth. Appl. Vol. **7**, No. 2, pp. 179-192
doi: 10.4208/nmtma.2014.y12038 May 2014

Simple Fourth-Degree Cubature Formulae with Few Nodes over General Product Regions

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Received 15 October 2012; Accepted (in revised version) 23 September 2013

Available online 16 May 2014

Abstract. A simple method is proposed for constructing fourth-degree cubature formulae over general product regions with no symmetric assumptions. The cubature formulae that are constructed contain at most $n^2 + 7n + 3$ nodes and they are likely the first kind of fourth-degree cubature formulae with roughly n^2 nodes for non-symmetric integrations. Moreover, two special cases are given to reduce the number of nodes further. A theoretical upper bound for minimal number of cubature nodes is also obtained.

AMS subject classifications: 65D30, 65D32

Key words: Fourth-degree cubature formula, cubature formula, product region, non-symmetric region, numerical integration.

1. Introduction

We are interested in the integration

$$I(f) = \int_{\Omega} f(\boldsymbol{x})\boldsymbol{\rho}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1.1}$$

over the product region

$$\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \tag{1.2}$$

with the non-negative weight function $\boldsymbol{\rho}(\boldsymbol{x})$ in the product form

$$\boldsymbol{\rho}(\boldsymbol{x}) = \rho_1(x_1) \cdots \rho_n(x_n), \tag{1.3}$$

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where a_i and b_i are finite or infinite numbers. For a general smooth function $f(\boldsymbol{x})$, such an integration is often numerically approximated by the following weighted sum

$$Q(f) = \sum_{j=1}^{N} w^{(j)} f(\boldsymbol{x}^{(j)}), \tag{1.4}$$

where $\boldsymbol{x}^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)}\right) \in \Omega \subset \mathbb{R}^n$ are N distinct cubature nodes for $j = 1, 2, \cdots, N$, and $w^{(j)} \in \mathbb{R}$ are cubature weights. Denote by $\mathcal{P}_m^n$ the space of the polynomials in n variables of degree no more than m. $Q(f)$ in (1.4) is said to be of degree m with respect to $I(f)$, if $Q(f) = I(f)$ for any $f \in \mathcal{P}_m^n$ and $Q(g) \neq I(g)$ for at least one $g \in \mathcal{P}_{m+1}^n$.

From the numerical points of view, people are interested in the cubature formula with a minimal number of nodes. Denote by $N_{min}^G(m,n)$ the minimal number of nodes of cubature formulae of degree m over general n-dimensional regions. Then one has the following general lower bound (see [3, Th. 9])

$$N_{\min}^G(m,n) \geq \dim \mathcal{P}_{[m/2]}^n, \tag{1.5}$$

where $[x]$ denotes the integer part of x. This lower bound is not very sharp for $n \geq 2$ and can be improved for odd degrees as follows:

$$N_{\min}^G(2k+1,n) \geq \dim \mathcal{P}_k^n + \frac{\sigma_l}{l}, \tag{1.6}$$

where, uniformly for any integer l satisfying $2 \leq l \leq n$, the constant

$$\begin{aligned} \sigma_l := \dim &\left\{ (f_1(\boldsymbol{x}), \cdots, f_l(\boldsymbol{x})) \in \mathcal{Z}_{k+1}^l : \sum_{i=1}^{l} x_i f_i(\boldsymbol{x}) \in \mathcal{P}_{k+1}^n \right\} \\ &- \dim \left\{ (f_1(\boldsymbol{x}), \cdots, f_l(\boldsymbol{x})) \in \mathcal{Z}_{k+1}^l : \sum_{i=1}^{l} x_i f_i(\boldsymbol{x}) \in \mathcal{Z}_{k+1} \right\}, \end{aligned}$$

and

$$\mathcal{Z}_{k+1} := \{ f(\boldsymbol{x}) \in \mathcal{P}_{k+1}^n : \; g(\boldsymbol{x}) \in \mathcal{P}_k^n \Rightarrow I(fg) = 0 \}.$$

See [2, 6, 7]. For centrally symmetric regions, one can get a better lower bound

$$N_{\min}^{CS}(2k+1,n) \geq 2 \dim \mathcal{Q}_k^n - \begin{cases} 1, & \text{if } k \text{ is even and } \mathbf{0} \text{ is a node}, \\ 0, & \text{others}, \end{cases} \tag{1.7}$$

where $\mathcal{Q}_{2k}^n$ is the subspace of $\mathcal{P}_{2k+1}^n$ generated by even polynomials and $\mathcal{Q}_{2k+1}^n$ is the subspace of $\mathcal{P}_{2k+1}^n$ generated by odd polynomials (see [7]), or explicitly (see [5])

$$N_{\min}^{CS}(2k+1,n) \geq \begin{cases} \dbinom{n+k}{n} + \displaystyle\sum_{s=1}^{n-1} 2^{s-n} \dbinom{s+k}{s}, & \text{if } k \text{ is odd}, \\ \dbinom{n+k}{n} + \displaystyle\sum_{s=1}^{n-1} (1-2^{s-n}) \dbinom{s+k-1}{s}, & \text{if } k \text{ is even}. \end{cases} \tag{1.8}$$

We also notice that the general lower bound in (1.5) is not improved for even degrees. Actually, people usually take $N_{\min}(2k+1,n)$ as $N_{\min}(2k,n)$ for $k \geq 2$, which is very reasonable for symmetric regions since the cubature formula of degree $2k$ over a symmetric region is also of degree $2k+1$. However, this does not hold for general regions. Until now, very few efforts were devoted to cubature formulae of even degrees.

