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Use of Shifted Laplacian Operators for Solving Indefinite Helmholtz Equations

Ira Livshits[1]

[1] *Department of Mathematical Sciences, Ball State University, Muncie IN, 47306, USA.*

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Abstract. A shifted Laplacian operator is obtained from the Helmholtz operator by adding a complex damping. It serves as a basic tool in the most successful multigrid approach for solving highly indefinite Helmholtz equations — a Shifted Laplacian preconditioner for Krylov-type methods. Such preconditioning significantly accelerates Krylov iterations, much more so than the multigrid based on original Helmholtz equations. In this paper, we compare approximation and relaxation properties of the Helmholtz operator with and without the complex shift, and, based on our observations, propose a new hybrid approach that combines the two. Our analytical conclusions are supported by two-dimensional numerical results.

AMS subject classifications: 65F10, 65N22, 65N55

Key words: Indefinite Helmholtz operator, multigrid, shifted Laplacian, ray correction.

1. Introduction

Considered here is a two-dimensional Helmholtz equation

$$Lu = \Delta u(x) + k^2(x)u(x) = f(x), \quad x \in \Omega \subset \mathbf{R}^2, \tag{1.1}$$

accompanied by the first-order Sommerfeld boundary conditions

$$\frac{\partial u(x)}{\partial n} - iku(x) = 0, \quad x \in \partial\Omega. \tag{1.2}$$

Use of standard discretization methods, considered on a sufficiently fine scale h, yields a system of linear equations

$$L^h u^h = f^h, \tag{1.3}$$

where $L^h \in \mathbf{C}^{n\times n}$ is a sparse matrix, with n, the number of discrete degrees of freedom, typically large.

Iterative methodologies applied to (1.3) include multigrid (MG) approach which is the focus of this paper. Multigrid is well known for its ability to deliver accurate solutions at optimal computational costs. The indefinite Helmholtz (HLM) operators, however, present many challenges for MG development, some of which are discussed later on and many are presented, for instance, in [1, 9]. Multigrid approaches for (1.1) notably include [2, 3, 5, 10, 11, 13, 14, 17]. The most practical multigrid method to date is the Shifted Laplacian (SL) approach suggested in [5, 6] and further developed in [4, 7, 8, 15, 16] among others. The SL employs differential operators $M = L + ik^2\beta I$ which are obtained from the Helmholtz operator by adding a complex shift. The positive constant β is typically chosen to be one half. Discretization on scale h yields

$$M^h = L^h + ik^2\beta I^h. \tag{1.4}$$

Both discrete operators L^H and M^H, $H = h, 2h, \cdots$ are computed using second-order central differences, resulting in five-point stencils.

The M^h-based multigrid preconditioners provide a significant acceleration of Krylov iterations applied for solving (1.3). This is due to the fact that all eigenvalues of the preconditioned operator, $(M^h)^{-1}L^h$ are located at the right half of the complex plane, i..e, for each such eigenvalue, its real part is positive. The goal of this paper is to investigate the advantages and shortcomings of employing the shifted Laplacian and the Helmholtz operators on different scales, $H = h, 2h, \cdots$, and to see if their combined use results in an improved preconditioning.

We compare two sets of operators $\{L_H\}_{H=h,2h,\cdots}$ and $\{M_H\}_{H=h,2h,\cdots}$ from two perspectives:

- Approximate qualities of coarse operators;
- The *smoothing* and *convergence* rates of relaxation schemes, especially for the near-kernel and physically smooth error components, the latter being comprised of eigenvectors of L^h with negative eigenvalues.

The remainder of the paper is organized as follows. The near-kernel components (nkc) of (1.1) are presented in Section 2, and the Shifted Laplacian preconditioning on different scales is discussed in Section 3. The accuracy, Section 4, and the relaxation, Section 5, properties of the Helmholtz and the Shifted Laplacian operators are compared, and a multigrid preconditioner based on a combination of the two is proposed, Section 6. Numerical experiments are presented in Section 7, and the concluding remarks are given in Section 8.

2. Error components and the Helmholtz operator

An efficient multigrid algorithm works in the following way: Each coarse operator A^H, $H = 2h, 4h, \cdots$ approximates the finest operator A^h for all components unreduced by the processing on preceding finer grids. A finer error $e^{H/2}$ with a large relative

residual, $\|r^{H/2}\| = \|A^{H/2}e^{H/2}\| \gg \|e^{H/2}\|$, is practically annihilated by a few relaxation sweeps applied to the residual equation $A^{H/2}e^{H/2} = r^{H/2}$. The remaining error, with a small relative residual, is accurately approximated on scale H, and so forth. This means in particular that error components with the smallest relative residuals, i.e., the near-kernel error components of A^h,

$$A^h e^h \approx 0, \tag{2.1}$$

have to be approximated on all, including the coarsest, scales. This works naturally when (2.1) are smooth, which is not the case for the Helmholtz operators with large wave numbers, for which the nkc are given by

$$e(x,y) = e^{i(\omega_1 x + \omega_2 y)}, \quad |\omega| = \sqrt{\omega_1^2 + \omega_2^2} \approx k. \tag{2.2}$$

Starting with some scale H, (2.2) are too oscillatory to be accurately represented by coarse scale operators.

