

A SURVEY OF OPEN CAVITY SCATTERING PROBLEMS*

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Abstract

This paper gives a brief survey of recent developments on mathematical modeling and analysis of the open cavity scattering problems, which arise in diverse scientific areas and have significant industrial and military applications. The scattering problems are studied for the two-dimensional Helmholtz equation corresponding to the transverse magnetic or electric polarization, and the three-dimensional time-harmonic and time-domain Maxwell equations. Since these problems are imposed in open domains, a key step of the analysis is to develop transparent boundary conditions and reformulate them equivalently into boundary value problems in bounded domains. The well-posedness of weak solutions are shown for the associated variational problems by using either the Lax–Milgram theorem or the Fredholm alternative.

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1. Introduction

The phenomenon of electromagnetic scattering by open cavities has received much attention by many researchers in the engineering and applied mathematics communities. An open cavity is a bounded domain embedded in the ground with its opening aligned with the ground surface. Given the cavity, the direct scattering problem is to determine how the wave is scattered by the cavity. The inverse problem is to answer what information can be extracted about the cavity from the measured wave field. The open cavity scattering problems arise in diverse scientific areas and have significant industrial and military applications. For instance, the radar cross section (RCS) measures the detectability of a target by a radar system. Deliberate control in the form of enhancement or reduction of the RCS of a target is highly important. The cavity RCS caused by jet engine inlet ducts or cavity-backed antennas can dominate the total RCS. A thorough understanding of the electromagnetic scattering characteristic of a target, particularly a cavity, is necessary for successful implementation of any desired control of its RCS.

The time-harmonic problems were introduced and studied firstly by engineers [22–25, 32, 34, 43]. Mathematical analysis of the problems were done in three fundamental papers [2–4], where transparent boundary conditions, based on the Fourier transform, were proposed on the opening. As the applied mathematics community has begun to work on these problems, there has been a rapid development of the theory, analysis, and computational techniques in this area. The mode matching method was developed to find analytic solutions for rectangular cavities [7, 14]. The analytic solutions provide a good understanding of the highly oscillatory

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nature of the large cavity problem, which is a perfect example for the long-standing high frequency scattering problems. Tremendous effort was made to develop various fast and accurate numerical methods to solve the large cavity problem [1, 11, 17, 18, 26, 27, 40, 41, 44, 45]. The challenging mathematical issue is to establish the stability estimates with explicit dependence on the high wavenumber [12, 13, 27], which help us gain a deeper understanding on high frequency problems. To include the analysis for more complex geometries, the overfilled cavity problems were investigated in [16, 31, 42], where the inhomogeneous medium filling the cavity interior was allowed to protrude into the space above the ground surface; the multiple cavity scattering problem was studied in [30], where the cavity was consisted of finitely many disjoint components.

The inverse cavity scattering problem is clearly challenging due to the nonlinearity and lack of stability, i.e., small variations in the data may give rise to large errors in the reconstructions. In the inverse problem community, it seems that more attention is paid on inverse medium, obstacle, or source scattering problems than on the inverse cavity scattering problem. Hence it is less studied. The results on uniqueness and local stability may be found in [6, 19, 28, 33]. Related optimal design problems can be found in [8–10], which was to design the shape of the cavity so as to minimize the RCS. We also refer to [5, 15] for the study on the electromagnetic field enhancement by interacting subwavelength cavities.

The time-domain electromagnetic scattering problems have attracted much attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity. Comparing with the time-harmonic problems, the time-domain problems are also less studied due to the additional challenge of the temporal dependence. The transient cavity scattering problems were examined in [20, 21, 35–39], where the focus was on temporal discretization and the analysis of the finite element method. A theoretical analysis can be found in [29] for the transient electromagnetic scattering from a three-dimensional open cavity.

The goal of this paper is to give a brief survey of recent developments on mathematical modeling and analysis of the open cavity scattering problems. Particular emphasis is on the formulation of the mathematical models, which include the two-dimensional Helmholtz equation corresponding to the transverse magnetic and electric polarizations and the three-dimensional time-harmonic and time-domain Maxwell equations. Since the problems are imposed in open domains, a key step of the analysis is to develop transparent boundary conditions and reformulate them equivalently into boundary value problems in bounded domains. The well-posedness of the weak solutions are presented for the associated variational problems by using either the Lax–Milgram theorem or the Fredholm alternative.

The paper is outlined as follows. In Section 2, the two-dimensional Helmholtz equations are introduced for the two fundamental polarizations. Section 3 and 4 are concerned with the three-dimensional time-harmonic and time-domain Maxwell equations, respectively. Topics are organized to present model problems, transparent boundary conditions, and well-posedness of weak solutions corresponding to each of these three sections. The paper is concluded with some general remarks and directions for future research in Section 5.

2. The Helmholtz Equation

We begin with a simpler model for the open cavity scattering problem and consider the two-dimensional Helmholtz equation by assuming that the structure is invariant along the z -axis.

2.1. Model problems

Let us first specify the problem geometry shown in Figure 2.1. Let $D \subset \mathbb{R}^2$ be the cross section of a z -invariant cavity with a Lipschitz continuous boundary $\partial D = S \cup \Gamma$. Here the cavity wall S is assumed to be a perfect electric conductor and the cavity opening Γ is aligned with the perfectly electrically conducting infinite ground surface Γ_g . The cavity is filled with some inhomogeneous medium, which may be characterized by the dielectric permittivity ε and the magnetic permeability μ . Let B_R^+ and Γ_R^+ be the half-disc and semi-circle above the ground surface with radius R . The exterior region $\Omega^e = \mathbb{R}_+^2 \setminus \bar{B}_R^+$ is filled with some homogeneous material with a constant permittivity ε_0 and a constant permeability μ_0 . Denote by $\Omega = B_R^+ \cup D$ the bounded domain in which our reduced boundary value problem will be formulated.

