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Generalized and Unified Families of Interpolating Subdivision Schemes

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Abstract. We present generalized and unified families of $(2n)$-point and $(2n-1)$-point p-ary interpolating subdivision schemes originated from Lagrange polynomial for any integers $n \geq 2$ and $p \geq 3$. Almost all existing even-point and odd-point interpolating schemes of lower and higher arity belong to this family of schemes. We also present tensor product version of generalized and unified families of schemes. Moreover error bounds between limit curves and control polygons of schemes are also calculated. It has been observed that error bounds decrease when complexity of the scheme decrease and vice versa. Furthermore, error bounds decrease with the increase of arity of the schemes. We also observe that in general the continuity of interpolating scheme do not increase by increasing complexity and arity of the scheme.

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Key words: Interpolating subdivision scheme, even-ary schemes, odd-ary schemes, Lagrange polynomial, parameters, error bounds, tensor product.

1. Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics concerned with algorithms for the design of smooth curves and surfaces and for their competent mathematical demonstration. Subdivision schemes have become a very celebrated research area in CAGD and become a very famous modeling tool of curves and surfaces because of its potential to handle arbitrary topology. To save a smooth object which is created by means of subdivision, one only requires storing a coarse approximation and the subdivision scheme constructing the object. This reality makes subdivision a useful tool in computer aided geometric design. In fact a subdivision

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scheme describes a curve from a primary arbitrary given control polygon by continuously subdividing them according to particular designed refining rules, such that the limiting curve can attain certain smoothness and continuity to meet the requirements of applications. In short one can develop complex smooth curves in a sensibly expected way from quite simple control polygons. Before giving the literature survey we first explain some basic terminologies:

- The number of points inserted at level $k+1$ between two consecutive points from level k is called *arity* of the scheme. If the number of points inserted are even then scheme is called *even-ary* scheme and if number of points inserted are odd then scheme is called *odd-ary* scheme.

- The number of points involved in the convex combination to insert a new point at next subdivision level is called *complexity* of the scheme. If the number of points involved are even then scheme is called *even-point* scheme otherwise it is called *odd-point* scheme.

The concept of subdivision has been first initiated by de Rham [17]. Later on, Deslauriers and Dubuc [2] presented b-ary 2N point schemes derived from polynomial interpolation. Dyn et al. [3] presented 4-point binary interpolating scheme with parameter. Brief review of higher arity schemes having even-point complexity is presented below. Ko et al. [11] introduced even point binary and ternary interpolating symmetric subdivision schemes. Mustafa and Khan [13] introduced a new 4-point C^3 quaternary approximating subdivision scheme. Lian [8] introduced 4-point and 6-point a-ary schemes. Lian [10] offered $2m$-point non-parametric interpolating even and odd-ary schemes for curve design. Zheng et al. [20] offered ternary even symmetric $2n$-point subdivision scheme. Zheng et al. [18] presented p-ary subdivision generalizing B-splines. Mustafa and Najma [14] unified all existing even-point interpolating and approximating schemes by offering general formula for the mask of (2b + 4)-point even-ary subdivision scheme.

Now we present brief review of higher arity schemes having odd-point complexity. Hassan and Dodgson [5] offered ternary and three-point univariate subdivision schemes. Hassan et al. [6] also presented 4-point ternary interpolating subdivision scheme. Lian [9] introduced 3-point and 5-point a-ary schemes. Lian [10] offered $(2m+1)$-point non-parametric interpolating odd-ary schemes for curve design. Zheng et al. [19] constructed $(2n-1)$-point ternary interpolatory subdivision schemes by using variation of constants. Aslam et al. [1] presented an explicit formula which unifies the mask of $(2n-1)$-point interpolating as well as approximating schemes. Mustafa et al. [16] presented an explicit formula for the mask of odd-points n-ary (for any odd $n \geq 3$) interpolating subdivision schemes. This formula unifies the schemes of [9, 10, 19] and many other schemes.

Zorin and Schröder [21] presented a unified framework for construction of an increasing sequence of alternating primal and dual quadrilateral subdivision schemes based on averaging approach. Starting with vertex split, they constructed variants of Doo-Sabin and Catmull-Clark schemes followed by novel schemes generalizing B-splines of bi-degree up to nine. Zhang and Wang [7] proposed a framework of uniform semi-stationary subdivision schemes for curve and surfaces. First they presented framework for curve scheme and then used tensor product approach to extend the curve case to surface as it is done in this paper.

The idea behind using Lagrange interpolants to construct subdivision schemes was initiated by Deslauriers and Dubuc [2]. Assuming for simplicity, that the initial input data are represented by a function $f(r)$, $r \in \mathbb{Z}$, the first step of the (n, p) Deslauriers and Dubuc scheme extends f to all integer multiple of $1/p$. In particular, between any two consecutive parameter values r and $r+1$, f is extended at parameter values $r+\frac{1}{p}, r+\frac{2}{p}, \cdots, r+\frac{p-1}{p}$ taking the values $L\big(r+\frac{1}{p}\big), L\big(r+\frac{2}{p}\big), \cdots, L\big(r+\frac{p-1}{p}\big)$ respectively, where L is the Lagrange polynomial of degree $2n-1$, interpolating f at parametric values $r-n+1, r-n+2, \cdots, r+n$. By applying this process iteratively, f is defined at all p-adic rationals and eventually, by continuity, at all real numbers. The schemes constructed by Lagrange interpolants, are well defined, symmetric and hold polynomial reproducing properties. Here question arises, is it possible the schemes constructed by Lagrange framework include schemes constructed by polynomials or by other frameworks? To answer this question, we choose Lagrange polynomial as base of our framework and primal parametrization for refinement of coarse polygon to refine polygon. Eventually, we have reached at the positive answer of above question. That is we have succeeded to develop the mechanism that generalize and unify all existing even-point, odd-point, even-ary and odd-ary parametric interpolating subdivision schemes.

The paper is structured as follows: In Section 2, basic results are presented which are helpful in construction of schemes in next sections. In Section 3, general odd-point odd-ary scheme is presented. Comparison with existing odd-point odd-ary schemes is also given. In Section 4, general even-point even-ary and odd-ary schemes are presented. Comparison with existing even-point even-ary and odd-ary schemes is also given. In Section 5, error bounds and continuities of the different schemes are discussed. Visual performance of the schemes is also given in this section. In the end of the paper, Section 6 is added which illustrate brief summary and conclusion of the paper.

2. Basic results

A general form of univariate p-ary subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ is defined by

$$f_{pi+\alpha}^{k+1} = \sum_{j\in\mathbb{Z}} a_{pj+\alpha} f_{i+j}^{k}, \quad \alpha = 0, 1, \cdots, p-1, \tag{2.1}$$

where $\mathbb{Z}$ be the set of integers and $p = 2, 3, \cdots$, stands for binary, ternary and so on. The set of coefficients $\{a_{pj+\alpha},\ \alpha = 0, 1, \cdots, p-1\}$ is called subdivision mask. This scheme is formally denoted by $f^{k+1} = Sf^k$. Tensor product of the scheme (2.1) is defined as follows.

$$f^{k+1}_{pi+\alpha,pj+\beta} = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_{pr+\alpha}a_{ps+\beta}f^k_{i+r,j+s}, \quad \alpha,\beta = 0,1,\cdots,p-1. \tag{2.2}$$

Schemes are different due to the mask and arity.