3. Shifted Laplacian preconditioner

One of the biggest shortcomings of the indefinite Helmholtz operator from the computational point of view is an abundance of eigenfunctions with negative eigenvalues, which, when erroneous, are not reduced or even amplified by standard linear iterative schemes. Preconditioning of L^h by M^h eliminates this particular issue. It is well established, e.g., [5], that on the finest scale h all eigenvalues of operator $(M^h)^{-1}L^h$ (and $L^h(M^h)^{-1}$) are located in the right half of the complex plane. This quality of the SL operators is not unique to small h. In fact, as Fig. 1 shows, for all scales $H = h, 2h, \cdots$, all eigenvalues of the HLM operator preconditioned by the SL operator have positive real parts, i.e., preconditioning of L^H by M^H will improve performance of the Krylov-based iterations for error components visible and updated on an arbitrary scale H. This is important because in the iterative multigrid cycle (versus the direct inversion of M^h on the finest grid), the preconditioning of L^h by the Shifted Laplacian operator for each error component is actually determined by the approximation of $(M^H)^{-1}$, where H is the scale where the error component in question is updated. For most error component L^H approximates L^h, and thus the action of $(M^H)^{-1}L^H$ plays the same role of the one of $(M^h)^{-1}L^h$ in one-grid solvers.

4. Approximation properties of the Helmholtz and the shifted Laplacian coarse operators

An approximation accuracy of a fine operator by a coarse operator is often measured by comparing *symbols* of the two for Fourier components visible on the coarser scale. Generally, a symbol of a linear operator A applied to $e^{i(\omega_1 x + \omega_2 y)}$ is defined as a complex coefficient $\tilde{A}(\omega_1, \omega_2)$:

$$Ae^{i(\omega_1 x + \omega_2 y)} = \tilde{A}(\omega_1, \omega_2)e^{i(\omega_1 x + \omega_2 y)}. \tag{4.1}$$

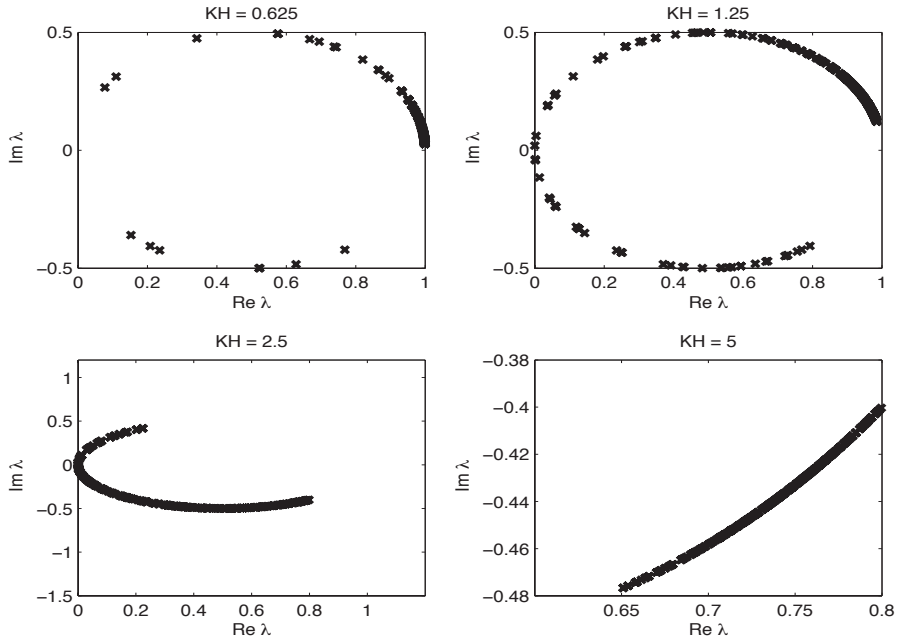

Figure 1: The eigenvalues of $(M^H)^{-1}L^H$ for different values of kH; the finest scale satisfies $kH = 0.625$, the coarsest – $kH = 5$.

