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# New Perturbation Bounds Analysis of a Kind of Generalized Saddle Point Systems

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**Abstract.** In this paper we consider new perturbation bounds analysis of a kind of generalized saddle point systems. We provide perturbation upper bounds for the solutions of generalized saddle point systems, which extend the corresponding results in [W.-W. Xu, W. Li, *New perturbation analysis for generalized saddle point systems*, Calcolo., 46(2009), pp. 25-36] to more general cases.

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**Key words**: Generalized saddle point system, perturbation bounds analysis.

## 1. Introduction

The saddle point system appears in scientific and engineering applications, such as, aeronautics, the mixed finite element solution of the Navier-Stokes, the Maxwell equations, electromagnetics and data fitting et. al. Numerical methods and perturbation bounds analysis for solving the saddle point system studied in some literatures. For details, please see [2-15] and the references therein. Recently, Xu et. al. in [1] considered perturbation bounds of the following generalized saddle point systems:

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A\in\mathscr{R}^{m\times m}$, $B\in\mathscr{R}^{n\times m}$, and $C\in\mathscr{R}^{n\times n}$, $n\le m$ (possibly $n\ll m$). This kind of system arises in many application problems, e.g., see [1]. As we know, a number of literatures deal with the solvers of the saddle point problem (1.1) with $C\neq 0$. Due to practical applications, perturbation analysis of the saddle point problem (1.1) should be discussed and the perturbation bounds and condition numbers for the system (1.1) are derived.

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In this paper we will extend System (1.1) to the more generalized saddle point system and consider perturbation upper bound for the solutions of this system:

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.2}$$

where $A\in\mathscr{R}^{m\times m}$, $B\in\mathscr{R}^{n\times m}$, $D\in\mathscr{R}^{m\times n}$ and $C\in\mathscr{R}^{n\times n}$, $x\in\mathscr{R}^{m}$, $y\in\mathscr{R}^{n}$, $n\le m$ (possibly $n\ll m$). Let $\mathscr{A}$ be the coefficient matrix of (1.2) and assume that $\mathscr{A}$ is nonsingular. The non-singularity conditions of $\mathscr{A}$ can be referred in Lemma 2.1 of [15]. Obviously, when $D=B^T$ in (1.2), System (1.2) reduces to System (1.1). We note that the perturbation bounds analysis for the solutions $x$ and $y$ of the system (1.2) have not discussed so far. By this motivation, we will consider this problem in the paper.

Let the perturbed system of (1.2) be as follows:

$$(\mathscr{A}+\Delta\mathscr{A})\begin{pmatrix} x+\Delta x \\ y+\Delta y \end{pmatrix}=\begin{pmatrix} A+\Delta A & D+\Delta D \\ B+\Delta B & C+\Delta C \end{pmatrix}\begin{pmatrix} x+\Delta x \\ y+\Delta y \end{pmatrix}=\begin{pmatrix} f+\Delta f \\ g+\Delta g \end{pmatrix}.$$

Throughout the paper, we always assume that

$$\begin{aligned} &\|\Delta A\|_F\le\epsilon\mathscr{D}_1, \qquad \|\Delta B\|_F\le\epsilon\mathscr{D}_2, \qquad \|\Delta C\|_F\le\epsilon\mathscr{D}_3,\\ &\|\Delta D\|_F\le\epsilon\sigma_1, \qquad \|\Delta f\|_2\le\epsilon\mathscr{D}_4, \qquad \|\Delta g\|_2\le\epsilon\mathscr{D}_5, \end{aligned} \tag{1.3}$$

and let

$$\delta=(\delta_1,\ \delta_2,\ \delta_3)^T, \qquad \hat{\delta}=(\hat{\delta}_1,\hat{\delta}_2)^T, \tag{1.4}$$

where

$$\begin{aligned} \epsilon>0, \quad &\delta_1=\sqrt{\mathscr{D}_1^2+\mathscr{D}_2^2}, \qquad \delta_2=\sqrt{\sigma_1^2+\mathscr{D}_3^2}, \qquad \delta_3=\sqrt{\mathscr{D}_4^2+\mathscr{D}_5^2},\\ &\hat{\delta}_1=\sqrt{\mathscr{D}_1^2+\mathscr{D}_2^2+\sigma_1^2+\mathscr{D}_3^2}, \qquad \hat{\delta}_2=\sqrt{\mathscr{D}_4^2+\mathscr{D}_5^2}. \end{aligned}$$

