

## A New Fourth-Order Compact Off-Step Discretization for the System of 2D Nonlinear Elliptic Partial Differential Equations

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**Abstract.** This paper discusses a new fourth-order compact off-step discretization for the solution of a system of two-dimensional nonlinear elliptic partial differential equations subject to Dirichlet boundary conditions. New methods to obtain the fourth-order accurate numerical solution of the first order normal derivatives of the solution are also derived. In all cases, we use only nine grid points to compute the solution. The proposed methods are directly applicable to singular problems and problems in polar coordinates, which is a main attraction. The convergence analysis of the derived method is discussed in detail. Several physical problems are solved to demonstrate the usefulness of the proposed methods.

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### 1. Introduction

We consider the system of two-dimensional (2D) nonlinear elliptic partial differential equations (PDE)

$$L\mathbf{u} \equiv \mathbf{A} \frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{B} \frac{\partial^2 \mathbf{u}}{\partial y^2} = \mathbf{f} \quad (1.1)$$

defined in the domain  $\Omega = \{(x, y) | 0 < x, y < 1\}$  with boundary  $\partial\Omega$ , where

$$\begin{aligned} \mathbf{A} &= \text{diag} \left( A^{(i)}(x, y) \right) \in \mathbb{R}^{n \times n}, & \mathbf{B} &= \text{diag} \left( B^{(i)}(x, y) \right) \in \mathbb{R}^{n \times n}, \\ \mathbf{u} &= \left[ u^{(1)}, u^{(2)}, \dots, u^{(n)} \right]^t, & \mathbf{f} &= \left[ f^{(1)}, f^{(2)}, \dots, f^{(n)} \right]^t, \end{aligned}$$

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and  $t$  denotes the transpose of the matrix. We consider  $i = 1(1)n$  throughout this paper. Each  $f^{(i)}$  is a function of  $x, y, u^{(1)}, u^{(2)}, \dots, u^{(n)}, u_x^{(1)}, u_x^{(2)}, \dots, u_x^{(n)}, u_y^{(1)}, u_y^{(2)}, \dots, u_y^{(n)}$ ; and the system (1.1) is subject to the Dirichlet boundary conditions given by

$$u^{(i)}(x, y) = u_0^{(i)}(x, y), \quad (x, y) \in \partial\Omega \quad (1.2)$$

where  $u_0^{(i)}$  are continuous functions in  $\partial\Omega$ . In addition, Eqs. (1.1) are assumed to satisfy the ellipticity conditions  $A^{(i)}B^{(i)} > 0$  in  $\Omega$ . Further,  $\forall (x, y) \in \Omega$  we assume that

- (i)  $u^{(i)}(x, y) \in C^6$ ;
- (ii)  $A^{(i)}(x, y), B^{(i)}(x, y) \in C^4$ ;
- (iii)  $f^{(i)}(x, y, u^{(1)}, u^{(2)}, \dots, u^{(n)}, u_x^{(1)}, u_x^{(2)}, \dots, u_x^{(n)}, u_y^{(1)}, u_y^{(2)}, \dots, u_y^{(n)})$  is differentiable, and for  $j = 1, 2, \dots, n$
- (iv)  $\partial f^{(i)} / \partial u^{(j)} \geq 0$ ;
- (v)  $|\partial f^{(i)} / \partial u_x^{(j)}| \leq C$  and  $|\partial f^{(i)} / \partial u_y^{(j)}| \leq D$ ,

where  $C$  and  $D$  are positive constants and  $C^m$  denotes the set of all functions of  $x$  and  $y$  with partial derivatives up to order  $m$  continuous in  $\Omega$  [3]. Conditions (iii), (iv) and (v) guarantee the existence and uniqueness of the solution of the given boundary value problem.