For a coarse correction, produced either by L^H or by M^H, to provide an adequate approximation to solution of (1.3), symbol ratios

$$\tau_H^{HLM}(\omega_1,\omega_2) = \frac{\tilde{L}^h(\omega_1,\omega_2)}{\tilde{L}^H(\omega_1,\omega_2)} = \frac{(2\cos\,\omega_1 h + 2\cos\,\omega_2 h - 4 + k^2h^2)\,H^2}{(2\cos\,\omega_1 H + 2\cos\,\omega_2 H - 4 + k^2H^2)\,h^2} \tag{4.2}$$

and

$$\tau_H^{SL}(\omega_1,\omega_2) = \frac{\tilde{L}^h(\omega_1,\omega_2)}{\tilde{M}^H(\omega_1,\omega_2)} = \frac{(2\cos\,\omega_1 h + 2\cos\,\omega_2 h - 4 + k^2h^2)\,H^2}{(2\cos\,\omega_1 H + 2\cos\,\omega_2 H - 4 + k^2H^2(1+i\beta))\,h^2} \tag{4.3}$$

should be close to one. Fig. 2 illustrates how values of (4.2) and (4.3) change when considered for increasingly larger H; it shows results for error components that are visible on each scale H, $0 \le \omega_{1,2}H \le \pi$. In particular, it demonstrates how well the error components that are oscillatory on scale H, with $\pi/2 \le \omega_{max}H \le \pi$, $\omega_{max} = \max\{|\omega_1|,|\omega_2|\}$, are approximated.

As Fig. 2 suggests, operators L^H and M^H have similar approximation accuracy for high-frequency components visible on the finest grids, both do not approximate the nkc components (seen as quarter of a ring in all figures, note that the presented range $[0,\pi]^2]$ is a quarter of the actual range $[-\pi,\pi]^2$) and perform differently for the smoothest components on intermediate and coarse scales. Symbols (4.2)-(4.3) have the following properties:

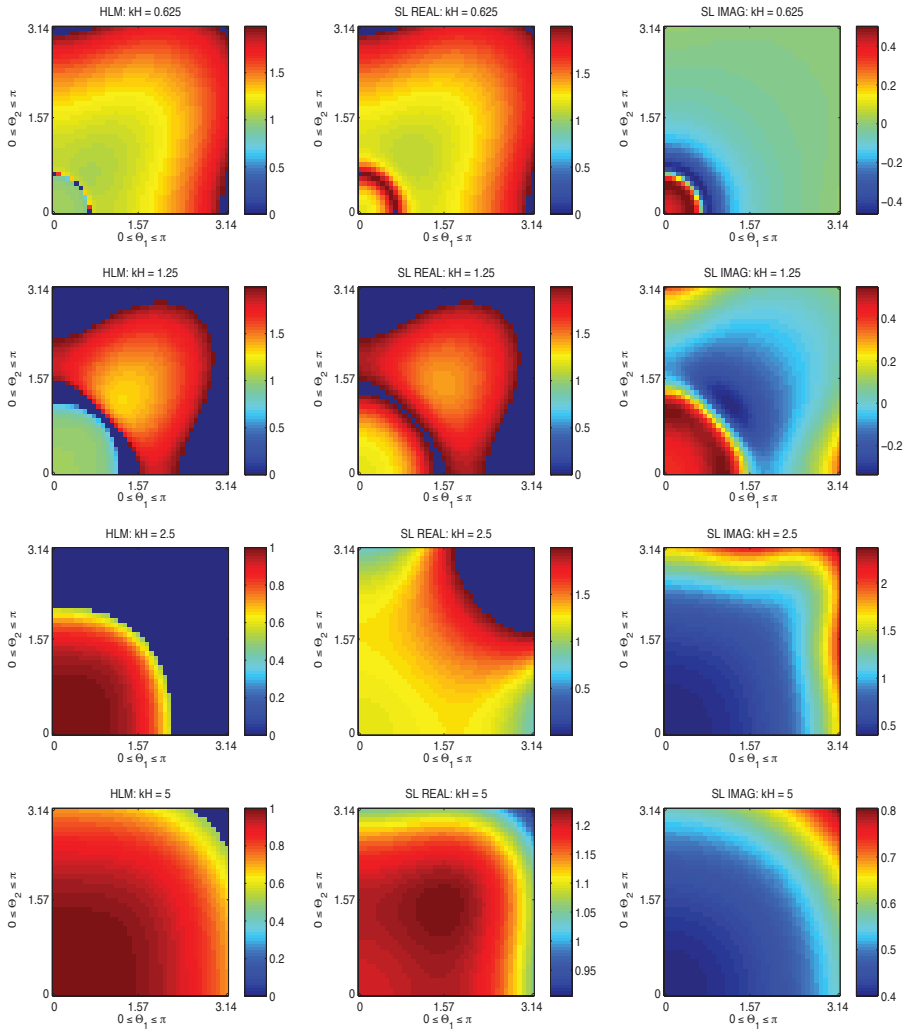

Figure 2: Symbol ratios for Fourier components with frequencies $\theta_1 = \omega_1 H, \theta_2 = \omega_2 H$ visible on scale H; the left column shows τ_H^{HLM}, the second and the third — the real and the imaginary parts of τ_H^{SL}; the first and the second columns are filtered to emphasize the regions with $0.5 \le \tau \le 2$, otherwise τ is set to zero, which is shown in dark blue. The imaginary part remains unchanged.