Here $\|\cdot\|_F$ denotes the Frobinus-norm.

The rest of the paper is organized as follows. In Section 2 we give some definitions, notations and useful lemmas to deduce the main results. In Section 3 we give perturbation bounds for the solutions of a kind of generalized saddle point systems. In Section 4 we give numerical examples to illustrate our results.

## 2. Preliminaries

We briefly give some useful lemmas in order to deduce our main results.

**Lemma 2.1.** *If $\mathscr{A}$ is nonsingular, then*

*i)* $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}=\mathscr{H}\theta+\mathscr{A}^{-1}(P,Q)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix},$

*ii)* $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}=\bar{\mathscr{H}}\bar{\theta}+\mathscr{A}^{-1}\Delta\mathscr{A}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix},$

*where*

$$\mathscr{H}=\mathscr{A}^{-1}\left(x^T\otimes I_{m+n},\ y^T\otimes I_{m+n},\ I_{m+n}\right),\ \ \theta=(\theta_1^T,\theta_2^T,\theta_3^T)^T,\ \ \theta_1=vec(P),\ \ \theta_2=vec(Q),$$
$$P=\begin{pmatrix}-\Delta A\\-\Delta B\end{pmatrix},\quad Q=\begin{pmatrix}-\Delta D\\-\Delta C\end{pmatrix},\quad \theta_3=\bar{\theta}_2=L=\begin{pmatrix}\Delta f\\\Delta g\end{pmatrix},$$
$$\bar{\mathscr{H}}=\mathscr{A}^{-1}\left((x^T,y^T)\otimes I_{m+n},I_{m+n}\right),\quad \bar{\theta}=\left(\bar{\theta}_1^T,\bar{\theta}_2^T\right)^T,\quad \bar{\theta}_1=vec(-\Delta\mathscr{A}).$$

*Proof.* Let $S_A=C-BA^{-1}D$ and we note that if $A$ is nonsingular, then $S_A$ is also nonsingular. Furthermore,

$$\mathscr{A}^{-1}=\begin{pmatrix}A^{-1}+A^{-1}DS_A^{-1}BA^{-1} & -A^{-1}DS_A^{-1}\\ -S_A^{-1}BA^{-1} & S_A^{-1}\end{pmatrix}. \tag{2.1}$$

It follows that

$$\begin{pmatrix}\Delta x\\\Delta y\end{pmatrix}=\mathscr{A}^{-1}\left(x^T\otimes I_{m+n},\ y^T\otimes I_{m+n},\ I_{m+n}\right)\begin{pmatrix}vec(P)\\vec(Q)\\L\end{pmatrix}-\mathscr{A}^{-1}\begin{pmatrix}\Delta A & \Delta D\\\Delta B & \Delta C\end{pmatrix}\begin{pmatrix}\Delta x\\\Delta y\end{pmatrix}.$$

Then the result follows immediately from the definitions of $\mathscr{H}$, $\theta$, $\bar{\mathscr{H}}$, $\bar{\theta}$, $P$, $Q$ and $L$ defined by this lemma. This completes the proof. $\square$

We may partition (2.1) into the following block matrix

$$\mathscr{A}^{-1}\equiv\begin{pmatrix}R\\T\end{pmatrix}, \tag{2.2}$$

i.e., $R=(A^{-1}+A^{-1}DS_A^{-1}BA^{-1},\ -A^{-1}DS_A^{-1})$, $T=(-S_A^{-1}BA^{-1},\ S_A^{-1})$. Then from (2.2) we have the following lemma.