Now we discuss some important identities related to the Lagrange interpolant. We may refer to [1,16] for more detail about the proofs of these identities. For the given n, we define Lagrange fundamental polynomials of degree $2n-1$, corresponding to nodes $\{j\}_{j=-n}^{n-1}$, by

$$L_j^{2n-1}(x) = \prod_{k=-n,k\neq j}^{n-1} \frac{(x-k)}{(j-k)}, \quad j=-n,-(n-1),\cdots,(n-1), \tag{2.3}$$

Lagrange fundamental polynomials of degree $2n-2$, corresponding to nodes $\{j\}_{j=-(n-1)}^{n-1}$, by

$$L_j^{2n-2}(x) = \prod_{k=-(n-1),k\neq j}^{n-1} \frac{(x-k)}{(j-k)}, \quad j=-(n-1),-(n-2),\cdots,(n-1), \tag{2.4}$$

and Lagrange fundamental polynomials of degree $2n-3$, corresponding to nodes $\{j\}_{j=-(n-2)}^{n-1}$, by

$$L_j^{2n-3}(x) = \prod_{k=-(n-2),k\neq j}^{n-1} \frac{(x-k)}{(j-k)}, \quad j=-(n-2),-(n-3),\cdots,(n-1). \tag{2.5}$$

By using algebraic operations, we get following expressions:

$$\prod_{k=-(n-1),\ j\neq k}^{n-1} (j-k) = (-1)^{n-j-1}(n+j-1)!(n-j-1)!, \tag{2.6}$$

where $j=-(n-1),\cdots,(n-1)$,

$$\prod_{k=-(n-2),\ j\neq k}^{n-1} (j-k) = (-1)^{n-j-1}(n+j-2)!(n-j-1)!, \tag{2.7}$$

where $j=-(n-2),\ldots,(n-1)$,

$$\prod_{k=-n,\ j\neq k}^{n-1} (j-k) = (-1)^{n-j-1}(n+j)!(n-j-1)!, \tag{2.8}$$

where $j = -n, \cdots, (n-1)$,

$$L_j^{2n-2}\left(\frac{-q}{2t-1}\right) = \frac{(-1)^{n+j-1}\prod_{k=-(n-2)}^{n}(q-(2t-1)+(2t-1)k)}{(2t-1)^{2n-2}(q+(2t-1)j)(n+j-1)!(n-j-1)!}, \tag{2.9}$$

where $j = -(n-1), \cdots, (n-1)$, $q = 1, 2, 3, \cdots, t-1$ and $t \geq 2$ (any integer),

$$\beta_{q,j} = L_j^{2n-3}\left(\frac{-q}{2t-1}\right) = \frac{(-1)^{n+j-2}\prod_{k=-(n-3)}^{n}(q-(2t-1)+(2t-1)k)}{(2t-1)^{2n-3}(q+(2t-1)j)(n+j-2)!(n-j-1)!}, \tag{2.10}$$

where $j = -(n-2), \cdots, n-1$, $q = 1, 2, 3, \cdots, t-1$ and $t \geq 2$ (any integer),

$$L_j^{2n-1}\left(\frac{-q}{2t}\right) = \frac{(-1)^{n+j}\prod_{k=-(n-1)}^{n}(q-2t+2tk)}{(2t)^{2n-2}(q+2tj)(n+j)!(n-j-1)!}, \tag{2.11}$$

where $j = -n, \cdots, n-1$, $q = 1, 2, 3, \cdots, 2t-1$, $t = \frac{p}{2}$ and $p \geq 3$ (any integer),

$$\theta_{q,j} = L_j^{2n-2}\left(\frac{-q}{2t}\right) = \frac{(-1)^{n+j-1}\prod_{k=-(n-2)}^{n}(q-2t+2tk)}{(2t)^{2n-2}(q+2tj)(n+j-1)!(n-j-1)!}, \tag{2.12}$$

where $j = -(n-1), \cdots, n-1$, $q = 1, 2, 3, \cdots, 2t-1$, $t = \frac{p}{2}$ and $p \geq 3$ (any integer),

$$\eta_j = \frac{L_j^{2n-2}\left(\frac{-q}{2t-1}\right) - L_j^{2n-3}\left(\frac{-q}{2t-1}\right)}{L_{-(n-1)}^{2n-2}\left(\frac{-q}{2t-1}\right)} = \frac{(-1)^{n+j-1}(2n-2)!}{(n+j-1)!(n-j-1)!}, \tag{2.13}$$

where $j = -(n-2), \cdots, (n-1)$ and

$$\xi_j = \frac{L_j^{2n-1}\left(\frac{-q}{2t}\right) - L_j^{2n-2}\left(\frac{-q}{2t}\right)}{L_{-n}^{2n-1}\left(\frac{-q}{2t}\right)} = \frac{(-1)^{n+j}(2n-1)!}{(n+j)!(n-j-1)!}, \tag{2.14}$$

where $j = -(n-1), \cdots, (n-1)$.

Remark 2.1. Justification for the evaluation of Lagrange polynomial at particular values of x: In the setting of primal parametrization, each p-ary refinement of coarse polygon of scheme (2.1) replaces the old data f_i^k by new data f_{pi}^{k+1} and f_{i+1}^k by f_{pi+1}^{k+1}. The sequence of control points $\{f_i^k\}$ is related, in a natural way, with the diadic mesh points $d_i^k = \frac{i}{p^k}$, $i \in Z$. In other words, p-ary refinement (2.1) defines a scheme whereby f_{pi}^{k+1} replaces f_i^k at the diadic mesh point $d_{pi}^{k+1} = d_i^k$ and $f_{p(i+1)}^{k+1}$ replaces f_{i+1}^k at the diadic mesh point $d_{p(i+1)}^{k+1} = d_{i+1}^k$, while $f_{pi+\alpha}^{k+1}$ is inserted at the new mesh point $d_{pi+\alpha}^{k+1} = \frac{1}{p}((p-\alpha)d_i^k + \alpha d_{i+1}^k)$ for $\alpha = 1, 2, \cdots, p-1$.

Therefore, we can select the value of x at $-\frac{q}{2t-1}$ ($q=1,2,\cdots,t-1$ and $t=\frac{1}{2}(p+1)$) and $-\frac{q}{2t}$ ($q=1,2,\cdots,2t-1$ and $t=\frac{p}{2}$) to establish the identities (2.9)-(2.14). In this paper, $x=-\frac{q}{2t-1}$ and $x=-\frac{q}{2t}$ have been selected. One can also select $x=\frac{q}{2t-1}$ and $x=\frac{q}{2t}$ to proof the above identities. The results of the above lemmas at $x=\frac{q}{2t-1}$ and $x=\frac{q}{2t}$ are same but the final mask of the scheme is obtained in reverse order. Negative values of x give a proper order of the mask, due to this reason negative values have been selected to prove the above identities.

3. $(2n-1)$-point p-ary interpolating scheme and comparison with existing schemes

In this section, we present $(2n-1)$-point p-ary interpolating subdivision scheme for any integer $n\geq 2$ and any odd integer $p\geq 3$. We will see that most of the existing odd-point interpolating schemes are special cases of our proposed scheme.

3.1. Odd-point odd-ary scheme

If an odd integer $p\geq 3$ stands for arity, $n\geq 2$ (any integer), $t=\frac{1}{2}(p+1)$ and $q=1,2,3,\cdots,t-1$, then the mask of following $(2n-1)$-point p-ary interpolating scheme

$$\begin{cases} f^{k+1}_{pi-q} = \sum\limits_{j=-(n-1)}^{n-1} a_{q,j} f^k_{i+j}, \\ f^{k+1}_{pi} = f^k_i, \\ f^{k+1}_{pi+q} = \sum\limits_{j=-(n-1)}^{n-1} a_{q,-j} f^k_{i+j}, \end{cases} \tag{3.1}$$

can be generated by

$$\begin{cases} a_{q,-(n-1)} = \omega_q, \\ a_{q,j} = \beta_{q,j} + \eta_j\omega_q, \quad j=-(n-2),\cdots,(n-1), \end{cases} \tag{3.2}$$

where ω_q is a free parameter, $\beta_{q,j}$ and η_j are defined by (2.10) and (2.13) respectively.