The present paper is concerned with solving the system (1.1) of 2D nonlinear elliptic PDE with variable coefficients by a new compact 9-point fourth-order off-step finite difference discretization. Such systems of equations arise in various important mathematical models in science and engineering. For linear elliptic problems, there has been considerable work done on the development of high order compact schemes and the convergence of relevant iterative solution methods — e.g. [5–10]. Ananthakrishnaiah & Saldanha [4] framed a 13-point fourth-order compact scheme for the solution of a scalar nonlinear elliptic PDE, which was later extended to a system of equations [18]. A variety of high order compact schemes have been developed for the solution of 2D steady state Navier-Stokes (N-S) equations in stream function vorticity form in Cartesian coordinates — e.g. [1, 2, 19–21].

One of the present authors previously proposed fourth-order difference methods for 2D nonlinear elliptic boundary value problems with variable coefficients using only 9 grid points of a single compact cell, with application to the singular problems involving Poisson equation and the Navier-Stokes equations in polar coordinates [12]. Subsequently, fourth-order accurate estimates were developed for the first order normal derivatives ( $\partial u / \partial n$ ) [13]. However, these methods could not be applied to singular elliptic problems directly, due to terms such as  $1/r_{l-1}$  in polar coordinates that create difficulties at  $l = 1$  where  $r_0 = 0$ . In such cases, a suitable difference approximation valid at  $r = 0$  or a suitable modification at the singular point is required. Consequently, Mohanty and Singh [14] derived a new compact fourth-order discretization for the solution of singularly perturbed 2D nonlinear elliptic problems and the estimates of  $(\partial u / \partial n)$ , in an approach referred to as *off-step* discretization.

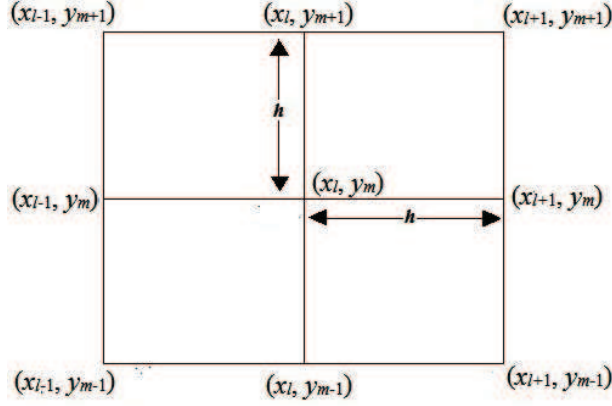


Figure 1: Single compact cell.

In this paper, we develop new fourth-order off-step discretizations for the solution of the system (1.1) and estimates of  $(\partial u^{(i)} / \partial n)$ , using 9 grid points of a single compact cell — cf. Fig. 1. In Section 2, we discuss the relevant proposed difference schemes, which are derived in Section 3. Under appropriate conditions, the fourth-order convergence of the method is established in Section 4. In Section 5, we consider numerical examples to illustrate and examine the accuracy of the methods for linear and nonlinear problems, including the singular problem mentioned earlier involving the 2D Poisson equation and steady state Navier-Stokes equations in polar coordinates.

## 2. Formulation of the Numerical Method

We adopt a rectangular grid on the domain  $\Omega$ , with spacing  $h > 0$  in both the  $x$  and  $y$  directions. The grid points are  $(x_l, y_m)$  for  $x_l = (l - 1)h$ ,  $y_m = (m - 1)h$ ;  $l, m = 1(1)N + 1$ , where  $Nh = 1$ . At each grid point  $(x_l, y_m)$ , let  $U_{l,m}^{(i)}$ ,  $A_{l,m}^{(i)}$  and  $B_{l,m}^{(i)}$  denote the exact values of  $u^{(i)}(x_l, y_m)$ ,  $A^{(i)}(x_l, y_m)$  and  $B^{(i)}(x_l, y_m)$  respectively. Similarly, at each grid point  $(x_l, y_m)$  we denote