**Lemma 2.2.** *Let $R$ and $T$ be given in* (2.2)*. Then it holds that*

*i)* $\Delta x=\mathscr{H}_x\theta+RP\Delta x+RQ\Delta y,\ \Delta y=\mathscr{H}_y\theta+TP\Delta x+TQ\Delta y,$

*ii)* $\Delta x=\bar{\mathscr{H}}_x\bar{\theta}+RP\Delta x+RQ\Delta y,\ \Delta y=\bar{\mathscr{H}}_y\bar{\theta}+TP\Delta x+TQ\Delta y,$

*where*

$$\begin{aligned}&\mathscr{H}_x=(H_{x1},\,H_{x2},\,H_{x3}),\quad H_{x1}=x^T\otimes R,\quad H_{x2}=y^T\otimes R,\\&H_{x3}=R;\quad \mathscr{H}_y=\left(H_{y1},\,H_{y2},\,H_{y3}\right),\quad H_{y1}=x^T\otimes T,\quad H_{y2}=y^T\otimes T,\\&H_{y3}=T,\quad \bar{\mathscr{H}}_x=(\bar{H}_{x1},\,\bar{H}_{x2}),\quad \bar{\mathscr{H}}_y=(\bar{H}_{y1},\,\bar{H}_{y2}),\quad \bar{H}_{x1}=(x^T,y^T)\otimes R,\\&\bar{H}_{x2}=R,\quad \bar{H}_{y1}=(x^T,y^T)\otimes T,\quad \bar{H}_{y2}=T\end{aligned}$$

*and $\theta$, $\bar{\theta}$, $P$, $Q$ are given by Lemma* 2.1.

*Proof.* The desired results follow from Lemma 2.1. $\square$

## 3. Perturbation Bounds

We note that $x$ and $y$ in (1.2) have different practical meanings. In this section, we will present the bounds on perturbations $\Delta x$ and $\Delta y$ for the generalized saddle point system (1.2).

**Theorem 3.1.** *Let $\epsilon$, $\delta$, $\delta_1$, $\delta_2$ be given in* (1.3) *and* (1.4)*, and let $R$, $T$ be given in* (2.2)*. Assume $\epsilon\delta_1\|R\|_2 \le \eta_1$, $\epsilon\delta_2\|T\|_2 \le \eta_2$, $0 < \eta_1, \eta_2 < 1$ and $\epsilon \ll 1$. Then we have*

$$\|\Delta x\|_2 \le \frac{1}{1-\eta_1}\epsilon\Gamma_x + \frac{1}{(1-\eta_1)(1-\eta_2)}\epsilon^2\delta_2\|Z\|_2\Gamma_y + \mathscr{O}(\epsilon^3),$$
$$\|\Delta y\|_2 \le \frac{1}{1-\eta_2}\epsilon\Gamma_y + \frac{1}{(1-\eta_1)(1-\eta_2)}\epsilon^2\delta_1\|Y\|_2\Gamma_x + \mathscr{O}(\epsilon^3),$$

*where $\Gamma_x = min\{\|\mathscr{H}_x\|_2\|\delta\|_2, \sqrt{\delta^T\tilde{H}_x\delta}\}$, $\Gamma_y = min\{\|\mathscr{H}_y\|_2\|\delta\|_2, \sqrt{\delta^T\tilde{H}_y\delta}\}$, $\mathscr{H}_x$, $\mathscr{H}_y$, $H_{xi}$, $H_{yi}$ are given in Lemma 2.2, and $\tilde{H}_x = (\tilde{h}_{ij}(x)) \in R^{3\times 3}$ is a matrix with entries $\tilde{h}_{ij}(x) = \|H_{xi}^T H_{xj}\|_2$, $\tilde{H}_y = (\tilde{h}_{ij}(y)) \in R^{3\times 3}$ is a matrix with entries $\tilde{h}_{ij}(y) = \|H_{yi}^T H_{yj}\|_2$, $i,j = 1,2,3$.*