The purpose of this paper is to propose an implementable scheme to construct fourth-degree cubature formulae over general product regions. Throughout this paper, the region (1.2) needs not to be symmetric. To get a fourth-degree cubature formula, we only need to deal with n one-dimensional moment problems, which simplifies the computation in a great deal and is numerically economical. We will show that our formulae need at most n^2+5n+3 nodes for $3 \leq n \leq 9$ and n^2+7n+3 nodes for $n \geq 10$, where n stands for the dimension. Compared with the well-known Smolyak formula, the number of the nodes is reduced by half.

This paper is organized as follows. In Section 2, we present our fourth-degree cubature formulae over general product regions. Also two special cases are given in which the number of nodes can be decreased further. Then the cubature formulae for Gamma and Beta weight functions are given in Section 3. These two special weight functions appear frequently in many applied areas. Finally, we compare our proposed scheme with some known cubature formulae in Section 4, showing the advantages of our scheme.

2. Cubature formulae of degree four

Assume that $\rho_i(x_i)$ is non-negative and normalized on $[a_i,b_i]$, i.e.,

$$\int_{a_i}^{b_i} \rho_i(x_i)\mathrm{d}x_i = 1, \quad i=1,\cdots,n.$$

Let $\varphi_i(x_i) := c_i x_i + d_i$ be the orthonormal polynomial of degree one with respect to $\rho_i(x_i)$ on $[a_i,b_i]$ and $(y_1,y_2,\cdots,y_n) = (\varphi_1(x_1),\cdots,\varphi_n(x_n))$. Then, the integration in (1.1) is transformed into

$$\widehat{I}(\widehat{f}) = \int_{\widehat{a}_1}^{\widehat{b}_1} \cdots \int_{\widehat{a}_n}^{\widehat{b}_n} \widehat{f}(y_1,\cdots,y_n)\widehat{\rho}_1(y_1)\cdots\widehat{\rho}_n(y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{2.1}$$

The normalization of ρ_i on $[a_i,b_i]$ and the assumption on $\varphi_i(x_i)$ yield that

$$\int_{\widehat{a}_i}^{\widehat{b}_i} \widehat{\rho}_i(y_i)\mathrm{d}y_i = 1, \quad \int_{\widehat{a}_i}^{\widehat{b}_i} y_i\widehat{\rho}_i(y_i)\mathrm{d}y_i = 0, \quad \int_{\widehat{a}_i}^{\widehat{b}_i} y_i^2\widehat{\rho}_i(y_i)\mathrm{d}y_i = 1, \quad i=1,2,\cdots,n. \tag{2.2}$$

Thus we come to that

$$\begin{aligned}
&\widehat{I}(1) = 1, \qquad \widehat{I}(y_{i_1}) = 0,\\
&\widehat{I}(y_{i_1}y_{i_2}) = \begin{cases} 1, & \text{for} \quad i_1 = i_2; \\ 0, & \text{for} \quad i_1 \neq i_2, \end{cases} \qquad \widehat{I}(y_{i_1}y_{i_2}y_{i_3}) = \begin{cases} \widehat{I}(y_{i_1}^3), & \text{for} \quad i_1 = i_2 = i_3; \\ 0, & \text{others}, \end{cases}
\end{aligned}$$

$$\widehat{I}(y_{i_1}y_{i_2}y_{i_3}y_{i_4}) = \begin{cases} 1, & \text{for} \quad i_1 = i_2 \text{ and } i_3 = i_4; \\ \widehat{I}(y_{i_1}^4), & \text{for} \quad i_1 = i_2 = i_3 = i_4; \\ 0, & \text{others.} \end{cases}$$

To derive a cubature formula of degree 4 with respect to $\widehat{I}(f)$, we begin with an auxiliary cubature formula $\mathcal{M}(f)$ of degree at least 4 with respect to some centrally symmetric integration $\mathcal{L}(f)$ which satisfies

$$\mathcal{L}(y_1^{k_1} \cdots y_n^{k_n}) = 0,$$

if at least one k_i is odd. Furthermore, we assume that

$$\mathcal{L}(1) = 1, \quad \mathcal{L}(y_i^2) = \lambda_1, \quad \mathcal{L}(y_i^4) = \lambda_2, \quad \text{and} \quad \mathcal{L}(y_i^2 y_j^2) = \lambda_3, \quad i \neq j,$$

and that the auxiliary cubature formula has the form:

$$\mathcal{M}(f) = \sum_{j=1}^{\widehat{N}} w^{(j)} f(\boldsymbol{y}^{(j)}). \tag{2.3}$$

Now we state the main result of this paper:

Theorem 2.1. *Let $w^{(j)}, \boldsymbol{y}^{(j)}$ be as assumed in (2.3), $j = 1, 2, \cdots, \widehat{N}$. There must exist a fourth-degree cubature formula with respect to $\widehat{I}(f)$ in the form*

$$\widehat{Q}(f) = \gamma \sum_{j=1}^{\widehat{N}} w^{(j)} f(\widehat{\gamma}\boldsymbol{y}^{(j)}) + \sum_{i=1}^{n} \sum_{j=1}^{l_i} w_i^{(j)} f\left(v_i^{(j)} \cdot \boldsymbol{e}_i\right) + w_0 f(0, \cdots, 0), \tag{2.4}$$

where γ be a positive number, $\widehat{\gamma} = (\lambda_3\gamma)^{-\frac{1}{4}}$, $l_i \leq 4$ and $\boldsymbol{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^n, i = 1, 2, \cdots, n$. Moreover, the total number of nodes is at most $\widehat{N} + 4n + 1$.