Remark 3.1. From the property of Lagrange fundamental polynomials and the construction of scheme, it is clear that the sum of mask coefficients of proposed $(2n-1)$-point p-ary interpolating scheme is one. It can also be proved by induction on n i.e. by substituting $n=2$ in (3.2), we get

$$\begin{aligned} a_{q,-1} &= \omega_q, \\ a_{q,0} &= \frac{q(q+2t-1)}{q(2t-1)} - 2\omega_q, \\ a_{q,1} &= -\frac{q(q+2t-1)}{(q+2t-1)(2t-1)} + \omega_q. \end{aligned}$$

This implies $\sum_{j=-(2-1)}^{2-1} a_{q,j} = 1$. For $n = 3$, we have

$$
\begin{aligned}
a_{q,-2} &= \omega_q, \\
a_{q,-1} &= \frac{(q-2t+1)q(q+2t-1)(q+4t-2)}{6(q-2t+1)(2t-1)^3} - 4\omega_q, \\
a_{q,0} &= -\frac{(q-2t+1)q(q+2t-1)(q+4t-2)}{2q(2t-1)^3} + 6\omega_q, \\
a_{q,1} &= \frac{(q-2t+1)q(q+2t-1)(q+4t-2)}{2(q+2t-1)(2t-1)^3} - 4\omega_q, \\
a_{q,2} &= -\frac{(q-2t+1)q(q+2t-1)(q+4t-2)}{6(q+4t-2)(2t-1)^3} + \omega_q.
\end{aligned}
$$

Again implies $\sum_{j=-(3-1)}^{3-1} a_{q,j} = 1$. Similarly for other values of n, $\sum_{j=-(n-1)}^{n-1} a_{q,j} = 1$.

3.2. Comparison with existing interpolating schemes

Here we see that existing odd-point interpolating schemes are special cases of our schemes generated by (3.1) and (3.2).

- By letting $p = 3$ and $a_{1,-(n-1)} = u$, we get $(2n-1)$-point ternary interpolating scheme of Aslam et al. [1].
- For $p = 3$, $n = 2$, $\omega_1 = b$ and $a = \omega_1 - \frac{1}{3}$, we get Hassan and Dodgson's 3-point ternary interpolating scheme [5].
- If $\{p = a, n = 2\}$ and $\{ p = a, n = 3\}$, we get 3-point and 5-point a-ary interpolating scheme of Lian [9] respectively.
- Let $p = a$ and $n = m + 1$, we get $(2m + 1)$-point a-ary interpolating scheme of Lian [10].
- By letting $\{p = 3, n = 2, \omega_1 = v_2, v_1 = -\frac{1}{3} + \omega_1\}$, $\{p = 3, n = 3, \omega_1 = v_2, v_1 = \frac{4}{81} + \omega_1\}$, $\{p = 5, n = 2, \omega_1 = \frac{3}{25}, \omega_2 = \frac{7}{25}\}$ and $\{p = 7, n = 2, \omega_1 = \frac{4}{49}, \omega_2 = \frac{9}{49}, \omega_3 = \frac{15}{49}\}$, we get 3-point ternary, 5-point ternary, 3-point quinary and 3-point septenary interpolating scheme of Mustafa et al. [16] respectively. Similarly we can easily derive the mask of other odd-point n-ary interpolating schemes of [16].
- For $p = 3$ and $\omega_1 = u$, we get $(2n-1)$-point ternary interpolating scheme of Zheng et al. [19].

3.3. Some new odd-point odd-ary schemes

Here we present some new 3-point ternary, quinary and septenary interpolating schemes generated by (3.1) and (3.2).

- By setting $p = 3$ and $n = 2$, we get following 3-point ternary scheme

$$\begin{cases} f_{3i-1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{4}{3} - 2\omega_1\right) f_i^k + \left(-\frac{1}{3} + \omega_1\right) f_{i+1}^k, \\ f_{3i}^{k+1} = f_i^k, \\ f_{3i+1}^{k+1} = \left(-\frac{1}{3} + \omega_1\right) f_{i-1}^k + \left(\frac{4}{3} - 2\omega_1\right) f_i^k + \omega_1 f_{i+1}^k. \end{cases} \tag{3.3}$$

- By taking $p = 5$ and $n = 2$, we get following 3-point quinary scheme

$$\begin{cases} f_{5i-2}^{k+1} = \omega_2 f_{i-1}^k + \left(\frac{7}{5} - 2\omega_2\right) f_i^k + \left(-\frac{2}{5} + \omega_2\right) f_{i+1}^k, \\ f_{5i-1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{6}{5} - 2\omega_1\right) f_i^k + \left(-\frac{1}{5} + \omega_1\right) f_{i+1}^k, \\ f_{5i}^{k+1} = f_i^k, \\ f_{5i+1}^{k+1} = \left(-\frac{1}{5} + \omega_1\right) f_{i-1}^k + \left(\frac{6}{5} - 2\omega_1\right) f_i^k + \omega_1 f_{i+1}^k, \\ f_{5i+2}^{k+1} = \left(-\frac{2}{5} + \omega_2\right) f_{i-1}^k + \left(\frac{7}{5} - 2\omega_2\right) f_i^k + \omega_2 f_{i+1}^k. \end{cases} \tag{3.4}$$

- By putting $p = 7$ and $n = 2$, we get 3-point septenary scheme

$$\begin{cases} f_{7i-3}^{k+1} = \omega_3 f_{i-1}^k + \left(\frac{10}{7} - 2\omega_3\right) f_i^k + \left(-\frac{3}{7} + \omega_3\right) f_{i+1}^k, \\ f_{7i-2}^{k+1} = \omega_2 f_{i-1}^k + \left(\frac{9}{7} - 2\omega_2\right) f_i^k + \left(-\frac{2}{7} + \omega_2\right) f_{i+1}^k, \\ f_{7i-1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{8}{7} - 2\omega_1\right) f_i^k + \left(-\frac{1}{7} + \omega_1\right) f_{i+1}^k, \\ f_{7i}^{k+1} = f_i^k, \\ f_{7i+1}^{k+1} = \left(-\frac{1}{7} + \omega_1\right) f_{i-1}^k + \left(\frac{8}{7} - 2\omega_1\right) f_i^k + \omega_1 f_{i+1}^k, \\ f_{7i+2}^{k+1} = \left(-\frac{2}{7} + \omega_2\right) f_{i-1}^k + \left(\frac{9}{7} - 2\omega_2\right) f_i^k + \omega_2 f_{i+1}^k, \\ f_{7i+3}^{k+1} = \left(-\frac{3}{7} + \omega_3\right) f_{i-1}^k + \left(\frac{10}{7} - 2\omega_3\right) f_i^k + \omega_3 f_{i+1}^k. \end{cases} \tag{3.5}$$

3.4. Tensor product odd-point odd-ary schemes

If an odd integer $p \geq 3$ stands for arity, $n \geq 2$ (any integer), $t = \frac{1}{2}(p+1)$, $\lambda = t+q-1$, $-(pn-t) \leq \alpha, \beta \leq (pn-t)$ and $q = 1, 2, 3, \cdots, t-1$, then tensor product $(2n-1)$-point p-ary interpolating scheme can be written as

$$\left\{ f_{pi+\alpha,pj+\beta}^{k+1} = \sum_{l=-(n-1)}^{n-1} \sum_{m=-(n-1)}^{n-1} a_{pl+\alpha} a_{pm+\beta} f_{i+l,j+m}^k, \right. \tag{3.6}$$

where

$$\begin{cases} a_0 = 1, & \\ a_{pj} = 0, & j = -(n-1), \cdots, (n-1), \ j \neq 0, \\ a_{\lambda-pn} = \omega_q, & \\ a_{\lambda+p(j-1)} = \beta_{p-q,j} + \eta_j \omega_q, & j = -(n-2), \cdots, (n-1), \\ a_j = a_{-j}, & j = -(pn-t), \cdots, (pn-t). \end{cases} \tag{3.7}$$

Also ω_q is a free parameter, $\beta_{p-q,j}$ is defined by (2.10) and η_j is defined by (2.13).

4. $2n$-point p-ary interpolating scheme and comparison with existing schemes

In this section, we present $2n$-point p-ary interpolating subdivision scheme for any integers $n \geq 2$ and $p \geq 4$. We will see that most of the existing even-point interpolating schemes are special cases of our proposed scheme.