*Proof.* It follows from Lemma 2.2 i) that

$$\Delta x = \mathscr{H}_x\theta + RP\Delta x + RQ\Delta y, \qquad \Delta y = \mathscr{H}_y\theta + TP\Delta x + TQ\Delta y.$$

Then

$$\|\Delta x\|_2 \le \|\mathscr{H}_x\theta\|_2 + \|R\|_2\|P\|_2\|\Delta x\|_2 + \|R\|_2\|Q\|_2\|\Delta y\|_2, \tag{3.1}$$

where $\theta$ is given by Lemma 2.2. Hence,

$$\|\mathscr{H}_x\theta\|_2 \le \|\mathscr{H}_x\|_2\left(\|P\|_F^2 + \|Q\|_F^2 + \|L\|_F^2\right)^{\frac{1}{2}} \le \epsilon\|\mathscr{H}_x\|_2\|\delta\|_2. \tag{3.2}$$

Meanwhile,

$$\begin{aligned}\|\mathscr{H}_x\theta\|_2^2 =& \theta^T\mathscr{H}_x^T\mathscr{H}_x\theta = \sum_{i=1}^{3}\sum_{j=1}^{3}\theta_i^T\mathscr{H}_{xi}^T\mathscr{H}_{xj}\theta_j \\ \le& \sum_{i=1}^{3}\sum_{j=1}^{3}\|\mathscr{H}_{xi}^T\mathscr{H}_{xj}\|_2\|\theta_i\|_2\|\theta_j\|_2 \\ \le& \epsilon^2\sum_{i=1}^{3}\sum_{j=1}^{3}\|\mathscr{H}_{xi}^T\mathscr{H}_{xj}\|_2\delta_i\delta_j := \epsilon^2\delta^T\tilde{H}_x\delta,\end{aligned} \tag{3.3}$$

where $\tilde{H}_x = (\tilde{h}_{ij}(x)) \in \mathbf{R}^{3\times 3}$ is the matrix with entries $\tilde{h}_{ij}(x) = \|H_{xi}^T H_{xj}\|_2$. Combining (3.1)-(3.3) we know

$$\|\mathscr{H}_x\theta\|_2 \le \epsilon\min\left\{\|\mathscr{H}_x\|_2\|\delta\|_2, \sqrt{\delta^T\tilde{H}_x\delta}\right\} := \epsilon\Gamma_x. \tag{3.4}$$

Substituting (3.4) into (3.1) leads to

$$\|\Delta x\|_2 \le \epsilon\Gamma_x + \epsilon\delta_1\|R\|_2\|\Delta x\|_2 + \epsilon\delta_2\|R\|_2\|\Delta y\|_2,$$

which together with the assumptions that $\epsilon\delta_1\|R\|_2 < 1$ yields

$$\|\Delta x\|_2 \le \frac{\epsilon(\Gamma_x + \delta_2\|R\|_2\|\Delta y\|_2)}{1-\epsilon\delta_1\|R\|_2}. \tag{3.5}$$

Similarly, we obtain

$$\|\Delta y\|_2 \le \frac{\epsilon(\Gamma_y + \delta_1\|T\|_2\|\Delta x\|_2)}{1-\epsilon\delta_2\|T\|_2}, \tag{3.6}$$

where $\Gamma_y = \min\{\|\mathscr{H}_y\|_2\|\delta\|_2, \sqrt{\delta^T\tilde{H}_y\delta}\}$. Substituting (3.6) into (3.5) gives

$$\begin{aligned}\|\Delta x\|_2 \le & \frac{\epsilon\Gamma_x}{1-\epsilon\delta_1\|R\|_2} + \frac{\epsilon^2\delta_2\|R\|_2\Gamma_y}{(1-\epsilon\delta_1\|R\|_2)(1-\epsilon\delta_2\|T\|_2)} \\ & + \frac{\epsilon^2\delta_1\delta_2\|T\|_2\|R\|_2\|\Delta x\|_2}{(1-\epsilon\delta_1\|R\|_2)(1-\epsilon\delta_2\|T\|_2)}.\end{aligned} \tag{3.7}$$