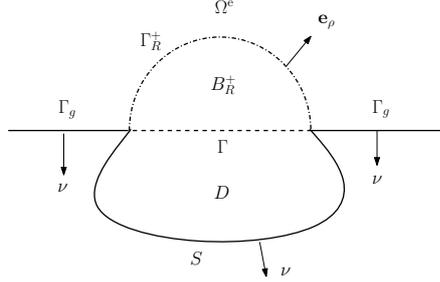


Fig. 2.1. A schematic diagram of the open cavity scattering problem geometry.

Since the structure is invariant in the z -axis, the problem can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). The three-dimensional Maxwell equations can be reduced to the two-dimensional Helmholtz equation. In the TM polarization, the magnetic field is transverse to the z -axis, the nonzero third component of the total electric field u satisfies

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } D \cup \mathbb{R}_+^2, \\ u = 0 & \text{on } S \cup \Gamma_g, \end{cases} \quad (2.1)$$

where $\kappa = \omega(\varepsilon\mu)^{1/2}$ is the wavenumber and ω is the angular frequency. In the TE polarization, the electric field is transverse to the z -axis, the nonzero third component of the total magnetic field u satisfies

$$\begin{cases} \nabla \cdot (\kappa^{-2} \nabla u) + u = 0 & \text{in } D \cup \mathbb{R}_+^2, \\ \partial_\nu u = 0 & \text{on } S \cup \Gamma_g, \end{cases} \quad (2.2)$$

where ν is the unit outward normal vector on $S \cup \Gamma_g$.

Let an incoming plane wave $u^{\text{inc}} = e^{i(\alpha x - \beta y)}$ be incident on the cavity from above, where $\alpha = \kappa_0 \sin \theta$, $\beta = \kappa_0 \cos \theta$, $\theta \in (-\pi/2, \pi/2)$ is the angle of incidence with respect to the positive y -axis, and $\kappa_0 = \omega(\varepsilon_0 \mu_0)^{1/2}$ is the free space wavenumber. Due to the perfectly electrically conducting ground surface, the reflected field in the TM polarization is $u^{\text{ref}} = -e^{i(\alpha x + \beta y)}$, while the reflected field in the TE polarization is $u^{\text{ref}} = e^{i(\alpha x + \beta y)}$. The total field can be split into the incident field, the reflected field, and the scattered field:

$$u = u^{\text{inc}} + u^{\text{ref}} + u^{\text{s}},$$

where the scattered field u^{s} is required to satisfy the Sommerfeld radiation condition:

$$\partial_\rho u^{\text{s}} - i\kappa_0 u^{\text{s}} = o(\rho^{-1/2}) \quad \text{as } \rho = |\mathbf{r}| \rightarrow \infty. \quad (2.3)$$

2.2. TM polarization

It can be verified that the scattered field u^s satisfies the Helmholtz equation

$$\Delta u^s + \kappa_0^2 u^s = 0 \quad \text{in } \Omega^e,$$

which can be written in the polar coordinates

$$\frac{\partial^2 u^s}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u^s}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u^s}{\partial \theta^2} + \kappa_0^2 u^s = 0, \quad \rho > R. \quad (2.4)$$

It follows from the radiation condition (2.3) that the solution of (2.4) has a Fourier series expansion

$$u^s(\rho, \theta) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 \rho)}{H_n^{(1)}(\kappa_0 R)} (a_n \sin(n\theta) + b_n \cos(n\theta)), \quad (2.5)$$

where $H_n^{(1)}$ is the Hankel function of the first kind with order n . Using the fact that $u = 0$ and $u^{\text{inc}} + u^{\text{ref}} = 0$ on Γ_g , we have $u^s(\rho, 0) = u^s(\rho, \pi) = 0$, which gives $b_n = 0$ in (2.5) and

$$u^s(\rho, \theta) = \sum_{n=1}^{\infty} \frac{H_n^{(1)}(\kappa_0 \rho)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\theta). \quad (2.6)$$

Evaluating (2.6) at $\rho = R$ and using the orthogonality of the sine functions, we obtain

$$a_n = \frac{2}{\pi} \int_0^\pi u^s(R, \theta) \sin(n\theta) d\theta.$$

Taking the partial derivative of (2.6) with respect to ρ and evaluating it at $\rho = R$ yields

$$\partial_\rho u^s(R, \theta) = \kappa_0 \sum_{n=1}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\theta).$$

Given any $u \in L^2_{\text{TM}}(\Gamma_R^+) = \{u \in L^2(\Gamma_R^+) : u(R, 0) = u(R, \pi) = 0\}$, it has the Fourier series expansion

$$u(R, \theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta), \quad a_n = \frac{2}{\pi} \int_0^\pi u(R, \theta) \sin(n\theta) d\theta.$$

We introduce a boundary operator

$$(\mathcal{B}_{\text{TM}} u)(R, \theta) = \kappa_0 \sum_{n=1}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\theta). \quad (2.7)$$

Define the trace functional space $H^s_{\text{TM}}(\Gamma_R^+) = \{u \in L^2_{\text{TM}}(\Gamma_R^+) : \|u\|_{H^s_{\text{TM}}(\Gamma_R^+)} < \infty\}$, where the $H^s_{\text{TM}}(\Gamma_R^+)$ norm is characterized by

$$\|u\|_{H^s_{\text{TM}}(\Gamma_R^+)}^2 = \sum_{n=1}^{\infty} (1 + n^2)^s |a_n|^2.$$

It is clear to note that the dual space of $H^s_{\text{TM}}(\Gamma_R^+)$ is $H^{-s}_{\text{TM}}(\Gamma_R^+)$ with respect to the scalar product in $L^2_{\text{TM}}(\Gamma_R^+)$ defined by

$$\langle u, v \rangle_{\Gamma_R^+} = \int_{\Gamma_R^+} u \bar{v} ds.$$

It is shown (cf. [42, Lemma 3.1]) that the operator $\mathcal{B}_{\text{TM}} : H_{\text{TM}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TM}}^{-1/2}(\Gamma_R^+)$ is continuous.