4.1. Even-point even-ary scheme

If an even integer $p \geq 4$ stands for arity, $n \geq 2$ (any integer) and $t = \frac{p}{2}$, then the mask of following $2n$-point p-ary interpolating scheme

$$\begin{cases} f_{pi}^{k+1} = f_i^k, \\ f_{pi+s}^{k+1} = \sum\limits_{j=-(n-1)}^{n} a_{p-s,-(1-j)} f_{i+j}^k, \quad s = 1,2,3,\cdots,t-1, \\ f_{pi+t}^{k+1} = \sum\limits_{j=0}^{n-1} b_j (f_{i-j}^k + f_{i+j+1}^k), \\ f_{pi+u}^{k+1} = \sum\limits_{j=-(n-1)}^{n} a_{u,-j} f_{i+j}^k, \qquad u = t+1, t+2, \cdots, 2t-1, \end{cases} \tag{4.1}$$

can be generated by

$$\begin{cases} a_{p-s,-n} = \omega_s, \\ a_{p-s,j} = \theta_{p-s,j} + \xi_j \omega_s, \quad s = 1,2,3,\cdots,t-1, \\ \qquad\qquad j = -(n-1),\cdots,(n-1), \end{cases} \tag{4.2}$$

$$\begin{cases} a_{u,-n} = \omega_{p-u}, \\ a_{u,j} = \theta_{u,j} + \xi_j \omega_{p-u}, \quad u = t+1, t+2, \cdots, 2t-1, \\ \qquad\qquad j = -(n-1),\cdots,(n-1), \end{cases} \tag{4.3}$$

$$\begin{cases} b_{n-1} = \gamma, \\ b_j = \theta_{t,j} + \xi_j \gamma, \quad j = 0,\cdots,n-2, \end{cases} \tag{4.4}$$

where ω_s, ω_{p-u} and γ are free parameters, $\theta_{p-s,j}$, $\theta_{u,j}$ and $\theta_{t,j}$ are defined by (2.12) and ξ_j is defined by (2.14).

Remark 4.1. The sum of mask coefficients defined in (4.2) is one. For example by substituting $n = 2$ in (4.2), we have

$$\begin{aligned} a_{p-s,-2} &= \omega_s, \\ a_{p-s,-1} &= \frac{(p-s-2t)(p-s)(p-s+2t)}{8t^2(p-s-2t)} - 3\omega_s, \\ a_{p-s,0} &= -\frac{(p-s-2t)(p-s)(p-s+2t)}{4t^2(p-s)} + 3\omega_q, \end{aligned}$$

$$a_{p-s,1} = \frac{(p-s-2t)(p-s)(p-s+2t)}{8t^2(p-s+2t)} - \omega_s.$$

This implies that $\sum_{j=-2}^{2-1} a_{p-s,j} = 1$. By substituting $n=3$ in (4.2), we get

$$\begin{aligned}
a_{p-s,-3} &= \omega_s, \\
a_{p-s,-2} &= \frac{(p-s-4t)(p-s-2t)(p-s)(p-s+2t)(p-s+4t)}{384t^4(p-s-4t)} - 5\omega_s, \\
a_{p-s,-1} &= -\frac{(p-s-4t)(p-s-2t)(p-s)(p-s+2t)(p-s+4t)}{96t^4(p-s-2t)} + 10\omega_s, \\
a_{p-s,0} &= \frac{(p-s-4t)(p-s-2t)(p-s)(p-s+2t)(p-s+4t)}{64t^4(p-s)} - 10\omega_s, \\
a_{p-s,1} &= -\frac{(p-s-4t)(p-s-2t)(p-s)(p-s+2t)(p-s+4t)}{96t^4(p-s+2t)} + 5\omega_s, \\
a_{p-s,2} &= \frac{(p-s-4t)(p-s-2t)(p-s)(p-s+2t)(p-s+4t)}{384t^4(p-s+4t)} - \omega_s.
\end{aligned}$$

Again implies $\sum_{j=-3}^{3-1} a_{p-s,j} = 1$. Similarly for other values of n, $\sum_{j=-n}^{n-1} a_{p-s,j} = 1$.

In the same way, we can easily show that the sum of mask coefficients defined in (4.3) and (4.4) is also one, i.e. $\sum_{j=-n}^{n-1} a_{u,j} = 1$ and $\sum_{-}j = 0^{n-1} b_j = 1$.

4.2. Some new even-point even-ary schemes

Here we present some new even-point even-arity interpolating schemes generated by (4.1)-(4.4).

- By setting $p=4$ and $n=2$, we get following 4-point quaternary scheme

$$\begin{cases}
f_{4i}^{k+1} = f_i^k, \\
f_{4i+1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{21}{32} - 3\omega_1\right) f_i^k + \left(\frac{7}{16} + 3\omega_1\right) f_{i+1}^k + \left(-\frac{3}{32} - \omega_1\right) f_{i+2}^k, \\
f_{4i+2}^{k+1} = \gamma(f_{i-1}^k + f_{i+2}^k) + \left(\frac{3}{4} + 3\gamma\right)(f_i^k + f_{i+1}^k), \\
f_{4i+3}^{k+1} = \left(-\frac{3}{32} - \omega_1\right) f_{i-1}^k + \left(\frac{7}{16} + 3\omega_1\right) f_i^k + \left(\frac{21}{32} - 3\omega_1\right) f_{i+1}^k + \omega_1 f_{i+2}^k.
\end{cases} \tag{4.5}$$

- By setting $p=4$ and $n=3$, we get following 6-point quaternary scheme

$$\begin{cases}
f_{4i}^{k+1} = f_i^k, \\
f_{4i+1}^{k+1} = \omega_1 f_{i-2}^k + \left(-\frac{77}{2048} - 5\omega_1\right) f_{i-1}^k + \left(\frac{385}{512} + 10\omega_1\right) f_i^k \\
\qquad + \left(\frac{385}{1024} - 10\omega_1\right) f_{i+1}^k + \left(-\frac{55}{512} + 5\omega_1\right) f_{i+2}^k + \left(\frac{35}{2048} - \omega_1\right) f_{i+3}^k, \\
f_{4i+2}^{k+1} = \gamma(f_{i-2}^k + f_{i+3}^k) + \left(-\frac{5}{32} + 5\gamma\right)(f_{i-1}^k + f_{i+2}^k) \\
\qquad + \left(\frac{45}{64} - 10\gamma\right)(f_i^k + f_{i+1}^k), \\
f_{4i+3}^{k+1} = \left(\frac{35}{2048} - \omega_1\right) f_{i-2}^k + \left(-\frac{55}{512} + 5\omega_1\right) f_{i-1}^k + \left(\frac{385}{1024} - 10\omega_1\right) f_i^k \\
\qquad + \left(\frac{385}{512} + 10\omega_1\right) f_{i+1}^k + \left(-\frac{77}{2048} - 5\omega_1\right) f_{i+2}^k + \omega_1 f_{i+3}^k.
\end{cases} \tag{4.6}$$

- By setting $p = 6$ and $n = 2$, we get following 4-point senary scheme

$$\begin{cases} f_{6i}^{k+1} = f_i^k, \\ f_{6i+1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{55}{72} - 3\omega_1\right) f_i^k + \left(\frac{11}{36} + 3\omega_1\right) f_{i+1}^k + \left(-\frac{5}{72} - \omega_1\right) f_{i+2}^k, \\ f_{6i+2}^{k+1} = \omega_2 f_{i-1}^k + \left(\frac{5}{9} - 3\omega_2\right) f_i^k + \left(\frac{5}{9} + 3\omega_2\right) f_{i+1}^k + \left(-\frac{1}{9} - \omega_2\right) f_{i+2}^k, \\ f_{6i+3}^{k+1} = \gamma(f_{i-1}^k + f_{i+2}^k) + \left(\frac{3}{4} + 3\gamma\right)(f_i^k + f_{i+1}^k), \\ f_{6i+4}^{k+1} = \left(-\frac{1}{9} - \omega_2\right) f_{i-1}^k + \left(\frac{5}{9} + 3\omega_2\right) f_i^k + \left(\frac{5}{9} - 3\omega_2\right) f_{i+1}^k + \omega_2 f_{i+2}^k, \\ f_{6i+5}^{k+1} = \left(-\frac{5}{72} - \omega_1\right) f_{i-1}^k + \left(\frac{11}{36} + 3\omega_1\right) f_i^k + \left(\frac{55}{72} - 3\omega_1\right) f_{i+1}^k + \omega_1 f_{i+2}^k. \end{cases} \tag{4.7}$$

4.3. Comparison with existing interpolating schemes

Here we see that existing even-point interpolating schemes are special cases of our schemes generated by (4.1)-(4.4).