$$A_{xl,m}^{(i)} = \frac{\partial A_{l,m}^{(i)}}{\partial x}, \quad A_{yl,m}^{(i)} = \frac{\partial A_{l,m}^{(i)}}{\partial y}, \quad A_{xxl,m}^{(i)} = \frac{\partial^2 A_{l,m}^{(i)}}{\partial x^2}, \quad \dots,$$

and the approximate solution value of  $u^{(i)}(x, y)$  by  $u_{l,m}^{(i)}$ . Thus at every grid point  $(x_l, y_m)$  each differential equation of the system (1.1) can be written

$$\begin{aligned} & A_{l,m}^{(i)} \frac{\partial^2 U_{l,m}^{(i)}}{\partial x^2} + B_{l,m}^{(i)} \frac{\partial^2 U_{l,m}^{(i)}}{\partial y^2} \\ &= f^{(i)}(x_l, y_m, U_{l,m}^{(1)}, U_{l,m}^{(2)}, \dots, U_{l,m}^{(n)}, U_{xl,m}^{(1)}, U_{xl,m}^{(2)}, \dots, U_{xl,m}^{(n)}, U_{yl,m}^{(1)}, U_{yl,m}^{(2)}, \dots, U_{yl,m}^{(n)}) \\ &\equiv F_{l,m}^{(i)}. \end{aligned}$$

For the fourth-order discretization of the system (1.1) subject to boundary conditions (1.2), we follow Chawla and Shivakumar [15] with the approximations

$$\overline{U}_{l\pm\frac{1}{2},m}^{(i)} = \frac{1}{2}(U_{l\pm 1,m}^{(i)} + U_{l,m}^{(i)}) \quad (2.1a)$$

$$\overline{U}_{l,m\pm\frac{1}{2}}^{(i)} = \frac{1}{2}(U_{l,m\pm 1}^{(i)} + U_{l,m}^{(i)}) \quad (2.1b)$$

$$\overline{U}_{xl,m}^{(i)} = \frac{1}{2h}(U_{l+1,m}^{(i)} - U_{l-1,m}^{(i)}) \quad (2.1c)$$

$$\overline{U}_{xl\pm\frac{1}{2},m}^{(i)} = \frac{1}{h}(\pm U_{l\pm 1,m}^{(i)} \mp U_{l,m}^{(i)}) \quad (2.1d)$$

$$\overline{U}_{xl,m\pm\frac{1}{2}}^{(i)} = \frac{1}{4h}(U_{l+1,m\pm 1}^{(i)} - U_{l-1,m\pm 1}^{(i)} + U_{l+1,m}^{(i)} - U_{l-1,m}^{(i)}) \quad (2.1e)$$

$$\overline{U}_{yl,m}^{(i)} = \frac{1}{2h}(U_{l,m+1}^{(i)} - U_{l,m-1}^{(i)}) \quad (2.1f)$$

$$\overline{U}_{yl\pm\frac{1}{2},m}^{(i)} = \frac{1}{4h}(U_{l\pm 1,m+1}^{(i)} - U_{l\pm 1,m-1}^{(i)} + U_{l,m+1}^{(i)} - U_{l,m-1}^{(i)}) \quad (2.1g)$$

$$\overline{U}_{yl,m\pm\frac{1}{2}}^{(i)} = \frac{1}{h}(\pm U_{l,m\pm 1}^{(i)} \mp U_{l,m}^{(i)}) \quad (2.1h)$$

$$\overline{U}_{xxl,m}^{(i)} = \frac{U_{l+1,m}^{(i)} - 2U_{l,m}^{(i)} + U_{l-1,m}^{(i)}}{h^2} \quad (2.1i)$$

$$\overline{U}_{xxl,m\pm 1}^{(i)} = \frac{U_{l+1,m\pm 1}^{(i)} - 2U_{l,m\pm 1}^{(i)} + U_{l-1,m\pm 1}^{(i)}}{h^2} \quad (2.1j)$$