Using the boundary operator (2.7), we obtain the transparent boundary condition for the TM polarization on Γ_R^+ :

$$\partial_\rho u^s = \mathcal{B}_{\text{TM}} u^s,$$

which can be equivalently written for the total field u :

$$\partial_\rho u = \mathcal{B}_{\text{TM}} u + f \quad \text{on } \Gamma_R^+, \quad (2.8)$$

where $f = \partial_\rho(u^{\text{inc}} + u^{\text{ref}}) - \mathcal{B}_{\text{TM}}(u^{\text{inc}} + u^{\text{ref}})$.

In the TM polarization, the open cavity scattering problem can be reduced to the following boundary value problem:

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } S, \\ \partial_\rho u = \mathcal{B}_{\text{TM}} u + f & \text{on } \Gamma_R^+, \end{cases} \quad (2.9)$$

which has the variational formulation: Find $u \in H_S^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } S\}$ such that

$$a_{\text{TM}}(u, v) = \langle f, v \rangle_{\Gamma_R^+} \quad \text{for any } v \in H_S^1(\Omega), \quad (2.10)$$

where the sesquilinear form

$$a_{\text{TM}}(u, v) = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) dx dy - \langle \mathcal{B}_{\text{TM}} u, v \rangle_{\Gamma_R^+}.$$

Since the sesquilinear form a_{TM} is not coercive in $H_S(\Omega)$, the Lax–Milgram theorem cannot be applied to show the well-posedness of the weak solution for the variational problem (2.10). The following result follows from the Fredholm alternative (cf. [42, Theorem 3.2]).

Theorem 2.1. *The variational problem (2.10) has a unique weak solution in $H_S^1(\Omega)$.*

2.3. TE polarization

Using the perfectly electrically conducting boundary condition for the TE polarization, we have $\partial_y u = 0$ and $\partial_y(u^{\text{inc}} + u^{\text{ref}}) = 0$ on Γ_g , which gives $\partial_\theta u^s(\rho, 0) = \partial_\theta u^s(\rho, \pi) = 0$. It follows from (2.5) that

$$u^s(\rho, \theta) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 \rho)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\theta), \quad \rho \geq R. \quad (2.11)$$

Evaluating (2.11) at $\rho = R$ and using the orthogonality of the cosine functions, we obtain

$$b_0 = \frac{1}{\pi} \int_0^\pi u^s(R, \theta) d\theta, \quad b_n = \frac{2}{\pi} \int_0^\pi u^s(R, \theta) \cos(n\theta) d\theta, \quad n \geq 1.$$

A simple calculation from (2.11) yields

$$\partial_\rho u^s(R, \theta) = \kappa_0 \sum_{n=0}^{\infty} \frac{H_n^{(1)'}(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\theta).$$

Given $u \in L^2_{\text{TE}}(\Gamma_R^+) = \{u \in L^2(\Gamma_R^+) : \partial_\theta u(R, 0) = \partial_\theta u(R, \pi) = 0\}$, it has the Fourier series expansion

$$u(R, \theta) = \sum_{n=0}^{\infty} b_n \cos(n\theta), \quad b_0 = \int_0^\pi u(R, \theta) d\theta, \quad b_n = \frac{2}{\pi} \int_0^\pi u(R, \theta) \cos(n\theta) d\theta, \quad n \geq 1.$$

We introduce a boundary operator

$$(\mathcal{B}_{\text{TE}}u)(R, \theta) = \kappa_0 \sum_{n=0}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\theta). \quad (2.12)$$

Define the trace functional space $H^s_{\text{TE}}(\Gamma_R^+) = \{w \in L^2_{\text{TE}}(\Gamma_R^+) : \|w\|_{H^s_{\text{TE}}(\Gamma_R^+)} < \infty\}$, where the $H^s_{\text{TE}}(\Gamma_R^+)$ norm is characterized by

$$\|u\|_{H^s_{\text{TE}}(\Gamma_R^+)}^2 = \sum_{n=0}^{\infty} (1 + n^2)^s |b_n|^2.$$

It can be shown (cf. [42, Lemma 3.3]) that the operator $\mathcal{B}_{\text{TE}} : H^{1/2}_{\text{TE}}(\Gamma_R^+) \rightarrow H^{-1/2}_{\text{TE}}(\Gamma_R^+)$ is continuous. Using the boundary operator (2.12), we obtain the transparent boundary condition for the TE polarization:

$$\partial_\rho u = \mathcal{B}_{\text{TE}}u + g \quad \text{on } \Gamma_R^+, \quad (2.13)$$

where $g = \partial_\rho(u^{\text{inc}} + u^{\text{ref}}) - \mathcal{B}_{\text{TE}}(u^{\text{inc}} + u^{\text{ref}})$.

In the TE polarization, the open cavity scattering problem can be reduced to the following boundary value problem:

$$\begin{cases} \nabla \cdot (\kappa^{-2} \nabla u) + u = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } S, \\ \partial_\rho u = \mathcal{B}_{\text{TE}}u + g & \text{on } \Gamma_R^+, \end{cases} \quad (2.14)$$

which has the variational formulation: Find $u \in H^1(\Omega)$ such that

$$a_{\text{TE}}(u, v) = \langle g, v \rangle_{\Gamma_R^+} \quad \text{for any } v \in H^1(\Omega), \quad (2.15)$$

where the sesquilinear form

$$a_{\text{TE}}(u, v) = \int_{\Omega} (\kappa^{-2} \nabla u \cdot \nabla \bar{v} - u \bar{v}) dx dy - \langle \mathcal{B}_{\text{TE}}u, v \rangle_{\Gamma_R^+}.$$

Similarly, the well-posedness of the weak solution can be shown from the Fredholm alternative (cf. [42, Theorem 3.4]).