- By putting $\{\omega_1 = -\frac{7}{128}, \gamma = -\frac{1}{16}\}$ in (4.5), $\{\omega_1 = \frac{77}{8192}, \gamma = \frac{3}{256}\}$ in (4.6) and $\{\omega_1 = -\frac{55}{1296}, \omega_2 = -\frac{5}{81}, \gamma = -\frac{1}{16}\}$ in (4.7), we get 4-point quaternary, 6-point quaternary and 4-point senary interpolating scheme of Lian [8] respectively.
- If $\{p = a, n = 2\}$ and $\{p = a, n = 3\}$, then from (4.1)-(4.4), we get 4-point and 6-point a-ary interpolating scheme of Lian [8] respectively.
- Let $p = a$ and $n = m$, then from (4.1)-(4.4) we get $2m$-point a-ary interpolating scheme of Lian [10].

4.4. Tensor product even-point even-ary schemes

If an even integer $p \geq 4$ stands for arity, $n \geq 2$ (any integer), $t = \frac{p}{2}$ and $0 \leq \alpha, \beta \leq p - 1$, then tensor product $(2n)$-point p-ary interpolating scheme can be written as

$$\left\{ f_{pi+\alpha,pj+\beta}^{k+1} = \sum_{l=-(n-1)}^{n} \sum_{m=-(n-1)}^{n} a_{pl+\alpha} a_{pm+\beta} f_{i+l,j+m}^k, \right. \tag{4.8}$$

where

$$\begin{cases} a_0 = 1, & \\ a_{pj} = 0, & j = -(n-1), \dots, n,\ j \neq 0, \\ a_{p(1-n)+s} = \omega_s, & \\ a_{p(1+j)+s} = \theta_{p-s,j} + \xi_j \omega_s, & s = 1, 2, 3, \cdots, t-1, \\ & j = -(n-1), \cdots, (n-1), \end{cases} \tag{4.9}$$

$$\begin{cases} a_{pn+u} = \omega_{p-u}, & \\ a_{p(1-j)+u} = \theta_{u,j} + \xi_{j-1}\omega_{p-u}, & u = t+1, t+2, \cdots, 2t-1, \\ & j = -(n-2), \cdots, n, \end{cases} \tag{4.10}$$

$$\begin{cases} a_{np+t} = \gamma, \\ a_{p(1+j)+t} = \theta_{t,j} + \xi_{j-1}\gamma, & j = 0, \cdots, n-2, \\ a_{p(1+j)+t} = a_{-pj+t}, & j = 0, \cdots, n-1. \end{cases} \tag{4.11}$$

Also ω_s, ω_{p-u} and γ are free parameters, $\theta_{p-s,j}$, $\theta_{u,j}$ and $\theta_{t,j}$ are defined by (2.12). ξ_j and ξ_{j-1} are defined by (2.14).

4.5. Even-point odd-ary scheme

If an odd integer $p \geq 3$ stands for arity, $n \geq 2$ (any integer) and $t = \frac{p}{2}$, then the mask of following $2n$-point p-ary interpolating scheme

$$\begin{cases} f_{pi}^{k+1} = f_i^k, \\ f_{pi+g}^{k+1} = \sum\limits_{j=-(n-1)}^{n} a_{p-g,-(1-j)} f_{i+j}^k, & g = 1, 2, 3, \cdots t - \frac{1}{2}, \\ f_{pi+h}^{k+1} = \sum\limits_{j=-(n-1)}^{n} a_{h,-j} f_{i+j}^k, & h = t + \frac{1}{2}, t + \frac{3}{2}, \cdots 2t - 1, \end{cases} \tag{4.12}$$

can be generated by

$$\begin{cases} a_{p-g,-n} = \omega_g, \\ a_{p-g,j} = \theta_{p-g,j} + \xi_j \omega_g, & j = -(n-1), \cdots, (n-1), \\ & g = 1, 2, 3, \cdots, t - \frac{1}{2}, \end{cases} \tag{4.13}$$

$$\begin{cases} a_{h,-n} = \omega_{p-h}, \\ a_{h,j} = \theta_{h,j} + \xi_j \omega_{p-h}, & j = -(n-1), \cdots, (n-1), \\ & h = t + \frac{1}{2}, t + \frac{3}{2}, \cdots, 2t - 1, \end{cases} \tag{4.14}$$

where ω_g and ω_{p-h} are free parameters, $\theta_{p-g,j}$ and $\theta_{h,j}$ are defined by (2.12) and ξ_j is defined by (2.14).

4.6. Some new even-point odd-ary schemes

Here we present some new even-point odd-ary interpolating schemes generated by (4.12)-(4.14).

- By setting $p = 3$ and $n = 2$, we get following 4-point ternary scheme

$$\begin{cases} f_{3i}^{k+1} = f_i^k, \\ f_{3i+1}^{k+1} = \omega_1 f_{i-1}^k + \left(\frac{5}{9} - 3\omega_1\right) f_i^k + \left(\frac{5}{9} + 3\omega_1\right) f_{i+1}^k + \left(-\frac{1}{9} - \omega_1\right) f_{i+2}^k, \\ f_{3i+2}^{k+1} = \left(-\frac{1}{9} - \omega_1\right) f_{i-1}^k + \left(\frac{5}{9} + 3\omega_1\right) f_i^k + \left(\frac{5}{9} - 3\omega_1\right) f_{i+1}^k + \omega_1 f_{i+2}^k. \end{cases} \tag{4.15}$$

- By taking $p=3$ and $n=3$, we get following 6-point ternary scheme

$$\begin{cases} f_{3i}^{k+1}=f_i^k, \\ f_{3i+1}^{k+1}=\omega_1 f_{i-2}^k+\left(-\frac{10}{243}-5\omega_1\right)f_{i-1}^k+\left(\frac{160}{243}+10\omega_1\right)f_i^k \\ \qquad +\left(\frac{40}{81}-10\omega_1\right)f_{i+1}^k+\left(-\frac{32}{243}+5\omega_1\right)f_{i+2}^k+\left(\frac{5}{243}-\omega_1\right)f_{i+3}^k, \\ f_{3i+2}^{k+1}=\left(\frac{5}{243}-\omega_1\right)f_{i-2}^k+\left(-\frac{32}{243}+5\omega_1\right)f_{i-1}^k+\left(\frac{40}{81}-10\omega_1\right)f_i^k \\ \qquad +\left(\frac{160}{243}+10\omega_1\right)f_{i+1}^k+\left(-\frac{10}{243}-5\omega_1\right)f_{i+2}^k+\omega_1 f_{i+3}^k. \end{cases} \tag{4.16}$$

- By putting $p=5$ and $n=2$, we get 4-point quinary scheme

$$\begin{cases} f_{5i}^{k+1}=f_i^k, \\ f_{5i+1}^{k+1}=\omega_1 f_{i-1}^k+\left(\frac{18}{25}-3\omega_1\right)f_i^k+\left(\frac{9}{25}+3\omega_1\right)f_{i+1}^k+\left(-\frac{2}{25}-\omega_1\right)f_{i+2}^k, \\ f_{5i+2}^{k+1}=\omega_2 f_{i-1}^k+\left(\frac{12}{25}-3\omega_2\right)f_i^k+\left(\frac{16}{25}+3\omega_2\right)f_{i+1}^k+\left(-\frac{3}{25}-\omega_2\right)f_{i+2}^k, \\ f_{5i+3}^{k+1}=\left(-\frac{3}{25}-\omega_2\right)f_{i-1}^k+\left(\frac{16}{25}+3\omega_2\right)f_i^k+\left(\frac{12}{25}-3\omega_2\right)f_{i+1}^k+\omega_2 f_{i+2}^k, \\ f_{5i+4}^{k+1}=\left(-\frac{2}{25}-\omega_1\right)f_{i-1}^k+\left(\frac{9}{25}+3\omega_1\right)f_i^k+\left(\frac{18}{25}-3\omega_1\right)f_{i+1}^k+\omega_1 f_{i+2}^k. \end{cases} \tag{4.17}$$

4.7. Comparison with existing interpolating schemes

Here we see that existing even-point odd-ary interpolating schemes are special cases of our schemes generated by (4.8)-(4.10).