Proof. Let

$$\widehat{\mathcal{M}}(f) = \gamma \sum_{j=1}^{\widehat{N}} w^{(j)} f(\widehat{\gamma}\boldsymbol{y}^{(j)}).$$

Then, remembering that $\mathcal{M}(f)$ is of degree at least 4, we obtain

$$\begin{aligned}
&\widehat{\mathcal{M}}(1) = \gamma, \\
&\widehat{\mathcal{M}}(y_{i_1}) = 0, \\
&\widehat{\mathcal{M}}(y_{i_1}y_{i_2}) = \gamma \sum_{j=1}^{\widehat{N}} w^{(j)} \left(\sqrt{\frac{1}{\lambda_3\gamma}} \cdot y_{i_1}^{(j)} y_{i_2}^{(j)} \right) = \sqrt{\frac{\gamma}{\lambda_3}} \sum_{j=1}^{\widehat{N}} w^{(j)} \left(y_{i_1}^{(j)} y_{i_2}^{(j)} \right) \\
&\qquad = \sqrt{\frac{\gamma}{\lambda_3}} \mathcal{L}(y_{i_1}y_{i_2}) = \begin{cases} \lambda_1 \sqrt{\frac{\gamma}{\lambda_3}}, & \text{for} \quad i_1 = i_2; \\ 0, & \text{for} \quad i_1 \neq i_2, \end{cases}
\end{aligned}$$

$$\widehat{\mathcal{M}}(y_{i_1}y_{i_2}y_{i_3})=0,$$

$$\widehat{\mathcal{M}}(y_{i_1}y_{i_2}y_{i_3}y_{i_4})=\gamma\sum_{j=1}^{\widehat{N}}w^{(j)}\left(\frac{1}{\lambda_3\gamma}\cdot y_{i_1}^{(j)}y_{i_2}^{(j)}y_{i_3}^{(j)}y_{i_4}^{(j)}\right)=\frac{1}{\lambda_3}\sum_{j=1}^{\widehat{N}}w^{(j)}\left(y_{i_1}^{(j)}y_{i_2}^{(j)}y_{i_3}^{(j)}y_{i_4}^{(j)}\right)$$

$$=\frac{1}{\lambda_3}\mathcal{L}(y_{i_1}y_{i_2}y_{i_3}y_{i_4})=\begin{cases}1, & \text{for } \ i_1=i_2 \text{ and } i_3=i_4;\\ \frac{\lambda_2}{\lambda_3}, & \text{for } \ i_1=i_2=i_3=i_4;\\ 0, & \text{others.}\end{cases}$$

Take $\widetilde{I}(f)=\widehat{I}(f)-\widehat{\mathcal{M}}(f)$. Now we construct the cubature formula $\widetilde{Q}(f)$ such that $\widetilde{Q}(f)=\widetilde{I}(f)$ for all $f(\boldsymbol{y})\in\mathcal{P}_4^n$. Introduce

$$\mathcal{R}_1=\left\{y_i^{k_i}\,:\,1\le i\le n,\ 0\le k_i\le 4\right\},\quad \mathcal{R}_2=\left\{\boldsymbol{y}^{\boldsymbol{k}}=y_1^{k_1}\cdots y_n^{k_n}\,:\,\boldsymbol{y}^{\boldsymbol{k}}\in\mathcal{P}_4^n,\ \boldsymbol{y}^{\boldsymbol{k}}\notin\mathcal{R}_1\right\}.$$

Noting that $\widetilde{I}(f)=0$ for $f\in\mathcal{R}_2$, we can impose on $\widetilde{Q}(f)$ the following structure

$$\widetilde{Q}(f)=\sum_{i=1}^{n}\sum_{j=1}^{l_i}w_i^{(j)}f(v_i^{(j)}\cdot\boldsymbol{e}_i)+w_0f(0,\cdots,0),$$

which satisfies that $\widetilde{Q}(f)=\widetilde{I}(f)=0$ for $f\in\mathcal{R}_2$. For $f\in\mathcal{R}_1$, it holds that

$$\widetilde{I}(1)=1-\gamma,\quad \widetilde{I}(y_i)=0,\quad \widetilde{I}(y_i^2)=1-\lambda_1\sqrt{\frac{\gamma}{\lambda_3}},\quad \widetilde{I}(y_i^3)=\widehat{I}(y_i^3),\quad \widetilde{I}(y_i^4)=\widehat{I}(y_i^4)-\frac{\lambda_2}{\lambda_3}.$$