Theorem 2.2. *The variational problem (2.15) has a unique weak solution in $H^1(\Omega)$.*

3. Time-harmonic Maxwell's Equations

In this section, we consider the electromagnetic open cavity scattering problem for the time-harmonic Maxwell equations and present the well-posedness of the solution for its variational formulation.

3.1. A model problem

Let us still use Figure 2.1 to illustrate the problem geometry. Denote by $D \subset \mathbb{R}^3$ the open cavity with a Lipschitz continuous boundary consisting of the perfectly electrically conducting cavity wall S and the opening Γ , which is aligned with the infinite perfectly electrically conductor ground plane Γ_g . The medium inside the cavity is characterized by the dielectric permittivity ε and the magnetic permeability μ . Let B_R^+ and Γ_R^+ be the half-ball and hemisphere above the ground plane, where the radius R is large enough to completely cover the possibly overfilled cavity. The exterior region $\Omega^e = \mathbb{R}_+^3 \setminus \bar{B}_R^+$ is filled with some homogeneous material with a constant permittivity ε_0 and a constant permeability μ_0 . From now on, we assume for simplicity that $\varepsilon_0 = \mu_0 = 1$. Denote $\Omega = B_R^+ \cup D$ with boundary $\partial\Omega = S \cup \Gamma_R^+$.

Consider the time-harmonic Maxwell equations in $\mathbb{R}_+^3 \cup D$:

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}, \quad \nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}, \quad (3.1)$$

where $\omega > 0$ is the angular frequency, \mathbf{E} and \mathbf{H} are the electric field and the magnetic field, respectively. Since the ground plane and the cavity wall are perfect electrical conductor, we have

$$\nu \times \mathbf{E} = 0 \quad \text{on } \Gamma_g \cup S, \quad (3.2)$$

where ν is the unit outward normal vector on Γ_g and S .

Let $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$ be the electromagnetic plane waves that are incident upon the cavity from the above, where

$$\mathbf{E}^{\text{inc}} = t e^{i\omega\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{H}^{\text{inc}} = s e^{i\omega\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{s} = \frac{\mathbf{q} \times \mathbf{t}}{\omega}, \quad \mathbf{t} \cdot \mathbf{q} = 0.$$

Here \mathbf{t} and \mathbf{s} are the polarization vectors, the propagation direction vector $\mathbf{q} = (\alpha_1, \alpha_2, -\beta) = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, -\cos\theta_1)$, and θ_1, θ_2 are incident angles satisfying $0 \leq \theta_1 < \pi/2$, $0 \leq \theta_2 < 2\pi$. Evidently, the incident fields $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$ satisfy the time-harmonic Maxwell equation (3.1) in Ω^e . Due to the perfectly electrically conducting ground plane, the reflected fields $(\mathbf{E}^{\text{ref}}, \mathbf{H}^{\text{ref}})$ can be explicitly written as

$$\mathbf{E}^{\text{ref}} = -t e^{i\omega\mathbf{q}^*\cdot\mathbf{x}}, \quad \mathbf{H}^{\text{ref}} = -\mathbf{t} \times \mathbf{q}^* e^{i\omega\mathbf{q}^*\cdot\mathbf{r}},$$

where $\mathbf{q}^* = (\alpha_1, \alpha_2, \beta)$. It is easy to verify that

$$\nu \times (\mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{ref}}) = 0 \quad \text{on } \Gamma_g.$$

The total electric and magnetic fields can be decomposed into the summation of the incident fields, the reflected fields, and the scattered fields:

$$\mathbf{E} = \mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{ref}} + \mathbf{E}^{\text{s}}, \quad \mathbf{H} = \mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{ref}} + \mathbf{H}^{\text{s}}.$$

The scattered fields $(\mathbf{E}^{\text{s}}, \mathbf{H}^{\text{s}})$ are required to satisfy the Silver–Müller radiation condition:

$$\mathbf{E}^{\text{s}} - \mathbf{H}^{\text{s}} \times \hat{\mathbf{r}} = o(|\mathbf{r}|^{-1}) \quad \text{as } |\mathbf{r}| \rightarrow \infty, \quad (3.3)$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

To make the survey self-contained, we collect some properties of the spherical harmonics and define some functional spaces in the following two sections. They will be used in subsequent analysis for both the time-harmonic and time-domain Maxwell equations, especially when introducing transparent boundary conditions.

3.2. Spherical harmonics on hemisphere

The spherical coordinates (ρ, θ, φ) are related to the Cartesian coordinates $\mathbf{r} = (x, y, z)$ by $x = \rho \sin \theta \cos \varphi$, $y = \rho \sin \theta \sin \varphi$, $z = \rho \cos \theta$, with the local orthonormal basis $\{\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$:

$$\begin{aligned}\mathbf{e}_\rho &= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \\ \mathbf{e}_\theta &= (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \\ \mathbf{e}_\varphi &= (-\sin \varphi, \cos \varphi, 0),\end{aligned}$$

where θ and φ are the Euler angles.

Let $\{Y_n^m(\theta, \varphi), |m| \leq n, n = 0, 1, 2, \dots\}$ be an orthonormal sequence of spherical harmonics of order n on the unit sphere. Explicitly, the spherical harmonics of order n is

$$Y_n^m(\theta, \varphi) = \left(\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} \right)^{1/2} P_n^{|m|}(\cos \theta) e^{im\varphi},$$

where the associated Legendre functions are

$$P_n^m(t) := (1-t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}, \quad m = 0, 1, \dots, n.$$