Similarly, we obtain

$$\begin{aligned}\|\Delta y\|_2 \le & \frac{\epsilon\Gamma_y}{1-\epsilon\delta_2\|T\|_2} + \frac{\epsilon^2\delta_1\|T\|_2\Gamma_x}{(1-\epsilon\delta_2\|T\|_2)(1-\epsilon\delta_1\|R\|_2)} \\ & + \frac{\epsilon^2\delta_1\delta_2\|T\|_2\|R\|_2\|\Delta x\|_2}{(1-\epsilon\delta_1\|R\|_2)(1-\epsilon\delta_2\|T\|_2)}.\end{aligned} \tag{3.8}$$

Since $\epsilon\delta_1\|R\|_2 \le \eta_1$, $\epsilon\delta_2\|T\|_2 \le \eta_2$, we have $1-\epsilon\delta_1\|R\|_2 \in [1-\eta_1, 1)$, $1-\epsilon\delta_2\|T\|_2 \in [1-\eta_2, 1)$. Then from (3.7) and (3.8) we obtain the desired results. ☐

By the similar technique and Lemma 2.2 ii) we have the following results.

**Theorem 3.2.** *Let $\epsilon$, $\hat{\delta}$, $\hat{\delta}_1$, $\hat{\delta}_2$ be given in* (1.3) *and* (1.4)*, and let $R$, $T$ be given in* (2.2)*. Assume $\epsilon\hat{\delta}_1\|R\|_2 \le \hat{\eta}_1$, $\epsilon\hat{\delta}_2\|T\|_2 \le \hat{\eta}_2$, $0 < \hat{\eta}_1, \hat{\eta}_2 < 1$ and $\epsilon \ll 1$. Then we have*

$$\begin{aligned}\|\Delta x\|_2 \le & \frac{1}{1-\hat{\eta}_1}\epsilon\hat{\Gamma}_x + \frac{1}{(1-\hat{\eta}_1)(1-\hat{\eta}_2)}\epsilon^2\hat{\delta}_2\|Z\|_2\hat{\Gamma}_y + \mathcal{O}(\epsilon^3), \\ \|\Delta y\|_2 \le & \frac{1}{1-\hat{\eta}_2}\epsilon\hat{\Gamma}_y + \frac{1}{(1-\hat{\eta}_1)(1-\hat{\eta}_2)}\epsilon^2\hat{\delta}_1\|Y\|_2\hat{\Gamma}_x + \mathcal{O}(\epsilon^3),\end{aligned}$$

*where $\hat{\Gamma}_x = min\{\|\bar{\mathscr{H}}_x\|_2\|\hat{\delta}\|_2, \sqrt{\hat{\delta}^T\hat{H}_x\hat{\delta}}\}, \hat{\Gamma}_y = min\{\|\bar{\mathscr{H}}_y\|_2\|\hat{\delta}\|_2, \sqrt{\hat{\delta}^T\hat{H}_y\hat{\delta}}\}$, $\bar{\mathscr{H}}_x$, $\bar{\mathscr{H}}_y$, $\bar{\mathscr{H}}_{xi}$, $\bar{\mathscr{H}}_{yi}$ are given in Lemma* 2.2*, and $\hat{H}_x = (\hat{h}_{ij}(x)) \in R^{2\times 2}$ is a matrix with entries $\hat{h}_{ij}(x) = \|\bar{H}_{xi}^T\bar{H}_{xj}\|_2$, $\hat{H}_y = (\hat{h}_{ij}(y)) \in R^{2\times 2}$ is a matrix with entries $\hat{h}_{ij}(y) = \|\bar{H}_{yi}^T\bar{H}_{yj}\|_2$, $i,j = 1,2$.*