Thus, in order to enforce that $\widetilde{Q}(f)$ is of degree 4, it requires that

$$\widetilde{Q}(1)=\sum_{j=1}^{l_1}w_1^{(j)}+\cdots+\sum_{j=1}^{l_n}w_n^{(j)}+w_0=\widetilde{I}(1), \tag{2.5a}$$

$$\widetilde{Q}(y_i^k)=\sum_{j=1}^{l_i}w_i^{(j)}\left(v_i^{(j)}\right)^k=\widetilde{I}(y_i^k),\quad k=1,2,3,4,\quad i=1,2,\cdots,n. \tag{2.5b}$$

Rewrite (2.5) as

$$\begin{cases}\sum_{j=1}^{l_i}w_i^{(j)}\left(v_i^{(j)}\right)^k=\widetilde{I}(y_i^k), \quad k=1,2,3,4,\\ \sum_{j=1}^{l_i}w_i^{(j)}=\theta_i,\end{cases}\quad i=1,2,\cdots,n, \tag{2.6a}$$

$$\theta_1+\theta_2+\cdots+\theta_n=\widetilde{I}(1)-w_0. \tag{2.6b}$$

Then (2.6a) constitute n one-dimensional moment problems, where θ_i are determined by (2.6b). For each i, given four distinct non-zero points $v_i^{(j)}$, $j=1,2,3,4$. Suppose that

$$P_4(y_i)=\sum_{j=1}^{4}f\left(v_i^{(j)}\right)E_i^{(j)}(y_i)+f(0)E_i^{(5)}(y_i)$$

is the corresponding fourth Lagrange interpolation formula. Let $l_i = 4$ and

$$w_i^{(j)} = \widetilde{I}\left(E_i^{(j)}(y_i)\right), \quad j = 1, 2, 3, 4. \tag{2.7}$$

Then (2.6a) is satisfied. Thus a fourth-degree cubature formula will be obtained by taking

$$w_0 = 1 - \gamma - \sum_{i=1}^{n}\sum_{j=1}^{4} w_i^{(j)} = 1 - \gamma - \sum_{i=1}^{n} \theta_i. \tag{2.8}$$

For this case, the total number of nodes is $N = \widehat{N} + 4n + 1$. The proof is completed. $\square$

From the proof above, we see that the total number of nodes N is closely related to the auxiliary formula $\mathcal{M}(f)$. In the following, we first consider a well-known cubature formula with respect to the integration over the n-dimensional sphere

$$U_n = \left\{ \boldsymbol{y} \in \mathbb{R}^n : \sum_{i=1}^{n} y_i^2 = 1 \right\}.$$

In this case, let $\mathcal{L}(f) = \frac{1}{V}\int_{U_n} f \mathrm{d}s$ and $V = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. It is easy to check that

$$\lambda_1 = \frac{1}{n}, \quad \lambda_2 = \frac{3}{n(n+2)}, \quad \text{and} \quad \lambda_3 = \frac{1}{n(n+2)}.$$

Take $\mathcal{M}(f)$ as the cubature formula of degree 5 over U_n for $n \geq 4$ given by Mysovskikh in [8]:

$$\mathcal{M}(f) = w_{\boldsymbol{p}} \sum_{j=1}^{n+1} \left[f(\boldsymbol{p}^{(j)}) + f(-\boldsymbol{p}^{(j)})\right] + w_{\boldsymbol{q}} \sum_{j=1}^{n(n+1)/2} \left[f(\boldsymbol{q}^{(j)}) + f(-\boldsymbol{q}^{(j)})\right], \tag{2.9}$$

where

$$\begin{aligned}
\boldsymbol{p}^{(j)} &= \left(p_1^{(j)}, p_2^{(j)}, \cdots, p_n^{(j)}\right), \quad j = 1, 2, \cdots, n+1, \\
p_i^{(j)} &= \begin{cases} -\sqrt{\frac{n+1}{n(n-i+2)(n-i+1)}}, & i < j; \\ \sqrt{\frac{(n+1)(n-j+1)}{n(n-j+2)}}, & i = j; \\ 0, & i > j, \end{cases} \\
\boldsymbol{q}^{(j)} &= \left(q_1^{(j)}, q_2^{(j)}, \cdots, q_n^{(j)}\right) \in \left\{ \sqrt{\frac{n}{2(n-1)}} \left(\boldsymbol{p}^{(k)} \right.\right. \\
&\qquad \left.\left. + \boldsymbol{p}^{(l)}\right) | k = 1, \cdots, l-1, \ l = 2, \cdots, n+1 \right\}, \\
w_{\boldsymbol{p}} &= -\frac{n(n-7)}{2(n+1)^2(n+2)}, \quad w_{\boldsymbol{q}} = \frac{2(n-1)^2}{n(n+1)^2(n+2)}.
\end{aligned} \tag{2.10}$$

Then we get the following result for $n \geq 4$.

