

On the Convergence of Two-Step Modulus-Based Matrix Splitting Iteration Methods for a Restricted Class of Nonlinear Complementarity Problems with H_+ -Matrices

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Abstract. We propose the two-step modulus-based matrix splitting iteration methods for a class of nonlinear complementarity problems. The corresponding convergence theory is established when the system matrix is an H_+ -matrix. Theoretical analysis gives the choice of parameter matrix involved based on the H -compatible splitting of the system matrix. Moreover, in actual implementation, the choices of iterative parameters for two-step modulus-based accelerated overrelaxation methods are studied. Numerical experiments show that the method is efficient and further verify the convergence theorems.

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1. Introduction

Given a matrix $A \in \mathbb{R}^{n \times n}$, a vector $q \in \mathbb{R}^n$ and a nonlinear mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the following nonlinear complementarity problem which aims to find n -dimensional real vectors z and w such that

$$z \geq 0, \quad w := Az + q + f(z) \geq 0, \quad z^T w = 0, \quad (1.1)$$

arises widely from many scientific computing and engineering applications, such as the network equilibrium problem, the contact problem and the free boundary problem with nonlinear source terms [1, 2]. In the free boundary problem, the function f is usually

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referred to the nonlinear source term. Moreover, f in problem (1.1) is assumed to be a diagonal differentiable mapping [3,4], which means the i th component of f is differentiable and only in terms of z_i , i.e.,

$$f_i = f_i(z_i), \quad i = 1, 2, \dots, n.$$

In problem (1.1), ' ≥ 0 ' is componentwise and the superscript ' T ' means the transpose of a vector.

If f is a linear function, the nonlinear complementarity problem (1.1) reduces to the linear complementarity problem. In the past decades, a number of efficient iteration methods have been proposed for solving linear complementarity problems, especially when the coefficient matrix is a real positive definite matrix or an H_+ -matrix. An excellent survey of the existing methods and the classification of matrices for linear complementarity problems can be found in [2]. By combining the modulus method [5–7] and the matrix splitting technique, Bai [8] presented the modulus-based matrix splitting iteration method for linear complementarity problems and obtained the convergence theorems when A is positive definite or an H_+ -matrix. Further, Zhang and Ren improved the convergence condition by weakening the H -compatible splitting of an H_+ -matrix to the H -splitting of it in [9]. Inspired by the work in [8], a series of modulus-based matrix splitting iteration methods were developed. For instance, the two-step modulus-based matrix splitting iteration method was proposed by Zhang and its convergence theory was proved when the system matrix is an H_+ -matrix [10, 11]. By putting another parameter diagonal matrix to the fixed-point formula, Li constructed a general modulus-based matrix splitting iteration scheme in [12]. Further, Xu and Liu [13] gave a modified general modulus-based matrix splitting method by replacing a positive diagonal matrix with a nonnegative one. See [14–21] for more variants of modulus-based matrix splitting iteration methods.

For the solution of large and sparse nonlinear complementarity problem, iteration methods also attract much attention of researchers. The parallel nonlinear multisplitting relaxation method and the Newton-type method were studied in [22, 23], respectively, and only local convergence theory for them was established. The Broyden-like method was proposed and its global and local superlinear convergence conditions were explored in [24]. Recently, Xia and Li firstly extended the modulus-based matrix splitting method to solve nonlinear complementarity problems (1.1) and also discussed the global convergence when A is positive definite or an H_+ -matrix [25]. The authors of this paper proposed the accelerated modulus-based matrix splitting iteration method for solving problem (1.1) and established the corresponding convergence theory both for positive definite system matrix and H_+ -matrix in [26]. Two-step modulus-based matrix splitting iteration methods were presented in [27], the convergence was only studied when A is positive definite. In this paper, we further study the two-step modulus-based matrix splitting iteration method and analyze the corresponding convergence conditions when A is an H_+ -matrix in detail. Numerical experiments verify the convergence theory and illustrate the efficiency of the method.

The rest of this paper is organized as follows. The two-step modulus-based matrix splitting iteration method for solving nonlinear complementarity problems (1.1) is proposed in

Section 2, while the convergence theory of this proposed method when A is an H_+ -matrix is established in Section 3. In Section 4, a number of numerical experiments are given and finally we derive some conclusions in Section 5.