- For $\{\omega_1=-\frac{5}{81}$ in (4.15)$\}$, $\{\omega_1=-\frac{1}{18}-\frac{1}{6}\mu$ in (4.15)$\}$, $\{\omega_1=\frac{8}{729}$ in (4.16)$\}$, $\{\omega_1=\frac{5}{243}-\omega$ in (4.16)$\}$ and $\{\omega_1=-\frac{6}{125}, \omega_2=-\frac{8}{125}$ in (4.17)$\}$, we get 4-point ternary [2,8], 4-point ternary [6], 6-point ternary [2,8], 6-point ternary [12] and 4-point quinary [10] interpolating schemes respectively.

- If $\{p=a,\ n=2\}$ and $\{p=a$ and $n=3\}$, then by (4.12)-(4.14), we get 4-point and 6-point a-ary interpolating scheme [8] respectively.

- Let $p=a$ and $n=m$, then by (4.12)-(4.14), we get $2m$-point a-ary interpolating scheme [10].

4.8. Tensor product even-point odd-ary scheme

If an odd integer $p\geq 3$ stands for arity, $n\geq 2$ (any integer), $t=\frac{p}{2}$ and $0\leq\alpha,\beta\leq p-1$, then tensor product $(2n)$-point p-ary interpolating scheme can be written as

$$\left\{ f_{pi+\alpha,pj+\beta}^{k+1}=\sum_{l=-(n-1)}^{n}\sum_{m=-(n-1)}^{n}a_{pl+\alpha}a_{pm+\beta}f_{i+l,j+m}^k, \right. \tag{4.18}$$

where

$$\begin{cases} a_0 = 1, & \\ a_{pj} = 0, & j = -(n-1), \cdots, n,\ j \neq 0, \\ a_{p(1-n)+g} = \omega_g, & \\ a_{p(1+j)+g} = \theta_{p-g,j} + \xi_j \omega_g, & g = 1, 2, 3, \cdots, t - \frac{1}{2}, \\ & j = -(n-1), \cdots, (n-1), \end{cases} \tag{4.19}$$

$$\begin{cases} a_{pn+h} = \omega_{p-h}, & \\ a_{p(1-j)+h} = \theta_{h,j} + \xi_{j-1}\omega_{p-h}, & h = t+1, t+2, \cdots, 2t-1, \\ & j = -(n-2), \cdots, n. \end{cases} \tag{4.20}$$

Also ω_g and ω_{p-h} are free parameters, $\theta_{p-g,j}$ and $\theta_{h,j}$ are defined by (2.12). ξ_j and ξ_{j-1} are defined by (2.14).

5. Continuity analysis, applications and error analysis of the schemes

5.1. Continuity analysis

Here we present a brief continuity analysis of one of the proposed scheme.

Theorem 5.1. *The 3-point ternary scheme defined by* (3.3) *is* C^1 *continuous for* $\frac{2}{9} < \omega_1 < \frac{1}{3}$.

Proof. The symbol $a(z)$ of the scheme (3.3) can be written as

$$\begin{aligned} a(z) = &\left(-\frac{1}{3} + \omega_1\right) z^{-4} + \omega_1 z^{-2} + \left(\frac{4}{3} - 2\omega_1\right) z^{-1} + 1 + \left(\frac{4}{3} - 2\omega_1\right) z \\ &+ \omega_1 z^2 + \left(-\frac{1}{3} + \omega_1\right) z^4. \end{aligned}$$

This implies

$$a(z) = \left(\frac{1 + z + z^2}{3z^2}\right)^2 b(z),$$

where

$$\begin{aligned} b(z) = &(-3 + 9\omega_1) + (6 - 18\omega_1)\, z + (-3 + 18\omega_1)\, z^2 + (6 - 18\omega_1)\, z^3 \\ &+ (-3 + 9\omega_1)\, z^4. \end{aligned}$$

Let S_b be the scheme corresponding to the symbol $b(z)$. Since

$$\left\|\frac{1}{3}S_b\right\|_\infty = \left(\frac{1}{3}\right) \max\left\{\sum_{j\in\mathbb{Z}} |b_{3j}|, \sum_{j\in\mathbb{Z}} |b_{3j+1}|, \sum_{j\in\mathbb{Z}} |b_{3j+2}|\right\},$$

then for $\frac{2}{9} < \omega_1 < \frac{1}{3}$, we have

$$\left\|\frac{1}{3}S_b\right\| = \left(\frac{1}{3}\right) \max\{|-3+9\omega_1| + |6-18\omega_1|, |-3+18\omega_1|\} < 1.$$

Therefore by [4, Corollary 4.11], the scheme S_a is C^1. □

Similarly we can easily make continuity analysis of rest of the proposed schemes by using Laurent polynomial formalism. Tables 1-3 show the parametric ranges of continuities of $(2n-1)$-point and $(2n)$-point p-ary interpolating schemes for $n = 2, 3, 4$ and 5. From these tables, we observe that continuity, in general, do not increase by increasing the complexity and arity of the schemes.

Table 1: The ranges of parameter for continuity of $(2n-1)$-point ternary scheme for $n = 2, 3, 4$ and 5.

Scheme	Parameters	Continuity
3-point ternary	$\frac{2}{9} < \omega_1 < \frac{1}{3}$	C^1
5-point ternary	$-\frac{5}{108} < \omega_1 < -\frac{7}{162}$	C^2
7-point ternary	$\frac{49}{5832} < \omega_1 < \frac{43}{4617}$	C^2
9-point ternary	$-\frac{116}{57591} < \omega_1 < -\frac{125}{1164}$	C^2

Table 2: The ranges of parametric continuity of $(2n)$-point ternary scheme for $n = 2, 3, 4$ and 5.

Scheme	Parameters	Continuity
4-point ternary	$-\frac{2}{27} < \omega_1 < -\frac{1}{15}$	C^2
6-point ternary	$\frac{14}{1215} < \omega_1 < \frac{13}{972}$	C^2
8-point ternary	$-\frac{287}{104976} < \omega_1 < -\frac{149}{65610}$	C^2
10-point ternary	$\frac{1121}{2361960} < \omega_1 < \frac{2231}{3779136}$	C^2

Table 3: The ranges of parameter for continuity of $(2n)$-point quaternary scheme for $n = 2, 3, 4$ and 5.