- For $kH \le 0.625$: all components with $\pi/2 < \omega_{max} H \le \pi$ are well approximated by L^H and M^H: $\mathrm{Re}(\tau_H^{SL}) \approx \tau_H^{HLM}$ are close to one, $\mathrm{Im}(\tau_H^{SL}) \le .1$.

- For $kH = 1.25$: all components with $\pi/2 < \omega_{max} H \le \pi$ are accurately approximated by L^H, they satisfy $(1+\alpha_1)k \le |\omega| \le 2(1+\alpha_1)k$, $\alpha_1 \approx .4$. The accuracy deteriorates as $|\omega|$ approaches k. M^H provides an accurate approximation for a smaller range of components, the ones with $(1+\beta_1) \le |\omega| \le 2(1+\alpha_1)k$, $\beta_1 \approx .8$. The growing imaginary part of τ_H^{SL} for smaller $|\omega|$ affects the approximation quality.

- For $kH = 2.5$: L^H provides an accurate approximation for $|\omega| \leq (1-\alpha_0)k$, with $\alpha_0 \approx .1$, and it does not approximate components with $|\omega| \geq k$. Shifted Laplacian operator M^H gives a rise to a wrong approximation for all components in question.

- For $kH \geq 5$: all components visible on scale H, $0 \leq |\omega| \leq 0.65k$, have an accurate approximation by L^H, but not by M^H (due to a large imaginary part).

To summarize, a sequence of coarse-grid Helmholtz operators $\{L^H\}_{H>h}$ accurately approximates the finest grid Helmholtz operator L^h for all Fourier components except (2.2) with ω satisfying $(1-\alpha_0)k \leq |\omega| \leq (1+\alpha_1)k$, where $\alpha_0 \approx .1$ and $\alpha_1 \approx .4$. Coarse-grid Shifted Laplacian operators $\{M^H\}_{H>h}$ approximate the finest-grid Helmholtz operator L^h for all oscillatory components, failing to approximate both (2.2) and (unlike L^H) smooth errors, i.e., for all ω with $0 \leq |\omega| \leq (1+\beta_1)k$, where $\beta_1 \approx .8$.

5. Relaxation for L^H and M^H

Next we discuss how different relaxation schemes perform when applied to the HLM and the SL operators on increasingly coarser scales H.

First iterative scheme considered here is the lexicographic Gauss-Seidel (lex-GS). When applied to L^H and M^H, one iteration changes an amplitude of an erroneous Fourier component $e^{i(\omega_1 x+\omega_2 y)}$ by a factor of $\mu(\theta_1,\theta_2)$, $\theta_1 = \omega_1 H$ and $\theta_2 = \omega_2 H$,

$$\mu(\theta_1,\theta_2) = \left| \frac{\exp(i\theta_1)+\exp(i\theta_2)}{\exp(-i\theta_1)+\exp(-i\theta_2)-c} \right|, \tag{5.1}$$

where $c = 4-k^2H^2$ and $c = 4-k^2(1+i\beta)H^2$ for the Helmholtz and the Shifted Laplacian operators, respectively. Typically, in predicting a convergence rate of a multigrid solver, the *smoothing* properties of the relaxation is the main parameter. It is measured by a *smoothing* rate, i.e., reduction of the oscillatory on the current scale error components, defined as

$$\overline{\mu} = \max_{\pi/2 \leq \max\{|\theta_1|,|\theta_2|\} \leq \pi} \mu(\theta_1,\theta_2). \tag{5.2}$$

For the Helmholtz operators, there is an additional phenomenon – a divergence of physically smooth error components corresponding to the eigenfunctions with negative eigenvalues. To monitor the divergence, an overall *convergence* rate is considered:

$$\underline{\mu} = \max_{0 \leq \max\{|\theta_1|,|\theta_2|\} \leq \pi} \mu(\theta_1,\theta_2). \tag{5.3}$$

Fig. 3 shows the values of (5.1) for L^H and M^H, obtained on increasingly coarser scale and for different frequencies. It confirms that

- For $kH = 0.625$, the smoothing rate are given by $\overline{\mu}^{HLM} \approx \overline{\mu}^{SL} \approx 0.7$, and the divergence is bounded by $\underline{\mu}^{HLM} \approx \underline{\mu}^{SL} \leq 1.1$; (for smaller values of kH, not depicted here, the smoothing rates are better and the divergence is very modest).