2. Two-step modulus-based matrix splitting iteration methods

Let Ω denote a positive diagonal matrix with suitable dimension and γ be a positive constant in the rest of this paper. For the nonlinear complementarity problem (1.1), by setting

$$z = \frac{1}{\gamma}(|x| + x), \quad w = \frac{1}{\gamma}\Omega(|x| - x).$$

Xia and Li reformulated it as an equivalent implicit fixed-point equation in [25]

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma(q + f(z)), \quad (2.1)$$

where $A = M - N$ is a splitting of A . The modulus-based matrix splitting iteration method is described as follows:

Algorithm 2.1 ([25]) The modulus-based matrix splitting iteration method.

Let $A = M - N$ be a splitting of matrix $A \in \mathbb{R}^{n \times n}$.

1. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, set $z^{(0)} = \frac{1}{\gamma}(|x^{(0)}| + x^{(0)})$;
2. For $k = 0, 1, \dots$, until the nonnegative iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ converges, compute $x^{(k+1)} \in \mathbb{R}^n$ by solving the following system

$$(\Omega + M)x^{(k+1)} = Nx^{(k)} + (\Omega - A)|x^{(k)}| - \gamma(q + f(z^{(k)})),$$

and set $z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)})$.

Different choices of matrix splitting can result in different modulus-based matrix splitting iteration methods. For example, set $A = D - L - U$, where D , $-L$, $-U$ are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of A , respectively. When

$$M = \frac{1}{\omega}(D - \beta L), \quad N = \frac{1}{\omega}[(1 - \omega)D + (\omega - \beta)L + \omega U],$$

Algorithm 2.1 becomes the modulus-based accelerated overrelaxation iteration method (**MAOR**), where $\omega > 0, \beta > 0$ are iterative parameters. The modulus-based successive overrelaxation iteration method (**MSOR**) and the modulus-based Gauss-Seidel iteration method (**MGS**) are the special cases when $\omega = \beta$ and $\omega = \beta = 1$, respectively.

By making full use of the information contained in matrix A , the following two-step modulus-based matrix splitting iteration method was set up in [27].

Algorithm 2.2 Two-step modulus-based matrix splitting iteration method.

Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of matrix $A \in \mathbb{R}^{n \times n}$.

1 Given an initial vector $x^{(0)} \in \mathbb{R}^n$, set $k := 0$ and $z^{(0)} = \frac{1}{\gamma}(|x^{(0)}| + x^{(0)})$;

2 Compute $x^{(k+1)} \in \mathbb{R}^n$ by solving the system

$$\begin{cases} (\Omega + M_1)x^{(k+\frac{1}{2})} = N_1x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma(q + f(z^{(k)})), \\ (\Omega + M_2)x^{(k+1)} = N_2x^{(k+\frac{1}{2})} + (\Omega - A)|x^{(k+\frac{1}{2})}| - \gamma(q + f(z^{(k+\frac{1}{2})})), \end{cases} \quad (2.2)$$

and set $z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)})$;

3 If $z^{(k+1)}$ satisfies the stopping rule, then stop; otherwise, set $k := k + 1$, go back to Step 1.

The corresponding implicit fixed-point equations of (2.2) are

$$\begin{cases} (\Omega + M_1)x = N_1x + (\Omega - A)|x| - \gamma(q + f(z)), \\ (\Omega + M_2)x = N_2x + (\Omega - A)|x| - \gamma(q + f(z)). \end{cases} \quad (2.3)$$

Algorithm 2.2 sets up a general framework of two-step modulus-based matrix splitting iteration methods for solving nonlinear complementarity problems (1.1). It is observed that the two-step modulus-based matrix splitting iteration method for linear complementarity problems [10] is a special case of Algorithm 2.2 when $f(z) \equiv 0$. New iteration methods can also be generated with suitable choices of matrix splittings, such as the two-step modulus-based Gauss-Seidel iteration method (**TMGS**) when

$$M_1 = D - L, \quad N_1 = U, \quad M_2 = D - U, \quad N_2 = L; \quad (2.4)$$

the two-step modulus-based successive overrelaxation iteration method (**TMSOR**) when

$$M_1 = \frac{1}{\omega}D - L, \quad N_1 = \frac{1}{\omega}[(1 - \omega)D + \omega U], \quad (2.5a)$$