Corollary 2.1. *There exists a fourth-degree cubature formula with respect to $\widehat{I}(f)$ for $n \geq 4$ in the form*

$$\widehat{Q}(f) = \widehat{w}_{\boldsymbol{p}} \sum_{j=1}^{n+1} [f(\widehat{\boldsymbol{p}}^{(j)}) + f(-\widehat{\boldsymbol{p}}^{(j)})] + \widehat{w}_{\boldsymbol{q}} \sum_{j=1}^{n(n+1)/2} [f(\widehat{\boldsymbol{q}}^{(j)}) + f(-\widehat{\boldsymbol{q}}^{(j)})]$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{l_i} w_i^{(j)} f\left(v_i^{(j)} \cdot \boldsymbol{e}_i\right) + w_0 f(0, \cdots, 0), \tag{2.11}$$

where l_i is no greater than 4, and

$$\widehat{\boldsymbol{p}}^{(j)} = \sqrt[4]{\frac{n(n+2)}{\gamma}} \cdot \boldsymbol{p}^{(j)}, \quad \widehat{\boldsymbol{q}}^{(j)} = \sqrt[4]{\frac{n(n+2)}{\gamma}} \cdot \boldsymbol{q}^{(j)}, \quad \widehat{w}_{\boldsymbol{p}} = \gamma w_{\boldsymbol{p}}, \quad \widehat{w}_{\boldsymbol{q}} = \gamma w_{\boldsymbol{q}}. \tag{2.12}$$

Here γ takes the same sense as that in Theorem 2.1.

Obviously, the total number of nodes is at most $n^2 + 7n + 3$ in this case.

We also mention another auxiliary cubature formula with respect to the integration over n-dimensional octahedron G_n:

$$G_n = \left\{ \boldsymbol{y} \in \mathbb{R}^n : \sum_{i=1}^{n} |y_i| \leq 1 \right\}.$$

Stroud [12] proposed a fifth-degree formula with n^2+n+2 nodes for $3 \leq n \leq 9$. If the formula is employed in Theorem 2.1, then we will get a cubature formula of degree 4 with at most $n^2 + 5n + 3$ nodes for $3 \leq n \leq 9$. Since there is no uniform expression of these auxiliary cubature formulae for different dimensions, we omit the details which have been tabled in [13].

Remark 2.1. In some cases, we can take some measures to reduce l_i. Denote by $A_i(f)$ the quadrature formula satisfying the one-dimensional moment problem (2.6a) for each i.

If each $A_i(f)$ is a Gaussian quadrature formula, i.e., $l_i = 3$, then the total number in (2.4) is $\hat{N}+3n+1$. Specially, if Mysovskikh's formulae and Stroud's formulae mentioned above are employed, the total numbers of nodes are $n^2 + 6n + 3$ and $n^2 + 4n + 3$, respectively. It is easy to check that the Gaussian quadrature formula exists if the matrix

$$H_i = \begin{pmatrix} \theta_i & \widetilde{I}(y_i) & \widetilde{I}(y_i^2) \\ \widetilde{I}(y_i) & \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) \\ \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) & \widetilde{I}(y_i^4) \end{pmatrix} = \begin{pmatrix} \theta_i & 0 & 1 - \lambda_1 \sqrt{\frac{\gamma}{\lambda_3}} \\ 0 & 1 - \lambda_1 \sqrt{\frac{\gamma}{\lambda_3}} & \widehat{I}(y_i^3) \\ 1 - \lambda_1 \sqrt{\frac{\gamma}{\lambda_3}} & \widehat{I}(y_i^3) & \widehat{I}(y_i^4) - \frac{\lambda_2}{\lambda_3} \end{pmatrix} \tag{2.13}$$

is positive (or negative) definite.

We also mention if all the moment problems are the same, i.e., all $A_i(f)$ are the same, then $l_i = 3$ is enough. In this case, the construction of cubature formula of one-dimensional moment problem can be rewritten as

$$\begin{cases} \sum_{j=1}^{3} w^{(j)} \left(v^{(j)}\right)^k = \widetilde{I}(y^k), \quad k = 1,2,3,4, \\ \sum_{j=1}^{3} w^{(j)} = \theta, \end{cases} \tag{2.14a}$$

$$n\theta = \widetilde{I}(1) - w_0. \tag{2.14b}$$

Once $v^{(j)}$ is given, then we totally have 5 equations and 5 unknowns ($w^{(j)}, j = 1,2,3, w_0, \gamma$), which usually gives a unique solution.

By taking the inverse transformation, we get the cubature formula with respect to $I(f)$ as follows:

$$\begin{aligned} Q(f) = \gamma \sum_{j=1}^{\widehat{N}} w^{(j)} f\left((\widehat{\gamma}\boldsymbol{y}^{(j)} - \boldsymbol{d})\boldsymbol{C}^{-1} \right) \\ + \sum_{i=1}^{n} \sum_{j=1}^{l_i} w_i^{(j)} f\left((v_i^{(j)} \cdot \boldsymbol{e}_i - \boldsymbol{d})\boldsymbol{C}^{-1} \right) + w_0 f\left(-\boldsymbol{d}\boldsymbol{C}^{-1} \right), \end{aligned} \tag{2.15}$$

where $\boldsymbol{d} = (d_1, d_2, \cdots, d_n)$ and $\boldsymbol{C}^{-1} = \mathrm{diag}(c_1^{-1}, c_2^{-1}, \cdots, c_n^{-1})$.

3. Numerical examples

In this section, we construct cubature formulae by choosing the auxiliary cubature formula as the one in (2.9). We consider two special cases for the weight function $\boldsymbol{\rho}(\boldsymbol{x})$, i.e., the Gamma function and Beta function as the weight functions. Such two special functions appear in many applied areas related to integration computations.