$$\overline{U}_{yy l,m}^{(i)} = \frac{U_{l,m+1}^{(i)} - 2U_{l,m}^{(i)} + U_{l,m-1}^{(i)}}{h^2} \quad (2.1k)$$

$$\overline{U}_{yy l\pm 1,m}^{(i)} = \frac{U_{l\pm 1,m+1}^{(i)} - 2U_{l\pm 1,m}^{(i)} + U_{l\pm 1,m-1}^{(i)}}{h^2}, \quad (2.1l)$$

and then define

$$\begin{aligned} \overline{F}_{l\pm\frac{1}{2},m}^{(i)} = f^{(i)} & \left( x_{l\pm\frac{1}{2}}, y_m, \overline{U}_{l\pm\frac{1}{2},m}^{(1)}, \overline{U}_{l\pm\frac{1}{2},m}^{(2)}, \dots, \overline{U}_{l\pm\frac{1}{2},m}^{(n)}, \overline{U}_{xl\pm\frac{1}{2},m}^{(1)}, \overline{U}_{xl\pm\frac{1}{2},m}^{(2)}, \dots, \right. \\ & \left. \overline{U}_{xl\pm\frac{1}{2},m}^{(n)}, \overline{U}_{yl\pm\frac{1}{2},m}^{(1)}, \overline{U}_{yl\pm\frac{1}{2},m}^{(2)}, \dots, \overline{U}_{yl\pm\frac{1}{2},m}^{(n)} \right) \end{aligned} \quad (2.2a)$$

$$\begin{aligned} \overline{F}_{l,m\pm\frac{1}{2}}^{(i)} = f^{(i)} & \left( x_l, y_{m\pm\frac{1}{2}}, \overline{U}_{l,m\pm\frac{1}{2}}^{(1)}, \overline{U}_{l,m\pm\frac{1}{2}}^{(2)}, \dots, \overline{U}_{l,m\pm\frac{1}{2}}^{(n)}, \overline{U}_{xl,m\pm\frac{1}{2}}^{(1)}, \overline{U}_{xl,m\pm\frac{1}{2}}^{(2)}, \dots, \right. \\ & \left. \overline{U}_{xl,m\pm\frac{1}{2}}^{(n)}, \overline{U}_{yl,m\pm\frac{1}{2}}^{(1)}, \overline{U}_{yl,m\pm\frac{1}{2}}^{(2)}, \dots, \overline{U}_{yl,m\pm\frac{1}{2}}^{(n)} \right). \end{aligned} \quad (2.2b)$$

Further, let

$$\begin{aligned} \overline{\overline{U}}_{l,m}^{(i)} = & U_{l,m}^{(i)} + \frac{h^2}{16A_{l,m}^{(i)}} \left( \overline{F}_{l+\frac{1}{2},m}^{(i)} + \overline{F}_{l-\frac{1}{2},m}^{(i)} \right) + \frac{h^2}{16B_{l,m}^{(i)}} \left( \overline{F}_{l,m+\frac{1}{2}}^{(i)} + \overline{F}_{l,m-\frac{1}{2}}^{(i)} \right) \\ & + \frac{h^2}{8} \left( 1 - \frac{A_{l,m}^{(i)}}{B_{l,m}^{(i)}} \right) \overline{U}_{xxl,m}^{(i)} + \frac{h^2}{8} \left( 1 - \frac{B_{l,m}^{(i)}}{A_{l,m}^{(i)}} \right) \overline{U}_{yy l,m}^{(i)} \end{aligned} \quad (2.3a)$$