$$M_2 = \frac{1}{\omega}D - U, \quad N_2 = \frac{1}{\omega}[(1 - \omega)D + \omega L]; \quad (2.5b)$$

and the two-step modulus-based accelerated overrelaxation iteration method (**TMAOR**) when

$$M_1 = \frac{1}{\omega}(D - \beta L), \quad N_1 = \frac{1}{\omega}[(1 - \omega)D + (\omega - \beta)L + \omega U], \quad (2.6a)$$

$$M_2 = \frac{1}{\omega}(D - \beta U), \quad N_2 = \frac{1}{\omega}[(1 - \omega)D + (\omega - \beta)U + \omega L]. \quad (2.6b)$$

3. Convergence theorems with an H_+ -matrix

First, necessary notations and terminology used throughout this paper are reviewed and most of them can be found in [2, 29, 30].

For two matrices $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{n \times m}$, we denote $A \geq B$ ($A > B$) to mean $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all $1 \leq i \leq n$, $1 \leq j \leq m$. Specially, when B is a zero matrix, A is said to be nonnegative (positive). Denote $|A| = (|a_{ij}|) \in \mathbb{R}^{n \times m}$ as the absolute value matrix of A . These notations can be easily specified to vectors in \mathbb{R}^n .

A Z -matrix is a matrix with nonpositive off-diagonal entries, an M -matrix is a nonsingular Z -matrix with nonnegative inverse. It is known that if $A \leq B$ with A being an M -matrix and B being a Z -matrix, then B is also an M -matrix [2].

An H -matrix is a matrix whose comparison matrix $\langle A \rangle$ is an M -matrix, where $\langle A \rangle = (\langle a \rangle_{ij})$ is defined by $\langle a \rangle_{ii} = |a_{ii}|$ for $i = 1, \dots, n$, and $\langle a \rangle_{ij} = -|a_{ij}|$ for $i \neq j$, $i, j = 1, \dots, n$. In particular, an H -matrix with positive diagonal entries is called an H_+ -matrix [28]. For an H -matrix A , it is nonsingular and $|A^{-1}| \leq \langle A \rangle^{-1}$; meanwhile, D is nonsingular and $\rho(|D|^{-1}|B|) < 1$, where $D, -B$ are the diagonal and off-diagonal parts of A , respectively, $\rho(\cdot)$ is the spectral radius of a matrix [29].

Now, we are ready to establish the convergence theory for Algorithm 2.2 when A is an H_+ -matrix and $A = M_1 - N_1 = M_2 - N_2$ are two H -compatible splittings of A , which means that $\langle A \rangle = \langle M_1 \rangle - |N_1| = \langle M_2 \rangle - |N_2|$.

Since $f(z)$ in problem (1.1) is supposed to be a diagonal differentiable function, its Jacobian J is a diagonal matrix, more specifically,

$$J := \text{diag} \left(\frac{df_1}{dz_1}, \dots, \frac{df_n}{dz_n} \right).$$

Further, as assumed in [27], J satisfies $0 \leq df_i/dz_i \leq \bar{j}_i$, $1 \leq i \leq n$, where $\bar{j}_i, i = 1, 2, \dots, n$, are all constants, thus

$$0 \leq J \leq \text{diag}(\bar{j}_1, \dots, \bar{j}_n) := \bar{J}. \tag{3.1}$$

Suppose x^* is the fixed point of (2.3), i.e.,

$$\begin{cases} (\Omega + M_1)x^* = N_1x^* + (\Omega - A)|x^*| - \gamma(q + f(z^*)), \\ (\Omega + M_2)x^* = N_2x^* + (\Omega - A)|x^*| - \gamma(q + f(z^*)), \end{cases} \tag{3.2}$$

subtracting (3.2) from (2.2) yields

$$\begin{cases} (\Omega + M_1) \left(x^{(k+\frac{1}{2})} - x^* \right) \\ \quad = N_1(x^{(k)} - x^*) + (\Omega - A) (|x^{(k)}| - |x^*|) - \gamma (f(z^{(k)}) - f(z^*)), \\ (\Omega + M_2) \left(x^{(k+1)} - x^* \right) \\ \quad = N_2 \left(x^{(k+\frac{1}{2})} - x^* \right) + (\Omega - A) \left(|x^{(k+\frac{1}{2})}| - |x^*| \right) - \gamma \left(f(z^{(k+\frac{1}{2})}) - f(z^*) \right). \end{cases} \tag{3.3}$$