We will construct three different kinds of formulae using different nodes for each weight function respectively. Here is the way to the choice of γ for each kind:

1. In Formula 1-1 (Formula 2-1), we can construct the cubature formulae with $n^2 + 7n + 3$ nodes theoretically for any positive γ. By the proof of Theorem 2.1 and Corollary 2.1, there must exist such cubature formulae. In Formula 1-1, we specify the value γ such that the corresponding cubature formula satisfies some special property.

2. In Formula 1-2 (Formula 2-2), the cubature formulae with $n^2 + 6n + 3$ nodes are constructed when $A_i(f)$ are Gaussian for all i. By Remark 2.1, the matrices H_i in (2.13) should be positive or negative. Hence there exists some range for γ. For example, H_i will be positive, if γ is chosen such that the following three inequalities hold

$$\theta_i > 0, \quad \begin{vmatrix} \theta_i & \widetilde{I}(y_i) \\ \widetilde{I}(y_i) & \widetilde{I}(y_i^2) \end{vmatrix} > 0, \quad \begin{vmatrix} \theta_i & \widetilde{I}(y_i) & \widetilde{I}(y_i^2) \\ \widetilde{I}(y_i) & \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) \\ \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) & \widetilde{I}(y_i^4) \end{vmatrix} > 0. \tag{3.1}$$

The first two inequalities imply that $\widetilde{I}(y_i^2) > 0$. As a result, a sufficient condition that the third inequality holds is $\widetilde{I}(y_i^4) > 0$. Consequently, the range of γ is

$$0 < \gamma < \frac{n}{n+2}\left(1 - \frac{(\widetilde{I}(y_i^3))^2}{\widetilde{I}(y_i^4)}\right)^2.$$

3. In Formula 1-3 (Formula 2-3), we also construct the cubature formulae with n^2+6n+3 nodes. The difference from Formula 1-2 (Formula 2-2) is that these formulae are constructed by a unique γ, which is determined by (2.14), see Remark 2.1.

Case 1. Gamma function as weight function over $\Omega = [0,\infty)^n$.

Set the Gamma weight function $\boldsymbol{\mu}(x) = \prod_{i=1}^n \mu_i(x_i)$ over $\Omega = [\,0,\infty)^n$, where

$$\mu_i(x_i) = \frac{1}{\Gamma(\alpha_i+1)} e^{-x_i} x_i^{\alpha_i}, \quad \alpha_i \geq 0.$$

The orthonormal polynomial of degree one with respect to $\mu_i(x_i)$ is

$$L^{\alpha_i}(x_i) = -\frac{1}{\sqrt{\alpha_i+1}} x_i + \sqrt{\alpha_i+1}.$$

Then, $\boldsymbol{d} = (\sqrt{\alpha_1+1}, \cdots, \sqrt{\alpha_n+1})$ and $\boldsymbol{C}^{-1} = -\mathrm{diag}(\sqrt{\alpha_1+1}, \cdots, \sqrt{\alpha_n+1})$.

Formula 1-1. n^2+7n+3 nodes.

For each i, given four distinct non-zero numbers $v_i^{(j)}, j = 1,2,3,4$. The corresponding values of $w_i^{(j)}$ and w_0 are given by

$$\begin{aligned}
w_i^{(1)} &= \frac{\tau_1 + \tau_2(v_i^{(2)} + v_i^{(3)} + v_i^{(4)}) + \tau_3(v_i^{(3)} v_i^{(4)} + v_i^{(2)}(v_i^{(3)} + v_i^{(4)}))}{v_i^{(1)}(v_i^{(1)} - v_i^{(2)})(v_i^{(1)} - v_i^{(3)})(v_i^{(1)} - v_i^{(4)})}, \\
w_i^{(2)} &= \frac{\tau_1 + \tau_2(v_i^{(1)} + v_i^{(3)} + v_i^{(4)}) - \tau_3(v_i^{(3)} v_i^{(4)} + v_i^{(1)}(v_i^{(3)} + v_i^{(4)}))}{v_i^{(2)}(v_i^{(2)} - v_i^{(1)})(v_i^{(2)} - v_i^{(3)})(v_i^{(2)} - v_i^{(4)})}, \\
w_i^{(3)} &= \frac{\tau_1 + \tau_2(v_i^{(1)} + v_i^{(2)} + v_i^{(4)}) + \tau_3(v_i^{(2)} v_i^{(4)} + v_i^{(1)}(v_i^{(2)} + v_i^{(4)}))}{v_i^{(3)}(v_i^{(3)} - v_i^{(1)})(v_i^{(3)} - v_i^{(2)})(v_i^{(3)} - v_i^{(4)})}, \\
w_i^{(4)} &= \frac{\tau_1 + \tau_2(v_i^{(1)} + v_i^{(2)} + v_i^{(3)}) - \tau_3(v_i^{(2)} v_i^{(3)} + v_i^{(1)}(v_i^{(2)} + v_i^{(3)}))}{v_i^{(4)}(v_i^{(4)} - v_i^{(1)})(v_i^{(4)} - v_i^{(2)})(v_i^{(4)} - v_i^{(3)})}, \\
w_0 &= 1 - \gamma - \sum_{i=1}^n \sum_{j=1}^4 w_i^{(j)},
\end{aligned}$$

where

$$\tau_1 = \frac{6}{\alpha_i + 1}, \quad \tau_2 = 2\sqrt{\frac{1}{\alpha_i + 1}}, \quad \tau_3 = 1 - \sqrt{\frac{(n+2)\gamma}{n}}.$$

Since $-\frac{d_i}{c_i} = \alpha_i + 1 \geq 1$ and $v_i^{(j)}$ can be chosen randomly, all the nodes will be inside Ω, if $(\pm\widehat{\boldsymbol{p}}^{(j)} - \boldsymbol{d})\boldsymbol{C}^{-1}$ and $(\pm\widehat{\boldsymbol{q}}^{(j)} - \boldsymbol{d})\boldsymbol{C}^{-1}$ are non-negative for all j. A sufficient condition for this requirement is that the parameter γ is chosen such that

$$\gamma \geq \max_i \frac{n(n+2)}{(\alpha_i + 1)^2}.$$

Formula 1-2. $n^2 + 6n + 3$ nodes.