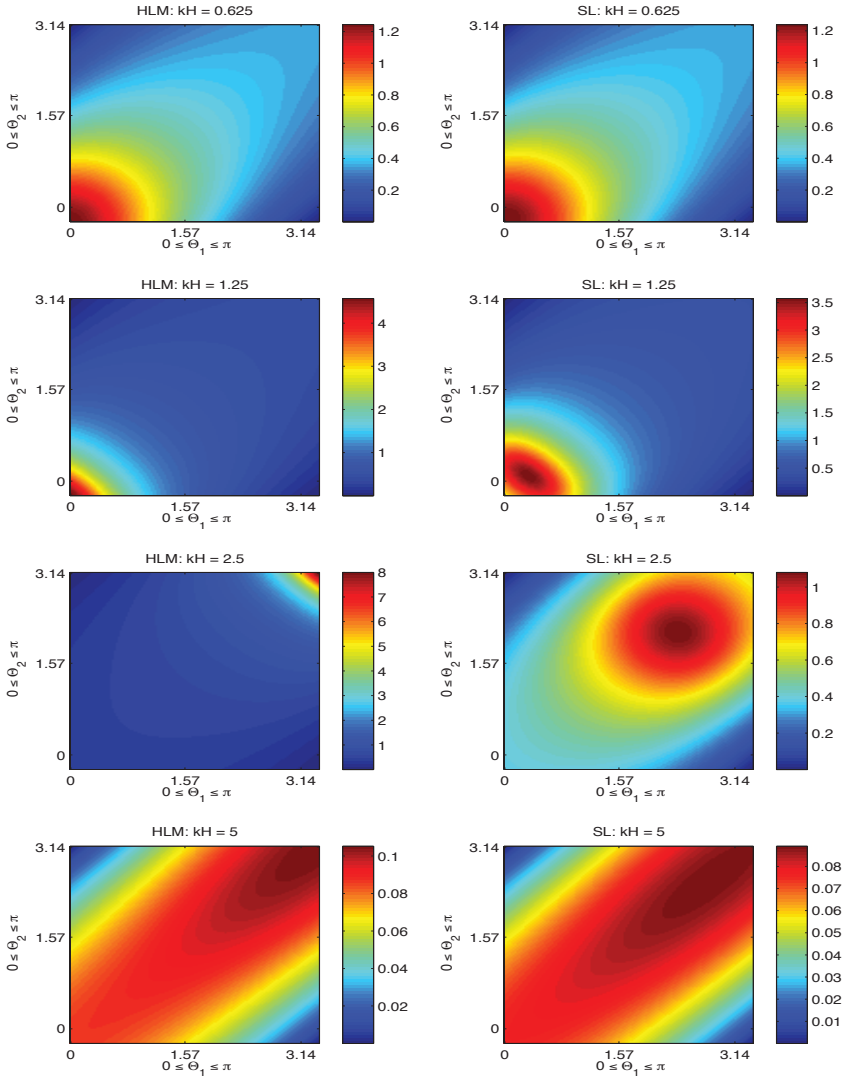

Figure 3: Reduction rates of the lex-GS applied to L^H (left column) and M^H (right column) on different scales H.

- For $kH = 1.25$, the divergence of smooth error components becomes large, with $\underline{\mu}^{HLM} \approx 4.5$ and $\underline{\mu}^{SL} \approx 3.5$; no error reduction for $|\omega| \approx k$. However, error components with $|\omega| \geq 1.3k$ for L^H and with $|\omega| \geq 1.8k$ for M^H are reduced by at least a factor of 0.7.

- For $kH = 2.5$, no convergence for physically oscillatory there near-kernel components, $\overline{\mu}^{HLM} \approx \overline{\mu}^{SL} \approx 1$. For smoother components, with $|\omega| \leq .9k$ for the HLM and ($|\omega| \leq .8k$) for the SL operators, the convergence rate $\mu(\omega_1 H, \omega_2 H)$ is below 0.7.

- For $kH = 5$, the overall convergence factors are given by $\underline{\mu}_H^{HLM} \approx .1$ and $\underline{\mu}_H^{SL} \approx .085$. making a couple of relaxation sweeps an equivalent to a direct solver. No coarser grids are needed.

Overall, the lexicographic Gauss-Seidel relaxation for both approaches performs well on scales with $kH \lessapprox 0.625$ and $kH \gtrapprox 2.5$. It fails to reduce near-kernel components (2.1) on any grid and causes a significant divergence of smooth error components when $kH \approx 1.25$.