As f is diagonal differentiable, by mean value theorem it is easy to get

$$\begin{aligned} f(z^{(k)}) - f(z^*) &= f\left(\frac{1}{\gamma}(|x^{(k)}| + x^{(k)})\right) - f\left(\frac{1}{\gamma}(|x^*| + x^*)\right) \\ &= \frac{1}{\gamma} J^{(k)} \cdot (|x^{(k)}| - |x^*| + x^{(k)} - x^*), \\ f(z^{(k+\frac{1}{2})}) - f(z^*) &= f\left(\frac{1}{\gamma}(|x^{(k+\frac{1}{2})}| + x^{(k+\frac{1}{2})})\right) - f\left(\frac{1}{\gamma}(|x^*| + x^*)\right) \\ &= \frac{1}{\gamma} J^{(k+\frac{1}{2})} \cdot (|x^{(k+\frac{1}{2})}| - |x^*| + x^{(k+\frac{1}{2})} - x^*), \end{aligned}$$

where $J^{(k)} := J(\xi^{(k)})$ and $J^{(k+\frac{1}{2})} := J(\xi^{(k+\frac{1}{2})})$ are the Jacobian of $f(z)$ at point $\xi^{(k)}$ and $\xi^{(k+\frac{1}{2})}$, respectively. Here, $\xi^{(k)}$ is a vector between $z^{(k)}$ and z^* , $\xi^{(k+\frac{1}{2})}$ is a vector between $z^{(k+\frac{1}{2})}$ and z^* . Then (3.3) becomes

$$\begin{cases} (\Omega + M_1)(x^{(k+\frac{1}{2})} - x^*) \\ \quad = (N_1 - J^{(k)})(x^{(k)} - x^*) + (\Omega - A - J^{(k)})(|x^{(k)}| - |x^*|), \\ (\Omega + M_2)(x^{(k+1)} - x^*) \\ \quad = (N_2 - J^{(k+\frac{1}{2})})(x^{(k+\frac{1}{2})} - x^*) + (\Omega - A - J^{(k+\frac{1}{2})})(|x^{(k+\frac{1}{2})}| - |x^*|). \end{cases} \quad (3.4)$$

Since A is an H_+ -matrix and $A = M_1 - N_1 = M_2 - N_2$ are two H -compatible splittings of A , it follows that

$$\langle A \rangle \leq \langle M_1 \rangle \leq \text{diag}(M_1), \quad \langle A \rangle \leq \langle M_2 \rangle \leq \text{diag}(M_2),$$

from which $\Omega + M_1$ and $\Omega + M_2$ are H_+ -matrices; meanwhile,

$$|(\Omega + M_1)^{-1}| \leq (\Omega + \langle M_1 \rangle)^{-1}, \quad |(\Omega + M_2)^{-1}| \leq (\Omega + \langle M_2 \rangle)^{-1}.$$

Multiplying $(\Omega + M_1)^{-1}$ from the left on both sides of the first equation in (3.4) and then taking absolute values on both sides, we get

$$\begin{aligned} |x^{(k+\frac{1}{2})} - x^*| &\leq |(\Omega + M_1)^{-1}| (|N_1 - J^{(k)}| + |\Omega - A - J^{(k)}|) |x^{(k)} - x^*| \\ &\leq (\Omega + \langle M_1 \rangle)^{-1} (|N_1| + J^{(k)} + |\Omega - A - J^{(k)}|) |x^{(k)} - x^*| \\ &= \mathcal{L}_1 |x^{(k)} - x^*|, \end{aligned}$$

where $\mathcal{L}_1 := (\Omega + \langle M_1 \rangle)^{-1} (|N_1| + J_1 + |\Omega - A - J_1|)$, $J_1 := J^{(k)}$. Similarly, it is obtained from the second equation in (3.4) that

$$\begin{aligned} |x^{(k+1)} - x^*| &\leq (\Omega + \langle M_2 \rangle)^{-1} (|N_2| + J^{(k+\frac{1}{2})} + |\Omega - A - J^{(k+\frac{1}{2})}|) |x^{(k+\frac{1}{2})} - x^*| \\ &= \mathcal{L}_2 |x^{(k+\frac{1}{2})} - x^*|, \end{aligned}$$