If the matrix H_i in (2.13) is positive definite, then we can get a formula with fewer nodes. For example, let $n = 15$, $\alpha_i = 1$ for $i = 1, 2, \cdots, 7$ and $\alpha_i = 2$ for $i = 8, 9, \cdots, 15$, and take $\theta_i = 2.1$ for $i = 1, 2, \cdots, 8$, $\theta_i = 3.1$ for $i = 8, \cdots, 15$ and $\gamma = 0.0551470588235$. Then we get a cubature formula with $n^2 + 6n + 3$ nodes. In Table 1, we list the values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 which are used in the final formula (2.15).

Table 1: Values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in the example of Formula 1-2.

i	j	$v_i^{(j)}$	$w_i^{(j)}$
$1, 2, \cdots, 7$	1	-2.495876187287017 $\mathrm{E}+2$	1.713099918086655 $\mathrm{E}-11$
	2	-2.058741674932151	0.163200783224578
	3	0.173476039836217	1.936799216758290
$8, 9, \cdots, 15$	1	-3.1258987655473015 $\mathrm{E}+2$	4.312589393482033 $\mathrm{E}-12$
	2	-1.6831643507920262	0.243904080210780
	3	0.143738398744674	2.856095919784908
		$w_0 =$	-3.855514705882353 $\mathrm{E}+1$

Formula 1-3. $n^2 + 6n + 3$ nodes.

If the integration is permutation symmetrical, we can also get a formula with n^2+6n+3 nodes. For example, assume that all $\alpha_i = 1$. Setting $v_i^{(j)} = \frac{3-2j}{\sqrt{2}}$ for $j = 1, 2, 3$ and then solving the system equations (2.14) will derive

$$w_i^{(1)} = -\frac{1}{2}, \quad w_i^{(2)} = -1, \quad w_i^{(3)} = \frac{1}{6}, \quad \gamma = \frac{n}{4(n+2)},$$
$$w_0 = 1 - \gamma - n\sum_{j=1}^{3} w_i^{(j)} = \frac{16n^2 + 41n + 24}{12(n+2)}.$$

Case 2. Beta function as weight function over $\Omega = [-1, 1]^n$.

Set the Beta weight function $\boldsymbol{\nu}(\boldsymbol{x}) = \prod_{i=1}^{n} \nu_i(x_i)$ over $\Omega = [-1,1]^n$, where

$$\nu_i(x_i) = \frac{1}{2^{\xi_i+\eta_i+1}B(\xi_i+1,\eta_i+1)}(1-x_i)^{\xi_i}(1+x_i)^{\eta_i}, \quad \xi_i \geq 0, \quad \eta_i \geq 0.$$

The orthonormal polynomial of degree one with respect to $\nu_i(x_i)$ is

$$J^{\xi_i,\eta_i}(x_i) = h_i((\xi_i+\eta_i+2)x_i + \xi_i - \eta_i),$$

where $h_i = \left(\frac{4(\xi_i+1)(\eta_i+1)}{\xi_i+\eta_i+3}\right)^{-1/2}$. Then

$$\boldsymbol{d} = (h_1(\xi_1-\eta_1), \cdots, h_n(\xi_n-\eta_n)), \quad \boldsymbol{C} = \mathrm{diag}(h_1(\xi_1+\eta_1+2), \cdots, h_n(\xi_n+\eta_n+2)).$$

In this case, we take the similar process of construction to the one in Case 1. We only list the results except for some necessary illustrations.

Formula 2-1. n^2+7n+3 nodes.

The specific expressions of $w_i^{(j)}$ are complicated for arbitrary given four nodes $v_i^{(j)}$. We give one example of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in Table 2 by taking $n = 15$, $\xi_i = 1$, $\eta_i = 2$, and $\gamma = 0.882352941176471$.

Table 2: Values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in the example of Formula 2-1.

i	j	$v_i^{(j)}$	$w_i^{(j)}$
$1,2,\cdots,15$	1	-3.0	-0.507936507936508 $\mathrm{E}-2$
	2	-1.75	0.348299319727891 $\mathrm{E}-1$
	3	0.75	0.162539682539683
	4	2.0	-0.380952380952381 $\mathrm{E}-1$
		$w_0 =$	-2.195278111244498

Formula 2-2. n^2+6n+3 nodes by taking the negative definite matrices.