Clearly, point-wise relaxations are usable on most scales. For the HLM operators on scales, $kH = 1.25$ and $kH = 2.5$, one needs a relaxation scheme that will yield a smaller divergence and better smoothing rates than traditional linear schemes. The same is true for the SL operators on scale $kH \approx 1.25$. Any converging (or modestly diverging) relaxation can be used on $kH \approx 2.5$, but only for the purposes of *preparing* error components to the Krylov iteration. Possible candidates for such schemes are the Kaczmarz relaxation scheme [12], or, in case of parallel processing, inspired by it, the point-wise Gauss-Seidel applied to normal operators $(L^H)^*(L^H)$ and $(M^H)^*M^H$, respectively. These relaxation schemes work, but they are both slow (many iterations are needed) and with larger operation count, both due to inflated (indirectly and directly, respectively) stencils. A cost of such relaxation is 13/5 times higher than the one for the GS applied to five-point stencils of L^H and M^H, and typically four pre- and four post-relaxation sweeps are needed to avoid a slow down due to relaxation on these scales. This results in an equivalent of 112/5 of regular iterations sweeps. We note, however, that this is an equivalent of 112/20 additional iterations on the next finer scale $kh = 0.625$, and only $112/80 \approx 1.4$ iterations on the target (in our experiments) scale $kh = 0.3125$.

Another, more economical, option is to apply the Gauss-Seidel relaxation to operators $(B^H)^*L^H$ (or $(B^H)^*M^H$), where B^H is an operator with a smaller stencil. Our choice for B^H is a discretization of one of the one-dimensional Shifted Laplacian operators

$$B_x = \partial_{xx} + k^2(1 + i\beta_0) \quad \text{and} \quad B_y = \partial_{yy} + k^2(1 + i\beta_0), \tag{5.4}$$

with $\beta_0 = 1$. Unlike M^H, the choice of larger complex dumping does not affect the approximation accuracy, and it does improve the convergence rates. The stencil's size of $(B^H)^*L^H$ and $(B^H)^*M^H$ is eleven, i.e., each iteration costs only 2.2 times more than a standard Gauss-Seidel applied to a five-point stencil. (It is only a small cost gain compared to the thirteen point stencil if the $B^H = L^H$, but it becomes more pronounced in three dimensions.) The reduction rate $\mu(\theta_1, \theta_2)$ of the new operators is given by

$$\mu(\theta_1, \theta_2) = \left| \frac{(2\cos\theta_2 + \overline{b} + c)\exp(i\theta_1) + \exp(2i\theta_1) + \overline{b}\exp(i\theta_2)}{(2\cos\theta_2 + \overline{b} + c)\exp(-i\theta_1) + \exp(-2i\theta_1) + \overline{b}\exp(-i\theta_2) + 2 + \overline{b}\,c} \right|, \tag{5.5}$$

where the values of c are the same as in (5.1) and $\overline{b} = -2 + k^2H^2(1 - i\beta_0)$. The values of (5.5) are given in Fig. 4 for $B_x^H L^H$ (left column) and $B_x^H M^H$ (right column). In