where $\mathcal{L}_2 := (\Omega + \langle M_2 \rangle)^{-1} (|N_2| + J_2 + |\Omega - A - J_2|)$, $J_2 := J^{(k+\frac{1}{2})}$. Therefore,

$$|x^{(k+1)} - x^*| \leq \mathcal{L}_2 \mathcal{L}_1 |x^{(k)} - x^*|.$$

Evidently, Algorithm 2.2 converges if $\rho(\mathcal{L}_2\mathcal{L}_1) < 1$. A sufficient condition for the convergence of Algorithm 2.2 can be given in the following theorem.

Theorem 3.1. *Let $A \in \mathbb{R}^{n \times n}$ be an H_+ -matrix, $A = M_1 - N_1 = M_2 - N_2$ be two H -compatible splittings of A , the Jacobian of $f(z)$ satisfies (3.1). Assume the positive diagonal matrix Ω satisfies $\Omega \geq D + \bar{J}$. Then, the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Algorithm 2.2 converges to the solution z^* of problem (1.1) for any initial vector $x^{(0)} \in \mathbb{R}^n$.*

Proof. To prove that $\{z^{(k)}\}_{k=0}^{+\infty}$ converges to the solution z^* of nonlinear complementarity problems (1.1), it is only needed to prove that $\{x^{(k)}\}_{k=0}^{+\infty}$ converges to x^* .

Since A is an H_+ -matrix, which means $\langle A \rangle$ is an M -matrix, from the property of an M -matrix, we know that there exists a positive vector $u > 0$ such that $\langle A \rangle u > 0$; see [29]. Thus

$$\begin{aligned} \mathcal{L}_1 u &= (\Omega + \langle M_1 \rangle)^{-1} (|N_1| + J_1 + |\Omega - A - J_1|) u \\ &= (\Omega + \langle M_1 \rangle)^{-1} (|N_1| + J_1 + \Omega - J_1 - \langle A \rangle) u \\ &= (\Omega + \langle M_1 \rangle)^{-1} (\Omega + \langle M_1 \rangle - \langle M_1 \rangle + |N_1| - \langle A \rangle) u \\ &= (\Omega + \langle M_1 \rangle)^{-1} (\Omega + \langle M_1 \rangle - 2\langle A \rangle) u \\ &= u - 2(\Omega + \langle M_1 \rangle)^{-1} \langle A \rangle u, \end{aligned}$$

where we use the facts that $\Omega \geq D + \bar{J} \geq D + J_1$ and $\langle A \rangle = \langle M_1 \rangle - |N_1|$. Combining $\langle A \rangle u > 0$ and $(\Omega + \langle M_1 \rangle)^{-1}$ is a nonnegative matrix without zero rows, it follows that

$$\mathcal{L}_1 u < u.$$

Note that the positive vector u is only dependent on the matrix A but not on the splittings $A = M_1 - N_1 = M_2 - N_2$, a similar argument applied to matrix \mathcal{L}_2 can lead to

$$\mathcal{L}_2 \mathcal{L}_1 u < \mathcal{L}_2 u < u,$$

which concludes that $\rho(\mathcal{L}_2\mathcal{L}_1) < 1$; see [30]. This completes the proof. \square

Remark 3.1. When $f(z) \equiv 0$, the above theorem is just the special case with $l = 1$ in [21] where the convergence property of the two-step modulus-based synchronous multisplitting iteration method for linear complementarity problems was studied.

From the above analysis, it is found that the convergence result of two-step modulus-based matrix splitting iteration methods for nonlinear complementarity problems (1.1) is similar to that for linear complementarity problems after dealing with the Jacobian matrix of the nonlinear term $f(z)$ properly. With analysis similar to Theorem 4.4 in [21] when $l = 1$, the following convergence theorem for two-step modulus-based accelerated overrelaxation methods can be obtained.