$$\begin{aligned} \overline{\overline{U}}_{xl,m}^{(i)} = & \overline{U}_{xl,m}^{(i)} + \frac{h}{4A_{l,m}^{(i)}} \left( \overline{F}_{l+\frac{1}{2},m}^{(i)} - \overline{F}_{l-\frac{1}{2},m}^{(i)} \right) + \frac{h}{8} \left( 1 - \frac{B_{l,m}^{(i)}}{A_{l,m}^{(i)}} \right) \left( \overline{U}_{yy l+1,m}^{(i)} - \overline{U}_{yy l-1,m}^{(i)} \right) \\ & - \frac{h^2 A_{xl,m}^{(i)}}{4A_{l,m}^{(i)}} \overline{U}_{xxl,m}^{(i)} - \frac{h^2 B_{xl,m}^{(i)}}{4A_{l,m}^{(i)}} \overline{U}_{yy l,m}^{(i)} \end{aligned} \quad (2.3b)$$

$$\begin{aligned} \overline{\overline{U}}_{yl,m}^{(i)} = & \overline{U}_{yl,m}^{(i)} + \frac{h}{4B_{l,m}^{(i)}} \left( \overline{F}_{l,m+\frac{1}{2}}^{(i)} - \overline{F}_{l,m-\frac{1}{2}}^{(i)} \right) + \frac{h}{8} \left( 1 - \frac{A_{l,m}^{(i)}}{B_{l,m}^{(i)}} \right) \left( \overline{U}_{xx l,m+1}^{(i)} - \overline{U}_{xx l,m-1}^{(i)} \right) \\ & - \frac{h^2 A_{yl,m}^{(i)}}{4B_{l,m}^{(i)}} \overline{U}_{xx l,m}^{(i)} - \frac{h^2 B_{yl,m}^{(i)}}{4B_{l,m}^{(i)}} \overline{U}_{yy l,m}^{(i)}, \end{aligned} \quad (2.3c)$$

and define

$$\begin{aligned} \overline{\overline{F}}_{l,m}^{(i)} = & f^{(i)} \left( x_l, y_m, \overline{\overline{U}}_{l,m}^{(1)}, \overline{\overline{U}}_{l,m}^{(2)}, \dots, \overline{\overline{U}}_{l,m}^{(n)}, \overline{\overline{U}}_{xl,m}^{(1)}, \overline{\overline{U}}_{xl,m}^{(2)}, \dots, \right. \\ & \left. \overline{\overline{U}}_{xl,m}^{(n)}, \overline{\overline{U}}_{yl,m}^{(1)}, \overline{\overline{U}}_{yl,m}^{(2)}, \dots, \overline{\overline{U}}_{yl,m}^{(n)} \right). \end{aligned} \quad (2.4)$$

Thus at each internal grid point  $(x_l, y_m)$  the system (1.1) is discretized by

$$\begin{aligned} L[U^{(i)}] \equiv & [I_1^{(i)} \delta_x^2 + I_2^{(i)} \delta_y^2 + I_3^{(i)} (2\delta_x^2 \mu_y \delta_y) + I_4^{(i)} (2\delta_y^2 \mu_x \delta_x) + I_5^{(i)} (\delta_x^2 \delta_y^2)] U_{l,m}^{(i)} \\ = & h^2 \left[ J_1^{(i)} \overline{F}_{l+\frac{1}{2},m}^{(i)} + J_2^{(i)} \overline{F}_{l-\frac{1}{2},m}^{(i)} + J_3^{(i)} \overline{F}_{l,m+\frac{1}{2}}^{(i)} + J_4^{(i)} \overline{F}_{l,m-\frac{1}{2}}^{(i)} - 2\overline{\overline{F}}_{l,m}^{(i)} \right] + \overline{\overline{T}}_{l,m}^{(i)} \end{aligned} \quad (2.5)$$