Here P_n is the Legendre polynomial of degree n . Define a sequence of rescaled spherical harmonics of order n :

$$X_n^m(\theta, \varphi) = \frac{\sqrt{2}}{R} Y_n^m(\theta, \varphi).$$

It is shown (cf. [31, Lemma 3.1]) that $\{X_n^m : |m| \leq n, n \in \mathbb{N}, m+n = \text{odd}\}$ forms a complete orthonormal basis in $L^2(\Gamma_R^+)$. For convenience, we take the following notation for double summations:

$$\begin{aligned}\sum_{|m| \leq n} u_n^m &:= \sum_{n=1}^{\infty} \sum_{m=-n}^n u_n^m, \\ \sum_{\substack{|m| \leq n \\ \text{odd}}} u_n^m &:= \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m+n=\text{odd}}}^n u_n^m, & \sum_{|m| \leq n}^{\text{even}} u_n^m &:= \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m+n=\text{even}}}^n u_n^m.\end{aligned}$$

Define two sequences of tangential fields

$$\mathbf{X}_n^m(\theta, \varphi) = \frac{1}{\sqrt{n(n+1)}} \nabla_{\Gamma_R^+} X_n^m(\theta, \varphi) \quad \text{and} \quad \mathbf{Y}_n^m(\theta, \varphi) = \mathbf{e}_\rho \times \mathbf{X}_n^m(\theta, \varphi),$$

where $\nabla_{\Gamma_R^+}$ is the tangential gradient on Γ_R^+ . It is shown (cf. [31, Lemma 3.2]) that the vector spherical harmonics $\{\mathbf{X}_n^m : |m| \leq n, n \in \mathbb{N}, m+n = \text{odd}\}$ and $\{\mathbf{Y}_n^m : |m| \leq n, n \in \mathbb{N}, m+n = \text{even}\}$ form a complete orthonormal basis in $\mathbf{L}_t^2(\Gamma_R^+) = \{\mathbf{u} \in L^2(\Gamma_R^+)^3 : \mathbf{e}_\rho \cdot \mathbf{u} = 0\}$.

3.3. Functional spaces

Given any $u \in L^2(\Gamma_R^+)$, it has the expansion

$$u(\theta, \varphi) = \sum_{|m| \leq n}^{\text{odd}} u_n^m X_n^m(\theta, \varphi).$$

Denote $H^s(\Gamma_R^+) = \{u \in L^2(\Gamma_R^+) : \|u\|_{H^s(\Gamma_R^+)} < \infty\}$, where the norm is characterized by

$$\|u\|_{H^s(\Gamma_R^+)}^2 = \sum_{|m| \leq n}^{\text{odd}} (1 + n(n+1))^s |u_n^m|^2.$$

Introduce three tangential trace spaces:

$$\begin{aligned} \mathbf{H}_t^s(\Gamma_R^+) &= \{\mathbf{u} \in \mathbf{H}^s(\Gamma_R^+), \mathbf{e}_\rho \cdot \mathbf{u} = 0, \mathbf{e}_\theta \times \mathbf{u}(\pi/2, \varphi) = 0\}, \\ \mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+) &= \{\mathbf{u} \in \mathbf{H}_t^{-1/2}(\Gamma_R^+), \text{curl}_{\Gamma_R^+} \mathbf{u} \in H^{-1/2}(\Gamma_R^+)\}, \\ \mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+) &= \{\mathbf{u} \in \mathbf{H}_t^{-1/2}(\Gamma_R^+), \text{div}_{\Gamma_R^+} \mathbf{u} \in H^{-1/2}(\Gamma_R^+)\}, \end{aligned}$$

where $\text{curl}_{\Gamma_R^+}$ and $\text{div}_{\Gamma_R^+}$ are the surface scalar curl and the surface divergence on Γ_R^+ .

For any tangential field $\mathbf{u} \in \mathbf{H}_t^s(\Gamma_R^+)$, it can be represented in the series expansion

$$\mathbf{u} = \sum_{|m| \leq n}^{\text{odd}} u_{1n}^m \mathbf{X}_n^m(\theta, \varphi) + \sum_{|m| \leq n}^{\text{even}} u_{2n}^m \mathbf{Y}_n^m(\theta, \varphi).$$

Using the series coefficients, the norm of the space $\mathbf{H}_t^s(\Gamma_R^+)$ can be characterized by

$$\|\mathbf{u}\|_{\mathbf{H}_t^s(\Gamma_R^+)}^2 = \sum_{|m| \leq n}^{\text{odd}} (1 + n(n+1))^s |u_{1n}^m|^2 + \sum_{|m| \leq n}^{\text{even}} (1 + n(n+1))^s |u_{2n}^m|^2;$$

the norm of the space $\mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+)$ can be characterized by

$$\|\mathbf{u}\|_{\mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+)}^2 = \sum_{|m| \leq n}^{\text{odd}} \frac{1}{\sqrt{1 + n(n+1)}} |u_{1n}^m|^2 + \sum_{|m| \leq n}^{\text{even}} \sqrt{1 + n(n+1)} |u_{2n}^m|^2;$$

and the norm of the space $\mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+)$ can be characterized by

$$\|\mathbf{u}\|_{\mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+)}^2 = \sum_{|m| \leq n}^{\text{odd}} \sqrt{1 + n(n+1)} |u_{1n}^m|^2 + \sum_{|m| \leq n}^{\text{even}} \frac{1}{\sqrt{1 + n(n+1)}} |u_{2n}^m|^2.$$

Clearly, the dual space of $\mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+)$ is $\mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+)$ under the scalar product in $L_t^2(\Gamma_R^+)$, which is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma_R^+} = \int_{\Gamma_R^+} \mathbf{u} \cdot \bar{\mathbf{v}} ds.$$

Introduce two functional spaces

$$\begin{aligned} \mathbf{H}(\text{curl}, \Omega) &= \{\mathbf{u} \in L^2(\Omega), \nabla \times \mathbf{u} \in L^2(\Omega)\}, \\ \mathbf{H}_S(\text{curl}, \Omega) &= \{\mathbf{u} \in \mathbf{H}(\text{curl}, \Omega), \nu \times \mathbf{u} = 0 \text{ on } S\}, \end{aligned}$$

which are Sobolev spaces with the norm:

$$\|\mathbf{u}\|_{\mathbf{H}(\text{curl}, \Omega)} = \left(\|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla \times \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