Scheme	Parameters	Continuity
4-point quaternary	$-\frac{5}{64} < \omega_1 < -\frac{1}{16}$, $\gamma = -\frac{1}{16}$	C^2
6-point quaternary	$\frac{43}{4096} < \omega_1 < \frac{59}{4096}$, $\gamma = \frac{3}{256}$	C^2
8-point quaternary	$-\frac{395}{131072} < \omega_1 < -\frac{267}{131072}$, $\gamma = -\frac{5}{2048}$	C^2
10-point quaternary	$\frac{7059}{16777216} < \omega_1 < \frac{11155}{16777216}$, $\gamma = \frac{35}{65536}$	C^2

In Fig. 1(a), the effect of parameters of 3-point ternary interpolating scheme on limit curve is shown. This figure is exposed to show the role of parameter when 3-point ternary interpolating scheme is applied on discrete data points. From this figure, we see that the behavior of the limiting curve acts as tightness/looseness when the value of parameter vary. There is very slight difference between error bounds of 4-point

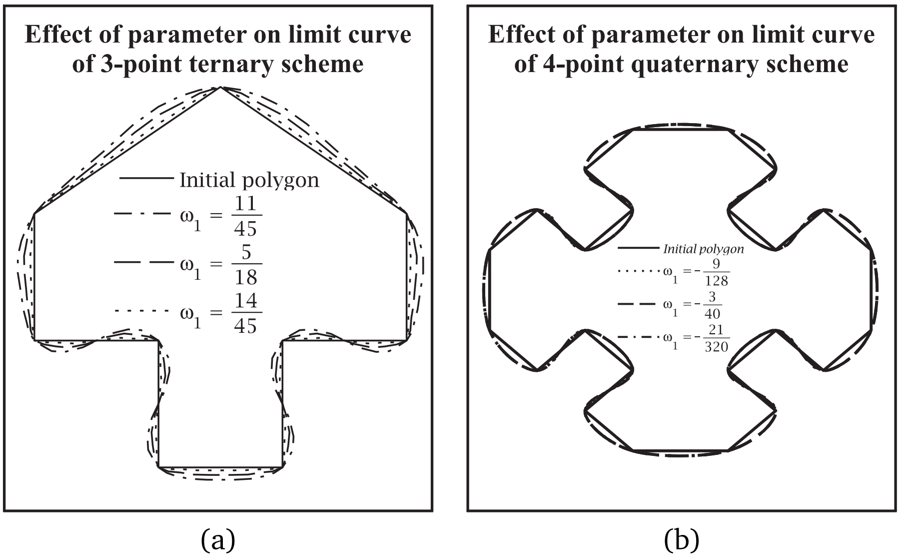
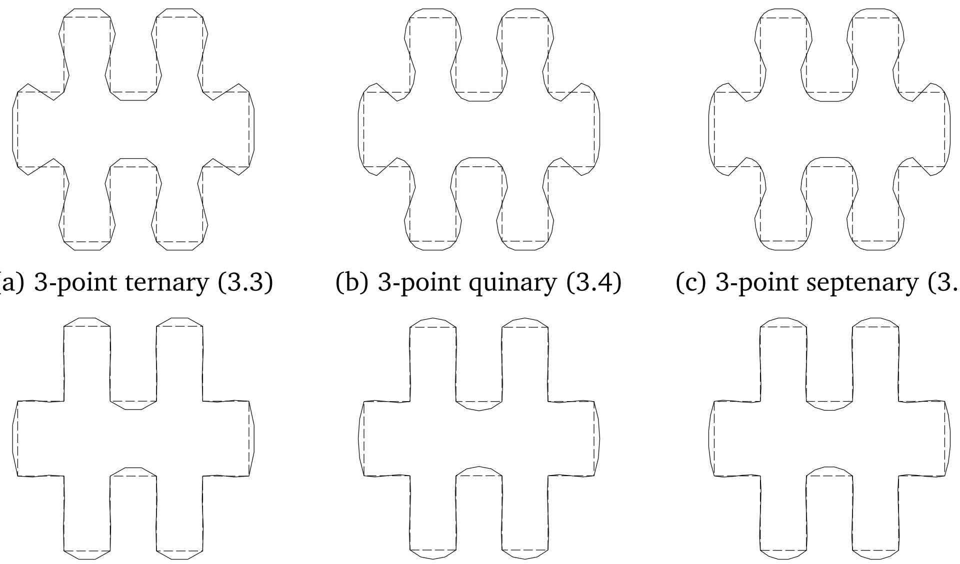


Figure 1: (a) Presents comparison among limiting curves generated by 3-point ternary scheme (3.3) and initial polygon, (b) Presents comparison among limiting curves generated by 4-point quaternary scheme (4.5) and initial polygon.

quaternary interpolating scheme at different values of parameter. So small effect of parameter on limiting curve generated by 4-point quaternary interpolating scheme is observed i.e. the limiting curve overlap for different values of parameter as shown in Fig. 1(b).

5.2. Applications

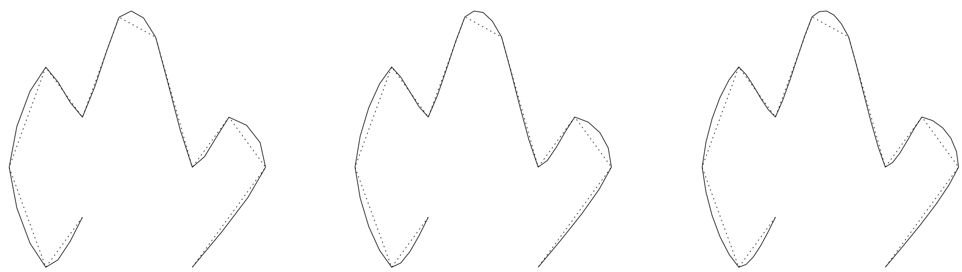
Here we give comparison among different schemes (w.r.t. arity and complexity) with the same set of initial control polygons to show their visual performances. We consider both close and open polygons cases to give comparison of visual behaviour of proposed schemes. In Fig. 2, initial close polygons are taken and represented by dashed lines while the solid curves are obtained by proposed schemes at first subdivision level. In Fig. 3, initial open polygons are taken and represented by dashed lines while the solid curves are obtained by proposed schemes at first subdivision level. From theses figures, we conclude that the higher arity schemes need less subdivision levels/iterations to produce smoother curves and converge to limit curve faster as compared to the lower arity schemes. The main purpose to give comparison at first level is to provide the clear visual differences among the refined polygons produced by different schemes. Fig. 6 shows the visual performance of 3-point tensor product ternary interpolating scheme with parametric value $w_1 = \frac{2}{9}$. In this figure 6(a), 6(e) and 6(i) show the initial polygons whereas 6(b), 6(f) and 6(j) are obtained at first iteration level, 6(c), 6(g) and 6(k) are obtained at second iteration level and 6(d), 6(h) and 6(l) show the smooth shading results produced by 3-point tensor product ternary interpolating scheme.



(a) 3-point ternary (3.3) (b) 3-point quinary (3.4) (c) 3-point septenary (3.5)

(d) 4-point ternary (4.15) (e) 4-point quaternary (4.5) (f) 4-point quinary (4.17)

Figure 2: Comparison among different subdivision schemes with same set of initial close control polygons: Dashed lines represent initial close control polygons while solid curves are generated by 3-point ternary (3.3), 3-point quinary (3.4), 3-point septenary (3.5), 4-point ternary (4.15), 4-point quaternary (4.5) and 4-point quinary (4.17) subdivision schemes at first subdivision level.



(a) 4-point ternary (4.15) (b) 4-point quaternary (4.5) (c) 4-point quinary (4.17)

Figure 3: Comparison among different subdivision schemes with same set of initial open control polygons: Dashed lines represent initial open control polygons while solid curves are generated by 4-point ternary (4.15), 4-point quaternary (4.5) and 4-point quinary (4.17) subdivision schemes at first subdivision level.