Theorem 3.2. *Let $A \in \mathbb{R}^{n \times n}$ be an H_+ -matrix with $A = D - L - U := D - B$, where D , $-L$ and $-U$ are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of A , respectively, the Jacobian of $f(z)$ satisfies (3.1). Assume that Ω is a positive diagonal matrix*

satisfying $\Omega \geq D + \bar{J}$. Then, for any initial vector, the two-step modulus-based accelerated overrelaxation iteration method for nonlinear complementarity problems (1.1) will converge when

$$0 < \beta \leq \omega < \frac{1}{\rho(D^{-1}|B|)}.$$

Remark 3.2. Notice that when A is an H_+ -matrix, $\rho(D^{-1}|B|) < 1$ always holds true, thus the upper bound for ω in Theorem 3.2 is larger than 1. Therefore, the two-step modulus-based successive overrelaxation method and the two-step modulus-based Gauss-Seidel method for nonlinear complementarity problems (1.1) are both convergent for arbitrary initial vectors.

4. Numerical experiments

Numerical experiments are given in this section to illustrate the efficiency of the proposed method and to verify the convergence theory established above. In all the following numerical experiments, the initial vector is chosen to be zero and $\gamma = 1$. Since the complementarity condition $z^T(Az + q + f(z)) = 0$ is equivalent to $\|\min(Az^{(k)} + q + f(z^{(k)}), z^{(k)})\|_2 = 0$, iterations are terminated when the norm of the residual vector (denoted by ‘RES’)

$$\text{RES}(z^{(k)}) := \|\min(Az^{(k)} + q + f(z^{(k)}), z^{(k)})\|_2$$

satisfies $\text{RES} \leq 10^{-5}$, or k reaches the maximal number of iteration steps, which is 1000 in our paper. All the computations are performed in MATLAB with double machine precision where the CPU is 2.40 GHz and the memory is 4.00 GB.

Example 4.1. Let m be a given positive integer, $n = m^2$. Choose A in (1.1) to be a block upper tridiagonal matrix as follows:

$$A = \begin{pmatrix} S & -I & -I & & \\ & S & -I & \ddots & \\ & & S & \ddots & -I \\ & & & \ddots & -I \\ & & & & S \end{pmatrix} \in \mathbb{R}^{n \times n},$$

where $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m}$ is a tridiagonal matrix. Let $q = (1, -1, \dots, 1, (-1)^{n-1})^T \in \mathbb{R}^n$ and

$$f(z) = \left(\sqrt{z_1^2 + 0.25}, \sqrt{z_2^2 + 0.25}, \dots, \sqrt{z_n^2 + 0.25} \right)^T \in \mathbb{R}^n.$$

The matrix A in Example 4.1 is an H_+ -matrix. In actual implementation, the parameter matrix Ω is chosen to be $D + I$ in Example 4.1 for both the modulus-based successive overrelaxation method and the two-step modulus-based successive overrelaxation method, where D is the diagonal matrix of A , I is the identity matrix.

Table 1: The optimal parameters ω^* for MSOR and TMSOR in Example 4.1.

m	ω		0.8	0.9	1.0	1.1*	1.2	1.3	1.4
256	MSOR	IT	23	21	20	19*	19	20	21
		CPU	0.234	0.203	0.203	0.187*	0.188	0.203	0.219
	TMSOR	IT	10	9	8	8*	8	8	9
		CPU	0.188	0.172	0.156	0.141*	0.156	0.156	0.172

Table 2: Numerical results for Example 4.1.

m		MGS	TMGS	MSOR	TMSOR
256	IT	20	8	19	8
	CPU	0.203	0.156	0.187	0.141
	RES	4.62e-06	4.87e-06	5.02e-06	3.30e-06
512	IT	20	8	19	8
	CPU	0.890	0.719	0.860	0.688
	RES	9.26e-06	7.36e-06	8.51e-06	4.78e-06
1024	IT	21	9	20	8
	CPU	3.798	3.078	3.625	2.782
	RES	7.76e-06	1.80e-06	6.36e-06	6.84e-06
2048	IT	22	9	21	8
	CPU	19.096	14.767	18.096	14.064
	RES	6.49e-06	2.69e-06	4.78e-06	9.76e-06

In Table 1, the number of iteration steps (denoted by ‘IT’) and the elapsed CPU time in seconds (denoted by ‘CPU’) are listed for the modulus-based successive overrelaxation iteration method and the two-step modulus-based successive overrelaxation iteration method when parameter ω varies from 0.8 to 1.4 with $m = 256$. The optimal parameters ω^* is chosen firstly to minimize the number of iteration steps. When the number of iteration steps are the same, then we choose ω^* to minimize the elapsed CPU time.