We suppose that the matrices H_i are negative definite in the following form

$$\theta_i < 0, \quad \begin{vmatrix} \theta_i & \widetilde{I}(y_i) \\ \widetilde{I}(y_i) & \widetilde{I}(y_i^2) \end{vmatrix} > 0, \quad \begin{vmatrix} \theta_i & \widetilde{I}(y_i) & \widetilde{I}(y_i^2) \\ \widetilde{I}(y_i) & \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) \\ \widetilde{I}(y_i^2) & \widetilde{I}(y_i^3) & \widetilde{I}(y_i^4) \end{vmatrix} < 0. \tag{3.2}$$

One example of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 is listed in Table 3 by taking $n = 15$, $\xi_i = 1$, $\eta_i = 2$, $\theta_i = -1.8$, and $\gamma = 3.529411764705882$.

Formula 2-3. n^2+6n+3 nodes by taking the same one-dimensional moment problems.

Table 3: Values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in the example of Formula 2-2.

i	j	$v_i^{(j)}$	$w_i^{(j)}$
$1, 2, \cdots, 15$	1	-0.616074472333992	-1.069400518584935
	2	0.901766804703215	-0.730599481413793
	3	2.585143076676308 $\mathrm{E}+2$	-1.272127915517571 $\mathrm{E}-12$
		$w_0 =$	2.447058823529412 $\mathrm{E}+1$

Table 4: Values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in the example of Formula 2-3.

i	j	$v_i^{(j)}$	$w_i^{(j)}$
$1, 2, \cdots, 15$	1	-3.0	0.7263542984726777 $\mathrm{E}-1$
	2	-1.75	-0.1979014794142954
	3	2.0	-0.6421064971660681 $\mathrm{E}-1$
		$w_0 =$	3.5196157661771905

One example of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 is listed in Table 4 by solving (2.14) when $n = 15$, $\xi_i = 1$, $\eta_i = 2$, and $v_i^{(j)} = -3, -1.75, 2$. In this case, $\gamma = 0.322534723077326$.

Remark 3.1. The conditions that the matrices in (2.13) are positive or negative definite are sufficient, but not necessary. Here we give one example that the matrices in (2.13) are neither positive nor negative definite. The example is given by choosing $n = 15, \alpha_i = 1, \theta_i = 0.2$, and $\gamma = 0.882352941176471$. In this case we also get a formula with $n^2 + 6n + 3$, in which the values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 are listed in Table 5.

Table 5: Values of $v_i^{(j)}$, $w_i^{(j)}$ and w_0 in the example of Remark 3.1.

i	j	$v_i^{(j)}$	$w_i^{(j)}$
$1, 2, \cdots, 15$	1	-4.814805767171235	0.780768647291474 $\mathrm{E}-2$
	2	0.759284815097199	0.284399952566790
	3	1.934200608514392	-0.922076390397053 $\mathrm{E}-1$
		$w_0 =$	-2.882352941176471

4. Comparisons and conclusions

Here we compare our proposed cubature formulae with those constructed by Smolyak's method, which is a combination method to construct high-dimensional cubature formulae. This method was already studied by Smolyak in [11]. The degrees of Smolyak formulae were studied in [4, 9, 10]. See also the survey paper [1] and the reference therein. We state the method and results very briefly. Assume that

$$T_j^i(f) = \sum_{k=1}^{N_i} w_k^i f(x_k^i), \quad j = 1, 2, \cdots, n,$$

are a sequence of quadrature formulae of degree m_i with respect to

$$\int_{a_j}^{b_j} \rho_j(x) f(x) \mathrm{d}x.$$

The Smolyak formula is then defined by

$$S(q,n) = \sum_{q-n+1 \le |\mathbf{i}| \le q} (-1)^{q-|\mathbf{i}|} \cdot \binom{n-1}{q-|\mathbf{i}|} \cdot (T_1^{i_1} \otimes \cdots \otimes T_n^{i_n}),$$

where $q \ge n$, $\mathbf{i} \in \mathbb{N}^n$ and $|\mathbf{i}| = i_1 + \cdots + i_n$.

To get a fourth-degree formula, one better way is to choose $m_1 = 1, m_2 = 3, m_3 = 4$ and $N_1 = 1, N_2 = 2, N_3 = 3$. The number of the nodes can be computed by the following recursive formula [10]

$$N(q,n) = N_1 N(q-1, n-1) + \sum_{s=2}^{q-n+1} \tau_s N(q-s, n-1),$$

where τ_s denote the number of nodes used by T_j^i but not by T_j^{i-1}. A rough estimation of $N(q,n)$ is given by [9] for large n and for fourth-degree case it can be written as $N(q,n) \approx 2n^2$, which is almost double the number of nodes of the formulae proposed by our scheme.

Now we end this paper with a brief conclusion. In this paper, we construct a kind of fourth-degree cubature formula with at most $n^2 + 5n + 3$ nodes for $3 \le n \le 9$ or $n^2 + 7n + 3$ nodes for $n \ge 10$, which may be the first kind of fourth-degree cubature formula with roughly n^2 nodes for non-symmetric integrations, where n stands for the dimension. Moreover, compared with the Smolyak formulae, the number of nodes of our proposed scheme is reduced by half. Importantly, the method only needs to deal with one-dimensional moment problems which is easy to apply.

Acknowledgments This research is supported by the National Natural Science Foundation of China (Grant Nos. 61033012, 11301153, 10771028) and "the Fundamental Research Funds for the Central Universities". The authors would like to thank the anonymous referee for the important comments and suggestions which make the paper much improved.

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