*Proof.* It follows from Lemma 2.2 ii) that

$$\Delta x = \bar{\mathscr{H}}_x\bar{\theta} + RP\Delta x + RQ\Delta y, \qquad \Delta y = \bar{\mathscr{H}}_y\bar{\theta} + TP\Delta x + TQ\Delta y.$$

Then

$$\|\Delta x\|_2 \le \|\bar{\mathscr{H}}_x\bar{\theta}\|_2 + \|R\|_2\|P\|_2\|\Delta x\|_2 + \|R\|_2\|Q\|_2\|\Delta y\|_2, \tag{3.9}$$

where $\bar{\theta}$ is given by Lemma 2.2. Hence,

$$\|\bar{\mathscr{H}}_x\bar{\theta}\|_2 \le \|\bar{\mathscr{H}}_x\|_2\|\bar{\theta}\|_2 \le \epsilon\|\bar{\mathscr{H}}_x\|_2\|\hat{\delta}\|_2. \tag{3.10}$$

Meanwhile,

$$\begin{aligned}\|\bar{\mathscr{H}}_x\bar{\theta}\|_2^2 =&\bar{\theta}^T\bar{\mathscr{H}}_x{}^T\bar{\mathscr{H}}_x\bar{\theta} = \sum_{i=1}^{2}\sum_{j=1}^{2}\bar{\theta}_i^T\bar{\mathscr{H}}_{xi}{}^T\bar{\mathscr{H}}_{xj}\bar{\theta}_j \\ \le&\sum_{i=1}^{2}\sum_{j=1}^{2}\|\bar{\mathscr{H}}_{xi}{}^T\bar{\mathscr{H}}_{xj}\|_2\|\bar{\theta}_i\|_2\|\bar{\theta}_j\|_2 \\ \le&\epsilon^2\sum_{i=1}^{2}\sum_{j=1}^{2}\|\bar{\mathscr{H}}_{xi}{}^T\bar{\mathscr{H}}_{xj}\|_2\hat{\delta}_i\hat{\delta}_j := \epsilon^2\hat{\delta}^T\hat{H}_x\hat{\delta},\end{aligned} \tag{3.11}$$

where $\hat{H}_x = (\hat{h}_{ij}(x)) \in \mathbf{R}^{2\times 2}$ is the matrix with entries $\hat{h}_{ij}(x) = \|\bar{H_{xi}}^T\bar{H_{xj}}\|_2$. Combining (3.10) and (3.11) we know

$$\|\bar{\mathscr{H}}_x\bar{\theta}\|_2 \le \epsilon\min\left\{\|\bar{\mathscr{H}}_x\|_2\|\hat{\delta}\|_2, \sqrt{\hat{\delta}^T\hat{H}_x\delta}\right\} := \epsilon\hat{\Gamma_x}. \tag{3.12}$$

Substituting (3.12) into (3.9) leads to

$$\|\Delta x\|_2 \le \epsilon\hat{\Gamma_x} + \epsilon\hat{\delta_1}\|R\|_2\|\Delta x\|_2 + \epsilon\hat{\delta_2}\|R\|_2\|\Delta y\|_2,$$

which together with the assumptions that $\epsilon\hat{\delta_1}\|R\|_2 < 1$ yields

$$\|\Delta x\|_2 \le \frac{\epsilon(\hat{\Gamma_x} + \hat{\delta_2}\|R\|_2\|\Delta y\|_2)}{1-\epsilon\hat{\delta_1}\|R\|_2}. \tag{3.13}$$