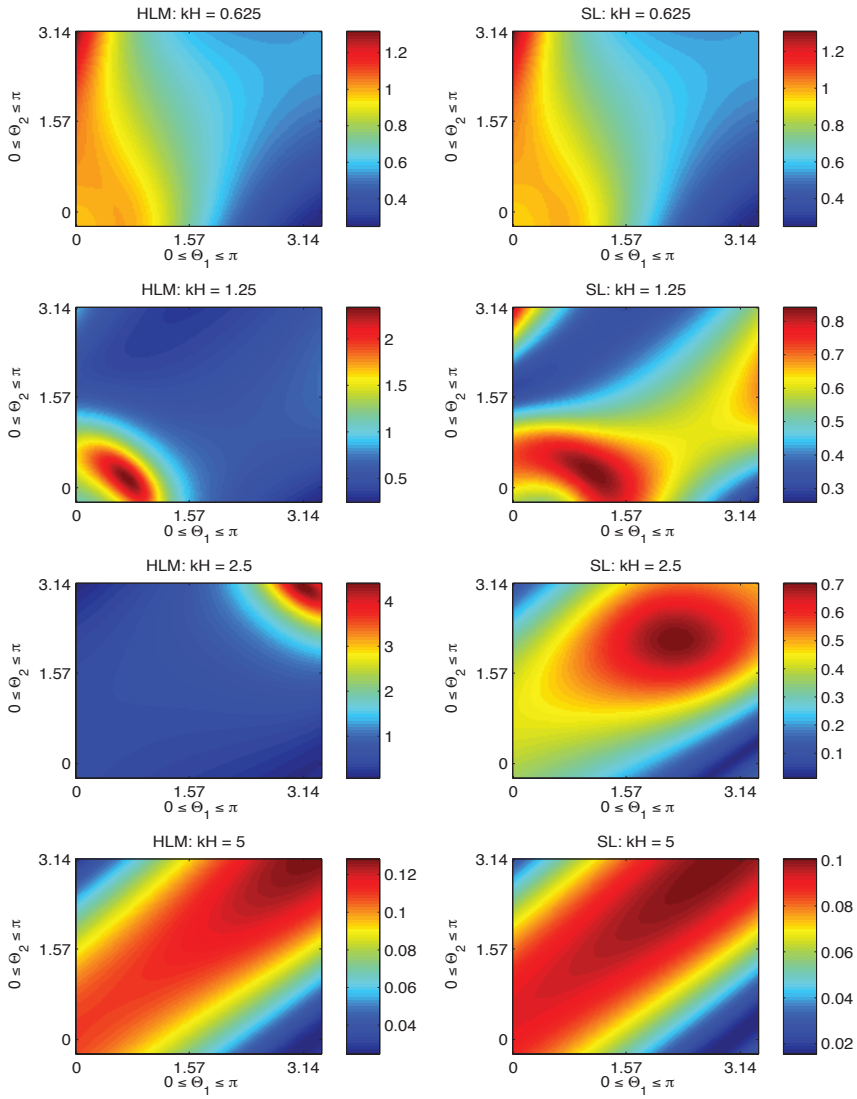

Figure 4: Reduction rates of the relaxation for $B_x^H L^H$ (left column) and $B_x^H M^H$ (right column), where B_x^H was obtained by discretizing operator B_x, (5.4).

our numerical experiments, this relaxation scheme is used on scales $kH = 1.25$ and $kH = 2.5$ for operators $B^H M^H$. Three pre- and two post relaxation sweeps is used for $kH \approx 1.25$. This is the cost of eleven Gauss-Seidel sweeps applied to L^H or M^H, equivalent of $11/4$ relaxation sweeps on the next finer scale and $11/16$ sweeps on the target $kH = 0.3125$. One pre- and one post iteration is used on scale $kH = 2.5$. As the Fig. 4 shows, the relaxation is convergent on both scales. The rates are not symmetric for θ_1 and θ_2. Therefore, the algorithms presented in the next Section alternate between B_x^H and B_y^H, the discretization of B_y.

6. Hybrid algorithm

We consider multigrid V-cycles constructed using the following elements:

- Operators L^H and/or M^H, results of the second-order finite-difference discretization of differential operators L and/or M, each with five-point stencils;
- A bilinear interpolation;
- A full weighting restriction;
- Gauss-Seidel iterations:
  - one pre- and post-smoothing steps on all scales except $kH \approx 1.25$ and $kH \approx 2.5$, applied either to L^H or to M^H;
  - three pre- and two post-smoothing steps on scale $kH \approx 1.25$, applied either to $(B^H)^T L^H$ or to $(B^H)^T M^H$;
  - one pre- and one post-smoothing step on scale $kH \approx 2.5$, applied either to $(B^H)^T L^H$ or to $(B^H)^T M^H$.

  The relaxation strategy depends on kH and not on k, i.e., the number of relaxation sweeps and their relative (to the fine-grid problem size) cost do not increase as k increases. Moreover, as discussed in Section 8, the relative cost becomes smaller when the target problem is considered on finer scale, with smaller kh.

On each scale, a coarse-grid operator is used in two capacities: (A) for relaxation and (B) for computing coarse grid residuals. Based on the analysis reported in Sections 2-5, in addition to the SL-based V-cycle preconditioner, we consider a new hybrid V-cycle (HYB-V) that

- Employs operators L^H, $H = h, 2h, \cdots$ to compute coarse residuals;
- Employs Gauss-Seidel relaxation of residual equations $L^H e^H = r^H$ on all scales except $kH \approx 1.25$ and $kH \approx 2.5$;
- Employs Gauss-Seidel relaxation of residual equations $(B^H)^* L^H e^H = (B^H)^* r^H$ on scales $kH \approx 1.25$ and $kH \approx 2.5$, where B^H alternates between B_x^H and B_y^H.

The motivation for the hybrid method comes from the observations reported in Sections 4 and 5 regarding performance of the SL and the HLM operators on intermediate and coarse scales. (On finer grids both act very similarly, and either one can be used.) The strength of the Shifted Laplacian approach, studied in detail in [5, 6], is the transformation rather than reduction of the near-kernel error components that mostly occurs on intermediate scales. This is the reason for employing M^H in relaxation there.

On coarse grids, however, Helmholtz operators L^H, give rise to an accurate approximation of smooth components, and, together with a fast convergence by Gauss-Seidel, allow for an efficient coarse-grid correction. Therefore, L^H is used in relaxation on the coarsest scale(s).

7. Numerical experiments and computational cost

In all numerical experiments, the computational domain is a unit square $[0,1]^2$, the right-hand-size f^h is homogeneous except at one point. The intergrid transfers used in the V-cycles are a linear interpolation and a full weighting. As Krylov iterations, the Bi-CGSTAB algorithm is used, as it showed to be more efficient than the GMRES algorithm. We note that each BI-CGSTAB iteration employs two multigrid cycles.