5.3. Error analysis

We have computed error bounds between limit curve and their control polygon after k-fold subdivision of $(2n-1)$-point and $(2n)$-point p-ary interpolating schemes for different values of n and p by using [15] (Eq. (18), with $\chi = 0.1$). The effect of parameters on error bounds between k-th level control polygon (i.e. $k = 1, 2, \cdots, 7$) and limit curves are shown graphically in Fig. 4. It is clear from Fig. 4 that as we increase value of parameter from left to right in the specified range (given in Tables 1-3) of paramet-

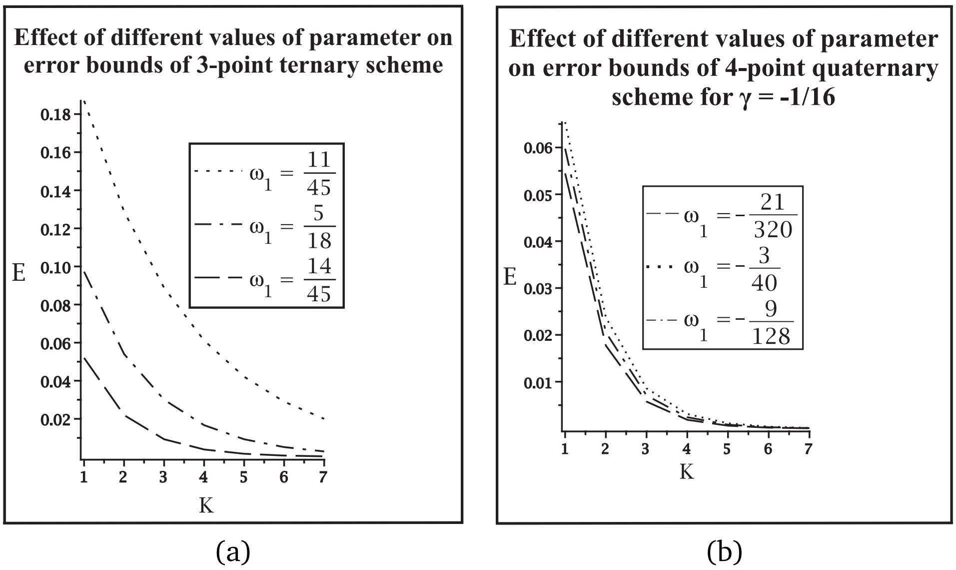


Figure 4: Significant effects of parameters on error bounds. K presents level of scheme and E presents the error bound.

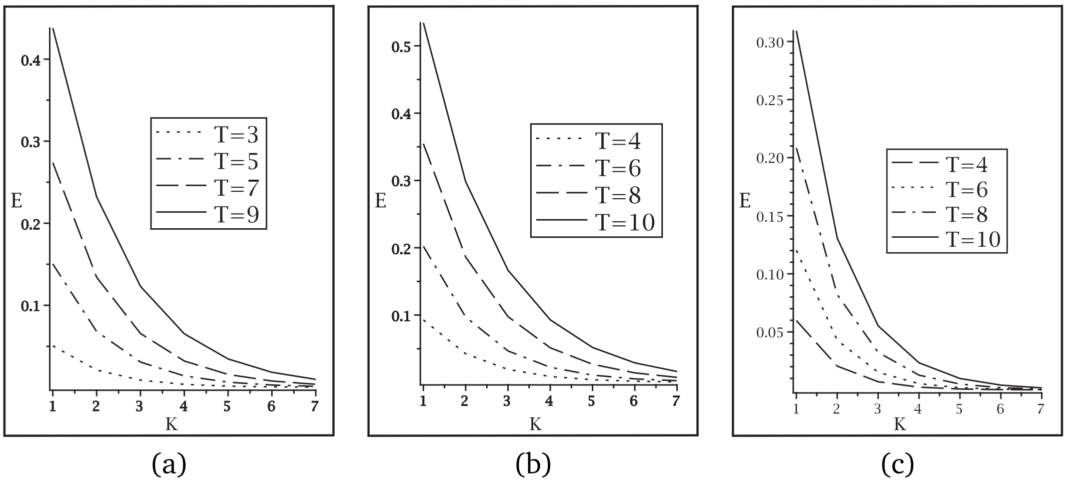


Figure 5: (a) Presents comparison among error bounds of odd-point ternary interpolating scheme, i.e., for T$= 3, 5, 7, 9$, (b) Presents comparison among error bounds of even-point ternary interpolating scheme, i.e., for T$= 4, 6, 8, 10$, (c) Presents comparison among error bounds of even-point quaternary interpolating scheme, i.e., for T$= 4, 6, 8, 10$. Here K presents subdivision level, E presents error bound and T presents complexity of subdivision scheme.

ric continuity of the scheme the error bounds decrease. Similar results can be obtained for $(2n-1)$-point and $2n$-point interpolating scheme for $n \geq 3$. In Fig. 5 graphical representation of error bounds of odd-point ternary, even-point ternary and even-point quaternary schemes is shown. We take mid-points of the parametric intervals given in

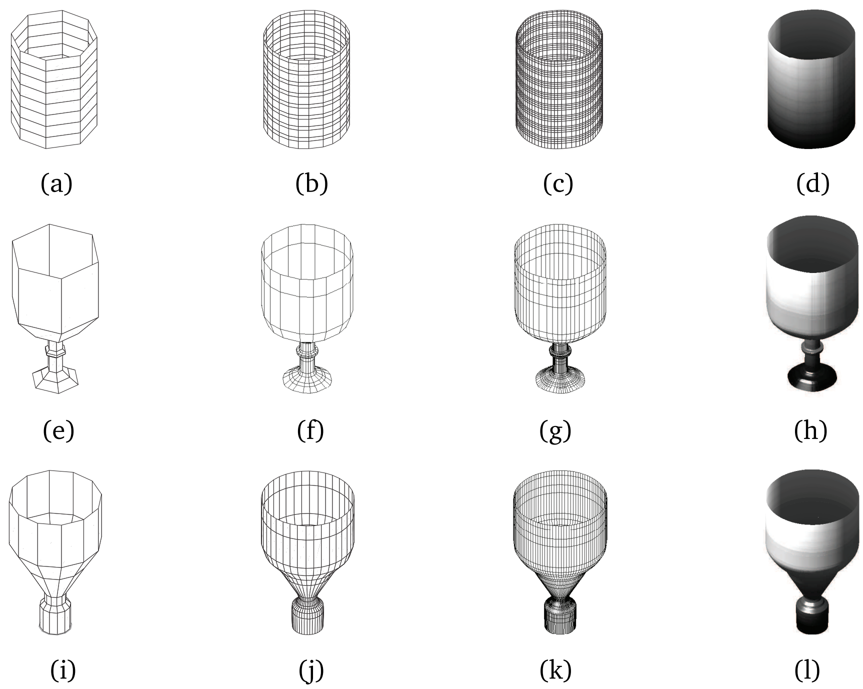


Figure 6: (a), (e) and (i) are initial polygons while (b), (f) and (j) are obtained at first subdivision level, (c), (g) and (k) are obtained at second subdivision level and (d), (h) and (l) are final shaded smooth results of 3-point tensor product ternary scheme.

Tables 1-3 for the continuity of schemes and then calculate error bounds at different subdivision levels. From Fig. 5 and in general, we have the following conclusions: Error bounds decrease with the increase of subdivision levels. Error bounds are directly proportional to the complexity of the schemes and decrease with the increase of arity of the schemes.

6. Conclusion

In this article, we offered $(2n)$-point and $(2n-1)$-point p-ary interpolating scheme for any integers $n \geq 2$ and $p \geq 3$. Moreover, 3-point and 4-point ternary interpolating scheme of Hassan et al. [5, 6], Jian-ao Lian's 3-point, 5-point, 4-point, 6-point, $(2m)$-point and $(2m+1)$-point a-ary interpolating schemes [8–10], $(2n-1)$-point ternary interpolating schemes of Zheng et al. [19], $(2n-1)$-point ternary interpolating scheme of Aslam et al. [1] and odd points n-ary interpolating scheme of Mustafa et al. [16] are also special cases of our family of scheme. We also have presented tensor product version of the proposed generalized and unified family of interpolating schemes. Furthermore, we have concluded that error bounds between limit curve and control polygon of subdivision scheme at k-th level decrease with the increase of arity of the scheme. We also observed that error bound is directly proportional to the complexity of the schemes. In general, we determine that continuity of interpolating schemes do not increase by increasing the complexity and arity of the schemes.

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