Similarly, we obtain

$$\|\Delta y\|_2 \le \frac{\epsilon(\hat{\Gamma_y} + \hat{\delta_1}\|T\|_2\|\Delta x\|_2)}{1-\epsilon\hat{\delta_2}\|T\|_2}, \tag{3.14}$$

where $\hat{\Gamma_y} = \min\{\|\bar{\mathscr{H}}_y\|_2\|\hat{\delta}\|_2, \sqrt{\hat{\delta}^T\hat{H}_y\hat{\delta}}\}$. Substituting (3.14) into (3.13) gives

$$\begin{aligned}\|\Delta x\|_2 \le&\frac{\epsilon\hat{\Gamma_x}}{1-\epsilon\hat{\delta_1}\|R\|_2} + \frac{\epsilon^2\hat{\delta_2}\|R\|_2\hat{\Gamma_y}}{(1-\epsilon\hat{\delta_1}\|R\|_2)(1-\epsilon\hat{\delta_2}\|T\|_2)} \\ &+ \frac{\epsilon^2\hat{\delta_1}\delta_2\|T\|_2\|R\|_2\|\Delta x\|_2}{(1-\epsilon\hat{\delta_1}\|R\|_2)(1-\epsilon\delta_2\|T\|_2)}.\end{aligned} \tag{3.15}$$

Similarly, we obtain

$$\begin{aligned}\|\Delta y\|_2 \le&\frac{\epsilon\hat{\Gamma_y}}{1-\epsilon\hat{\delta_2}\|T\|_2} + \frac{\epsilon^2\hat{\delta_1}\|T\|_2\hat{\Gamma_x}}{(1-\epsilon\hat{\delta_2}\|T\|_2)(1-\epsilon\hat{\delta_1}\|R\|_2)} \\ &+ \frac{\epsilon^2\hat{\delta_1}\hat{\delta_2}\|T\|_2\|R\|_2\|\Delta x\|_2}{(1-\epsilon\delta_1\|R\|_2)(1-\epsilon\hat{\delta_2}\|T\|_2)}.\end{aligned} \tag{3.16}$$

Since $\epsilon\hat{\delta_1}\|R\|_2 \le \hat{\eta_1}, \epsilon\hat{\delta_2}\|T\|_2 \le \hat{\eta_2}$, we have $1-\epsilon\hat{\delta_1}\|R\|_2 \in [1-\hat{\eta_1}, 1),\ 1-\epsilon\hat{\delta_2}\|T\|_2 \in [1-\hat{\eta_2}, 1)$. Then from (3.15) and (3.16) we obtain the desired results. $\square$

**Remark 3.1.** If in (2.2) $D = B^T$, then

$$\begin{aligned} &\sigma_1 = \mathscr{D}_2, \qquad \delta_2 = \sqrt{\sigma_1^2 + \mathscr{D}_3^2} = \sqrt{\mathscr{D}_2^2 + \mathscr{D}_3^2}, \\ &R = \left(A^{-1} + A^{-1}B^T S_A^{-1} B A^{-1},\ -A^{-1}B^T S_A^{-1}\right). \end{aligned}$$

Then the perturbation bounds of $\|\Delta x\|_2$ and $\|\Delta y\|_2$ in Theorem 3.1 reduce to the results in Theorem 3.2 of [1]. Therefore, Theorem 3.1 extends the scope of generalized saddle point systems in [1] about perturbation bounds of $\|\Delta x\|_2$ and $\|\Delta y\|_2$.

## 4. Numerical Example

In this section we will present numerical experiments for a model problem arising from the Navier-Stokes equations. The model problem involves a stabilized finite element discretization of the Navier-Stokes equations. We use the software toolkit for a two-dimensional leaky lid- driven cavity problem. Using this toolkit, we can easily apply the analysis from this paper to the stabilized Navier Stokes problem (Oseen case). We assess perturbation bounds of $\|\Delta x\|_2$ and $\|\Delta y\|_2$ in Theorems 3.1 and 3.2.