First, the algorithms are tested for (1.1) with a constant k, considered on $\Omega = [0,1]^2$, and the results are presented in Table 1. Initial approximations x^0 are zero in all experiments; iterations are performed until the initial residual $\|r^0\| = \|f\|$ is reduced by a factor of 10^7. In Tables 1-2, right-hand-sides are homogeneous except at the center of Ω, where $f(.5,.5) = 1$.

Clearly, the relaxation regiment on scales $kH \approx 1.25, 2.5$ needs fine-tuning, both with the type of relaxation (e.g., different types of B^H, in particular the size of complex shift, β_0) are to be optimized, as well as the number of relaxation sweeps. This is a subject of future work.

In Table 2, performance of the SL-V and the HYB-V methods is tested when considered on increasingly finer h; both show an improved convergence when computing increasingly accurate solutions.

Next, the algorithms are tested for (1.1) with heterogeneous medium — a wedge problem shown in Fig. 5; numerical experiments presented in Table 3. Again, the hybrid preconditioner performs better than the algorithm based strictly on the shifted Laplacian operator.

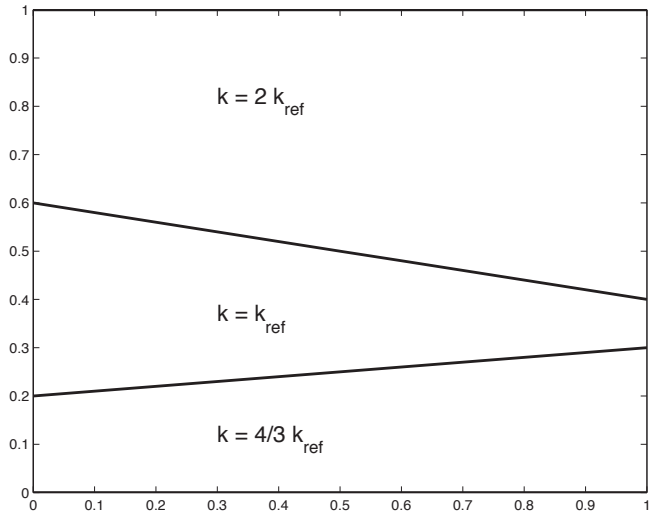

Figure 5: Wave number distribution in the wedge problem The point source is located at the middle of the upper boundary: $f(0.5,1.) = 1$, and it is zero elsewhere.

Table 1: The number of the Bi-CGSTAB iterations for the SL-V and the HYB-V preconditioners for different values of constant wave numbers; in all experiments $kh = 0.3125$. Given in parenthesis is the number of the Bi-CGSSTAB if the number of post-iterations on scale $kH \approx 1.25$ is increases from two to three, and on $kH \approx 2.5$ — from one to two.

k	40	50	80	100	160
h	$1/64$	$1/80$	$1/128$	$1/160$	$1/240$
SL-V	21	24	35.5	45	68.5 (67.5)
HYB-V	14.5	17.5	27.5	34	55.5 (50)

Table 2: The number of the Bi-CGSTAB iterations for the SL-V and the HYB-V preconditioners; $k = 40$, kh ranges from 0.625 to 0.078125.

h	1/64	1/128	1/256	1/512
SL-V	22	21	20.5	19
HYB-V	18	15.5	14.5	13.5

Table 3: The number of Bi-CGSTAB iterations for the SL-V and the HYB-V preconditioners, applied to the Helmholtz equation with discontinuous $k(x, y)$ defined in Fig. 5; in all experiments $k_{ref}h \approx 0.2344$; the value of k in the table varies from 15 to 480.

k_{ref}	15	30	60	120	240
SL-V	12	20	41	77	121
HYB-V	9	14	23	44	72

Computational cost of the SL-V and the HYB-V preconditioners is close to the one of a standard multigrid $V(1, 1)$ cycle applied to any complex discrete operator with a five point stencil; the only difference is the cost of additional relaxation sweeps applied to operators with larger stencils on scales $kH = 1.25$ and $kH = 2.5$, which is discussed in Section 5. While the absolute cost of these iterations remains the same for a given (1.1), its relative fraction in the overall costs becomes smaller when (1.3) is discretized on finer scale h.

8. Conclusions and future work

Analysis of approximation and relaxation properties of the Helmholtz and the Shifted Laplacian operators showed that the use of one or another is beneficial depending on the considered scale. On the finer grids both L^H and M^H perform equally well; on the intermediate grids the SL operator allows a better preconditioning of the nkc for the Krylov iteration, and on the coarsest scale the Helmholtz operator gives a more accurate approximation for the smoothest error components. In addition, a new type of relaxation is proposed to replace linear relaxation scheme on the intermediate grids where the latter are strongly divergent. The hybrid multigrid algorithm HYB-V built on these observations show better convergence than the SL-V solver.

Our next step is to test these findings in the context of the original Shifted Laplacian algorithm [5], including its three dimensional implementation.

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