3.4. Reduced problem

Given a tangential field $\mathbf{u} \in \mathbf{L}_t^2(\Gamma_R^+)$, it has the expansion

$$\mathbf{u} = \sum_{|m| \leq n}^{\text{odd}} \alpha_n^m \mathbf{X}_n^m + \sum_{|m| \leq n}^{\text{even}} \beta_n^m \mathbf{Y}_n^m.$$

We introduce a boundary operator

$$\mathcal{B}_{\text{TH}} \mathbf{u} = \sum_{|m| \leq n}^{\text{odd}} \frac{i\omega R \alpha_n^m}{1 + r_n(\omega R)} \mathbf{X}_n^m + \sum_{|m| \leq n}^{\text{even}} \frac{(1 + r_n(\omega R)) \beta_n^m}{i\omega R} \mathbf{Y}_n^m, \quad (3.4)$$

where

$$r_n(t) = \frac{t (h_n^{(1)})'(t)}{h_n^{(1)}(t)}. \quad (3.5)$$

Here $h_n^{(1)}$ is the special Hankel function of the first kind with order n . It is shown (cf. [31, Lemma 4.1]) that the operator $\mathcal{B}_{\text{TH}} : \mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+) \rightarrow \mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+)$ is continuous.

Given a vector field $\mathbf{u} \in \Gamma_R^+$, denote by $\mathbf{u}_{\Gamma_R^+} = -\mathbf{e}_\rho \times (\mathbf{e}_\rho \times \mathbf{u})$ the tangential component of \mathbf{u} on Γ_R^+ . With the use of the boundary operator, we obtain the following transparent boundary condition

$$(\nabla \times \mathbf{E}) \times \mathbf{e}_\rho = i\omega \mathcal{B}_{\text{TH}} \mathbf{E}_{\Gamma_R^+} + \mathbf{f}, \quad (3.6)$$

where

$$\mathbf{f} = i\omega \left((\mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{ref}}) \times \mathbf{e}_\rho - \mathcal{B}_{\text{TH}}(\mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{ref}}) \right).$$

Under the help of the transparent boundary condition (3.6), we may eliminate the magnetic field from the Maxwell equations and derive a boundary value problem for the electric field:

$$\begin{cases} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = 0 & \text{in } \Omega, \\ \nu \times \mathbf{E} = 0 & \text{on } S, \\ (\nabla \times \mathbf{E}) \times \mathbf{e}_\rho - i\omega \mathcal{B}_{\text{TH}} \mathbf{E}_{\Gamma_R^+} = \mathbf{f} & \text{on } \Gamma_R^+. \end{cases} \quad (3.7)$$

which is equivalent to find $\mathbf{E} \in \mathbf{H}_S(\text{curl}, \Omega)$ such that

$$a_{\text{TH}}(\mathbf{E}, \mathbf{w}) = \langle \mathbf{f}, \mathbf{w} \rangle_{\Gamma_R^+} \quad \text{for all } \mathbf{w} \in \mathbf{H}_S(\text{curl}, \Omega), \quad (3.8)$$

where the sesquilinear form

$$a_{\text{TH}}(\mathbf{E}, \mathbf{w}) = \int_{\Omega} \mu^{-1} (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{w}}) - \omega^2 \int_{\Omega} \varepsilon \mathbf{E} \cdot \bar{\mathbf{w}} - i\omega \langle \mathcal{B}_{\text{TH}} \mathbf{E}_{\Gamma_R^+}, \mathbf{w} \rangle_{\Gamma_R^+}.$$

It is more sophisticated to prove the well-posedness of the variational problem (3.8) for Maxwell's equations than the variational problem (2.10) or (2.15) for the Helmholtz equation. The proof of the following theorem (cf. [31, Theorem 5.2]) is based on a combination of several techniques including a unique continuation for Maxwell's equations, a Hodge decomposition of $\mathbf{H}_S(\text{curl}, \Omega)$, and a compact embedding result.

Theorem 3.1. *The variational problem (3.8) has a unique weak solution in $\mathbf{H}_S(\text{curl}, \Omega)$.*

4. Time-domain Maxwell's Equations

In this section, we consider the open cavity scattering problem for the time-domain Maxwell equations. An initial-boundary value problem is formulated by using a time-domain transparent boundary condition.

4.1. A model problem

The problem geometry is the same as the one for the time-harmonic Maxwell equations. Let $D \subset \mathbb{R}^3$ be an open cavity which is enclosed by the cavity wall S and the opening Γ aligned with the ground plane Γ_g .

Consider the system of time-domain Maxwell equations in $\mathbb{R}_+^3 \cup D$ for $t > 0$:

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, t) + \mu \partial_t \mathbf{H}(\mathbf{r}, t) = \mathbf{0}, \\ \nabla \times \mathbf{H}(\mathbf{r}, t) - \varepsilon \partial_t \mathbf{E}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t), \end{cases} \quad (4.1)$$

where \mathbf{E} is the electric field, \mathbf{H} is the magnetic field, and \mathbf{J} is the electric current density which is assumed to be compactly supported in D . The system is constrained by the initial conditions:

$$\mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0 \quad \text{in } \mathbb{R}_+^3 \cup D, \quad (4.2)$$

where \mathbf{E}_0 and \mathbf{H}_0 are also assumed to be compactly supported in D . Assuming that the cavity wall and the ground plane are perfectly electrical conducting, we have

$$\nu \times \mathbf{E} = \mathbf{0} \quad \text{on } \Gamma_g \cup S, \quad t > 0, \quad (4.3)$$

where ν is the unit outward normal vector on $\Gamma_g \cup S$. In addition, we impose the Silver–Müller radiation condition:

$$\hat{\mathbf{r}} \times (\partial_t \mathbf{E} \times \hat{\mathbf{r}}) + \hat{\mathbf{r}} \times \partial_t \mathbf{H} = o(|\mathbf{r}|^{-1}), \quad \text{as } |\mathbf{r}| \rightarrow \infty, \quad t > 0, \quad (4.4)$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

For any $s = s_1 + is_2$ with $s_1, s_2 \in \mathbb{R}, s_1 > 0$, define by $\check{\mathbf{u}}(s)$ the Laplace transform of the vector field \mathbf{u} , i.e.,

$$\check{\mathbf{u}} = \mathcal{L}(\mathbf{u})(s) = \int_0^\infty e^{-st} \mathbf{u}(t) dt.$$

The Laplace transform is our key tool to analyze the time-domain Maxwell equations.