From Table 1, it is seen that for Example 4.1, the optimal parameter $\omega^* = 1.1$ for both the modulus-based successive overrelaxation iteration method and the two-step modulus-based successive overrelaxation iteration method when $m = 256$. From experiments, it is found that this phenomenon also happens when m is increasing. In the following, we choose $\omega^* = 1.1$ for both the modulus-based successive overrelaxation iteration method and the two-step modulus-based successive overrelaxation iteration method.

In Table 2, the number of iteration steps, the elapsed CPU time in seconds and the residual for four methods are listed respectively when m is varying.

From Table 2, it is observed that with the same dimension, the number of iteration steps for two-step modulus-based matrix splitting method is less than half of that for modulus-based matrix splitting method, and the two-step modulus-based matrix splitting method costs less CPU time. Meanwhile, the CPU time increases when the problem size $n = m^2$ increases for all methods, while the number of the iteration steps changes few.

Table 3: The optimal parameters ω^* for MSOR and TMSOR in Example 4.2.

m	ω		0.8	0.9	1.0	1.1	1.2*	1.3	1.4
256	MSOR	IT	24	22	20	19	17*	18	20
		CPU	0.312	0.281	0.235	0.235	0.218*	0.235	0.235
	TMSOR	IT	11	10	9	8	8*	8	9
		CPU	0.265	0.234	0.203	0.203	0.188*	0.203	0.219

Table 4: Numerical results for Example 4.2.

m		MGS	TMGS	MSOR	TMSOR
256	IT	20	9	17	8
	CPU	0.235	0.203	0.218	0.188
	RES	7.63e-06	5.48e-06	9.76e-06	1.94e-06
512	IT	21	10	18	8
	CPU	1.109	1.015	0.969	0.812
	RES	6.73e-06	1.66e-06	7.44e-06	7.13e-06
1024	IT	22	10	19	8
	CPU	4.735	4.094	4.141	3.484
	RES	5.86e-06	3.37e-06	5.62e-06	7.47e-06
2048	IT	23	10	20	9
	CPU	22.206	18.830	19.502	16.533
	RES	5.07e-06	6.79e-06	4.23e-06	1.54e-06

Example 4.2. Let m be a given positive integer, $n = m^2$. Choose $A = \hat{A} + 4I$ in (1.1), where \hat{A} is the same as the matrix A defined in Example 4.1. Let $q = (1, -1, \dots, (-1)^{n-1})^T \in \mathbb{R}^n$ and

$$f(z) = (-\operatorname{arccot}(z_1 + 1), -\operatorname{arccot}(z_2 + 1), \dots, -\operatorname{arccot}(z_n + 1))^T \in \mathbb{R}^n.$$

The matrix A in Example 4.2 is also an H_+ -matrix. We choose $\Omega = D + \frac{1}{2}I$ for the modulus-based successive overrelaxation method and the two-step modulus-based successive overrelaxation method in Example 4.2.

In Table 3, the number of iteration steps and the elapsed CPU time in seconds are listed for the modulus-based successive overrelaxation iteration method and the two-step modulus-based successive overrelaxation iteration method when parameter ω varies from 0.8 to 1.4 with $m = 256$.

From Table 3, it is seen that for the successive overrelaxation iteration methods both the number of iteration steps and the elapsed CPU time are minimized when $\omega^* = 1.2$ with $m = 256$. Note that the optimal parameters ω^* may change a little when m is increasing. In the following, we choose the optimal parameters ω^* experimentally for both the modulus-based successive overrelaxation iteration method and the two-step modulus-based successive overrelaxation iteration method.

In Table 4, the number of iteration steps, the elapsed CPU time in seconds and the residual for four methods are listed respectively when m is varying.

From Table 4, it is clear that the CPU time increases when the problem size $n = m^2$ increases for all methods, while the number of the iteration steps changes few. Further, the number of iteration steps for all two-step modulus-based matrix splitting methods is less than half of that for modulus-based matrix splitting methods, and also two-step modulus-based matrix splitting methods cost less CPU time.

5. Conclusions

Two-step modulus-based matrix splitting iteration methods for a class of nonlinear complementarity problems were proposed and their convergence theories were studied when the system matrix is an H_+ -matrix. Numerical experiments further verified the convergence properties of the proposed method.

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