**Example 4.1.** For our experiments, we choose a $16 \times 16$ grid, viscosity parameter $\nu = 0.1$, and stabilization parameter $\beta = 0.25$. After removing the constant pressure mode, the system has 705 unknowns. Since multigrid cycles are actually matrix splittings, we use a number of multigrid V-cycles to define the splitting of the (1,1) block. For example, we set $\epsilon = 10^{-8}$, $\|\Delta A\|_F \le \epsilon$, $\|\Delta B\|_F \le \epsilon$, $\|\Delta C\|_F \le \epsilon$, $\|\Delta D\|_F \le \epsilon$, $\|\Delta f\|_2 \le \epsilon$, $\|\Delta g\|_2 \le \epsilon$ and $x = (1, \cdots, 0)^T$. All the runs were done in MATLAB 7.0 on a CPU 1.86GHZ and 1022MB memory computer. We denote perturbation bounds of $\|\Delta x\|_2, \|\Delta y\|_2$ from Theorem 3.1 by $d_1, d_2$ and the ones from Theorem 3.2 by $\hat{d}_1, \hat{d}_2$. From Tables 1 and 2 we can see the perturbation bounds of $\|\Delta x\|_2$ are in the order of $10^{-8}$ or so. Also, sometimes the bounds derived by Theorem 3.1 are smaller than the ones derived by Theorem 3.2 and sometimes the bounds derived by Theorem 3.2 are smaller than the ones derived by Theorem 3.1.

Table 1: Perturbation bounds of $\|\Delta x\|_2$ and $\|\Delta y\|_2$ by Theorem 3.1.

| $y$ | $d_1$ | $d_2$ |
|---|---|---|
| $(1,0,0,\cdots,0,0)^T$ | $1.4059e-8+\mathcal{O}(e-16)$ | $1.7924e-8+\mathcal{O}(e-16)$ |
| $(0,1,0,\cdots,0,0)^T$ | $2.1104e-8+\mathcal{O}(e-16)$ | $2.4820e-8+\mathcal{O}(e-16)$ |
| $(0,0,0,\cdots,0,1)^T$ | $1.8596e-8+\mathcal{O}(e-16)$ | $1.6713e-8+\mathcal{O}(e-16)$ |
| $(1,1,0,\cdots,0,0)^T$ | $2.2010e-8+\mathcal{O}(e-16)$ | $2.5036e-8+\mathcal{O}(e-16)$ |
| $(1,1,0,\cdots,0,0)^T$ | $2.3952e-8+\mathcal{O}(e-16)$ | $2.0048e-8+\mathcal{O}(e-16)$ |

Table 2: Perturbation bounds of $\|\Delta x\|_2$ and $\|\Delta y\|_2$ by Theorem 3.2.

| $y$ | $\hat{d}_1$ | $\hat{d}_2$ |
|---|---|---|
| $(1,0,0,\cdots,0,0)^T$ | $1.6402e-8+\mathcal{O}(e-16)$ | $1.7217e-8+\mathcal{O}(e-16)$ |
| $(0,1,0,\cdots,0,0)^T$ | $2.9238e-8+\mathcal{O}(e-16)$ | $2.8347e-8+\mathcal{O}(e-16)$ |
| $(0,0,0,\cdots,0,1)^T$ | $3.1405e-8+\mathcal{O}(e-16)$ | $3.5363e-8+\mathcal{O}(e-16)$ |
| $(0,1,0,\cdots,0,1)^T$ | $2.4502e-8+\mathcal{O}(e-16)$ | $2.4713e-8+\mathcal{O}(e-16)$ |
| $(1,1,0,\cdots,0,0)^T$ | $3.5519e-8+\mathcal{O}(e-16)$ | $3.7765e-8+\mathcal{O}(e-16)$ |
| $(0,0,\cdots,0,1,1)^T$ | $3.2154e-8+\mathcal{O}(e-16)$ | $3.49036e-8+\mathcal{O}(e-16)$ |

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