4.2. Reduced problem

Since \mathbf{J} is supported in D , the Maxwell equations (4.1) reduce to

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = \mathbf{0}, \quad \nabla \times \mathbf{H} - \partial_t \mathbf{E} = \mathbf{0} \quad \text{in } \Omega^e, \quad t > 0. \quad (4.5)$$

Let $\check{\mathbf{E}}(\mathbf{r}, s) = \mathcal{L}(\mathbf{E})$ and $\check{\mathbf{H}}(\mathbf{r}, s) = \mathcal{L}(\mathbf{H})$ be the Laplace transforms of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ with respect to t , respectively. Recall that

$$\mathcal{L}(\partial_t \mathbf{E}) = s\check{\mathbf{E}} - \mathbf{E}_0, \quad \mathcal{L}(\partial_t \mathbf{H}) = s\check{\mathbf{H}} - \mathbf{H}_0.$$

Taking the Laplace transform of (4.5), and noting that $\mathbf{E}_0, \mathbf{H}_0$ are supported in D , we obtain the time-harmonic Maxwell equations with complex parameters:

$$\nabla \times \check{\mathbf{E}} + s\check{\mathbf{H}} = \mathbf{0}, \quad \nabla \times \check{\mathbf{H}} - s\check{\mathbf{E}} = \mathbf{0} \quad \text{in } \Omega^e, \quad s_1 > 0. \quad (4.6)$$

Given a tangential field $\mathbf{u} \in \mathbf{L}_t^2(\Gamma_R^+)$, it has the expansion

$$\mathbf{u} = \sum_{|m| \leq n}^{\text{odd}} \alpha_n^m \mathbf{X}_n^m + \sum_{|m| \leq n}^{\text{even}} \beta_n^m \mathbf{Y}_n^m.$$

We define a boundary operator

$$\mathcal{B}_{\text{TD}} \mathbf{u} = - \sum_{|m| \leq n}^{\text{odd}} \frac{sR}{1 + r_n(isR)} \alpha_n^m \mathbf{X}_n^m - \sum_{|m| \leq n}^{\text{even}} \frac{(1 + r_n(isR))}{sR} \beta_n^m \mathbf{Y}_n^m, \quad (4.7)$$

where $r_n(t)$ is defined in (3.5). It can be shown (cf. [29, Lemma 2.1]) that the operator $\mathcal{B}_{\text{TD}} : \mathbf{H}^{-1/2}(\text{curl}, \Gamma_R^+) \rightarrow \mathbf{H}^{-1/2}(\text{div}, \Gamma_R^+)$ is continuous.

With the aid of the frequency domain boundary operator, we obtain the following transparent boundary condition imposed upon the hemisphere Γ_R^+ in the s -domain:

$$\mathcal{B}_{\text{TD}} \check{\mathbf{E}}_{\Gamma_R^+} = \check{\mathbf{H}} \times \mathbf{e}_\rho, \quad (4.8)$$

which maps the tangential component of the electric field to the tangential trace of the magnetic field. Taking the inverse Laplace transform of (4.8) yields the transparent boundary condition in the time-domain:

$$\mathcal{T}_{\text{TD}} \mathbf{E}_{\Gamma_R^+} = \mathbf{H} \times \mathbf{e}_\rho, \quad \text{where } \mathcal{T}_{\text{TD}} := \mathcal{L}^{-1} \circ \mathcal{B}_{\text{TD}} \circ \mathcal{L}. \quad (4.9)$$

Equivalently, we may eliminate the magnetic field and obtain an alternative transparent boundary condition in the s -domain:

$$s^{-1}(\nabla \times \check{\mathbf{E}}) \times \mathbf{e}_\rho + \mathcal{B}_{\text{TD}} \check{\mathbf{E}}_{\Gamma_R^+} = 0 \quad \text{on } \Gamma_R^+. \quad (4.10)$$

Correspondingly, by taking the inverse Laplace transform of (4.10), we may derive an alternative transparent boundary condition in the time domain:

$$(\nabla \times \mathbf{E}) \times \mathbf{e}_\rho + \mathcal{C}_{\text{TD}} \mathbf{E}_{\Gamma_R^+} = 0 \quad \text{on } \Gamma_R^+, \quad \text{where } \mathcal{C}_{\text{TD}} = \mathcal{L}^{-1} \circ s \mathcal{B}_{\text{TD}} \circ \mathcal{L}. \quad (4.11)$$

Using the transparent boundary condition (4.11), we may consider the following equivalent initial-boundary value problem:

$$\left\{ \begin{array}{ll} \nabla \times \mathbf{E} + \mu \partial_t \mathbf{H} = 0, \quad \nabla \times \mathbf{H} - \varepsilon \partial_t \mathbf{E} = \mathbf{J} & \text{in } \Omega, \quad t > 0, \\ \mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0 & \text{in } \Omega, \\ \nu \times \mathbf{E} = 0 & \text{on } S, \quad t > 0, \\ \mathcal{T}_{\text{TD}} \mathbf{E}_{\Gamma_R^+} = \mathbf{H} \times \mathbf{e}_\rho & \text{on } \Gamma_R^+, \quad t > 0. \end{array} \right. \quad (4.12)$$

To show the well-posedness of the reduced problem (4.12), we make some assumptions for the initial and boundary data:

$$\mathbf{E}_0, \mathbf{H}_0 \in \mathbf{H}(\text{curl}, \Omega), \quad \mathbf{J} \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)), \quad \mathbf{J}|_{t=0} = 0.$$

The following result (cf. [29, Theorem 4.2]) shows the well-posedness of stability of the time-domain Maxwell equations. The proof is based on the Lax-Milgram theorem and an abstract inversion theorem of the Laplace transform.

Theorem 4.1. *The problem (4.12) has a unique solution (\mathbf{E}, \mathbf{H}) such that*

$$\begin{aligned}\mathbf{E} &\in \mathbf{L}^2(0, T; \mathbf{H}_S(\text{curl}, \Omega)) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)), \\ \mathbf{H} &\in \mathbf{L}^2(0, T; \mathbf{H}(\text{curl}, \Omega)) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)).\end{aligned}$$

Moreover, they satisfy

$$\begin{aligned}& \max_{t \in [0, T]} \left(\|\partial_t \mathbf{E}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(\Omega)} + \|\partial_t \mathbf{H}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{H}\|_{\mathbf{L}^2(\Omega)} \right) \\ & \lesssim \|\mathbf{E}_0\|_{\mathbf{H}(\text{curl}, \Omega)} + \|\mathbf{H}_0\|_{\mathbf{H}(\text{curl}, \Omega)} + \|\mathbf{J}\|_{\mathbf{H}^1(0, T; \mathbf{L}^2(\Omega))}.\end{aligned}$$

Finally we present an a priori stability estimate for the electric field with a minimum regularity requirement for the data and explicit dependence on the time variable. Eliminating the magnetic field in (4.1) and using the transparent boundary condition (4.11), we consider the initial-boundary value problem in a bounded domain:

$$\begin{cases} \varepsilon \partial_t^2 \mathbf{E} = -\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \mathbf{F} & \text{in } \Omega, \quad t > 0, \\ \mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \partial_t \mathbf{E}|_{t=0} = \mathbf{E}_1 & \text{in } \Omega, \\ \nu \times \mathbf{E} = 0 & \text{on } S, \quad t > 0, \\ (\nabla \times \mathbf{E}) \times \mathbf{e}_\rho + \mathcal{C}_{\text{TD}} \mathbf{E}_{\Gamma_R^+} = 0 & \text{on } \Gamma_R^+, \quad t > 0, \end{cases} \quad (4.13)$$

where

$$\mathbf{F} = \partial_t \mathbf{J}, \quad \mathbf{E}_1 = \varepsilon^{-1} (\nabla \times \mathbf{H}_0 - \mathbf{J}_0).$$

The variational problem is to find $\mathbf{E} \in \mathbf{H}_S(\text{curl}, \Omega)$ for all $t > 0$ such that

$$\begin{aligned} \int_{\Omega} \varepsilon \partial_t^2 \mathbf{E} \cdot \bar{\mathbf{w}} \, d\mathbf{r} &= - \int_{\Omega} \mu^{-1} (\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{w}}) \, d\mathbf{r} \\ &\quad - \langle \mathcal{C}_{\text{TD}} \mathbf{E}_{\Gamma_R^+}, \mathbf{w}_{\Gamma_R^+} \rangle_{\Gamma_R^+} - \int_{\Omega} \mathbf{F} \cdot \bar{\mathbf{w}} \, d\mathbf{r} \quad \text{for all } \mathbf{w} \in \mathbf{H}_S(\text{curl}, \Omega). \end{aligned} \quad (4.14)$$

By taking a special test function, we can show the following stability estimate (cf. [29, Theorem 4.4]).

Theorem 4.2. *Let $\mathbf{E} \in \mathbf{H}_S(\text{curl}, \Omega)$ be the solution of (4.14) for any $t > 0$. Given $\mathbf{E}_0, \mathbf{E}_1 \in \mathbf{L}^2(\Omega)$ and $\mathbf{F} \in \mathbf{L}^1(0, T; \mathbf{L}^2(\Omega))$ for any $T > 0$, there holds*

$$\begin{aligned} \|\mathbf{E}\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))} &\lesssim \|\mathbf{E}_0\|_{\mathbf{L}^2(\Omega)} + T \|\mathbf{E}_1\|_{\mathbf{L}^2(\Omega)} + T \|\mathbf{F}\|_{\mathbf{L}^1(0, T; \mathbf{L}^2(\Omega))}, \\ \|\mathbf{E}\|_{\mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))} &\lesssim T^{1/2} \|\mathbf{E}_0\|_{\mathbf{L}^2(\Omega)} + T^{3/2} \|\mathbf{E}_1\|_{\mathbf{L}^2(\Omega)} + T^{3/2} \|\mathbf{F}\|_{\mathbf{L}^1(0, T; \mathbf{L}^2(\Omega))}.\end{aligned}$$

5. Discussions

A brief survey is given on the recent developments of mathematical modeling and analysis for the scattering by open cavities, which offer rich and challenging mathematical problems. The governing models are introduced for the two-dimensional Helmholtz equation and the three-dimensional Maxwell equations. A key step is to develop transparent boundary conditions which help to reduce the problems from open domains into bounded domains.

We point out some future directions along the line of this research. We assume that the infinite ground plane is perfectly electrically conducting throughout the paper. Results are

very rare for the impedance boundary condition of the infinite ground plane. It is well studied on how to apply the perfectly matched layer (PML) techniques to truncated the unbounded domain for the time-harmonic problems [31, 46, 47]. It is challenging to consider the PML for the time-domain problems. Computationally, the variational approach leads naturally to a class of finite element methods. As a time-dependent problem, a fast and accurate marching scheme shall be developed to deal with the temporal convolution in the transparent boundary condition for the time-domain Maxwell equations.

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