for  $\overline{\overline{T}}_{l,m}^{(i)} = O(h^6)$ , where  $l, m = 2(1)N$  and

$$\begin{aligned} K_1^{(i)} &= A_{xl,m}^{(i)} / A_{l,m}^{(i)}, \quad K_2^{(i)} = B_{yl,m}^{(i)} / B_{l,m}^{(i)}, \\ I_1^{(i)} &= 6A_{l,m}^{(i)} + \frac{1}{2}h^2 \left( A_{xxl,m}^{(i)} + A_{yy l,m}^{(i)} - 2K_1^{(i)} A_{xl,m}^{(i)} - 2K_2^{(i)} A_{yl,m}^{(i)} \right), \\ I_2^{(i)} &= 6B_{l,m}^{(i)} + \frac{1}{2}h^2 \left( B_{xxl,m}^{(i)} + B_{yy l,m}^{(i)} - 2K_1^{(i)} B_{xl,m}^{(i)} - 2K_2^{(i)} B_{yl,m}^{(i)} \right), \end{aligned}$$

$$\begin{aligned}
I_3^{(i)} &= \frac{1}{2}h \left( A_{yl,m}^{(i)} - K_2^{(i)} A_{l,m}^{(i)} \right), \\
I_4^{(i)} &= \frac{1}{2}h \left( B_{xl,m}^{(i)} - K_1^{(i)} B_{l,m}^{(i)} \right), \\
I_5^{(i)} &= \frac{1}{2} \left( A_{l,m}^{(i)} + B_{l,m}^{(i)} \right), \\
J_1^{(i)} &= 2 - hK_1^{(i)}, \quad J_2^{(i)} = 2 + hK_1^{(i)}, \\
J_3^{(i)} &= 2 - hK_2^{(i)}, \quad J_4^{(i)} = 2 + hK_2^{(i)},
\end{aligned}$$

with  $\delta_x U_l = (U_{l+1/2} - U_{l-1/2})$  and  $\mu_x U_l = (U_{l+1/2} + U_{l-1/2})/2$  the central and average difference operators in  $x$ -direction, etc..

Let us now consider the fourth-order numerical methods for the estimates of  $(\partial u^{(i)}/\partial x)$  and  $(\partial u^{(i)}/\partial y)$ . If  $U_{xl,m}^{(i)}, U_{yl,m}^{(i)}$  denote the exact and  $u_{xl,m}^{(i)}, u_{yl,m}^{(i)}$  the approximate solutions of  $(\partial u^{(i)}/\partial x)$ ,  $(\partial u^{(i)}/\partial y)$  respectively at the grid point  $(x_l, y_m)$ , following Stephenson [16] we obtain

$$\begin{aligned}
U_{xl,m}^{(i)} &= \frac{1}{2h} (2\mu_x \delta_x) U_{l,m}^{(i)} + \frac{1}{6A_{l,m}^{(i)}} (A_{xl,m}^{(i)} \delta_x^2 + B_{xl,m}^{(i)} \delta_y^2) U_{l,m}^{(i)} + \frac{B_{l,m}^{(i)}}{12hA_{l,m}^{(i)}} (2\delta_y^2 \mu_x \delta_x) U_{l,m}^{(i)} \\
&\quad - \frac{h}{6A_{l,m}^{(i)}} \left( \bar{F}_{l+\frac{1}{2},m}^{(i)} - \bar{F}_{l-\frac{1}{2},m}^{(i)} \right) + \bar{T}_{l,m}^{(i,x)}, \quad l, m = 2(1)N, \quad (2.6)
\end{aligned}$$

$$\begin{aligned}
U_{yl,m}^{(i)} &= \frac{1}{2h} (2\mu_y \delta_y) U_{l,m}^{(i)} + \frac{1}{6B_{l,m}^{(i)}} (A_{yl,m}^{(i)} \delta_x^2 + B_{yl,m}^{(i)} \delta_y^2) U_{l,m}^{(i)} + \frac{A_{l,m}^{(i)}}{12hB_{l,m}^{(i)}} (2\delta_x^2 \mu_y \delta_y) U_{l,m}^{(i)} \\
&\quad - \frac{h}{6B_{l,m}^{(i)}} \left( \bar{F}_{l,m+\frac{1}{2}}^{(i)} - \bar{F}_{l,m-\frac{1}{2}}^{(i)} \right) + \bar{T}_{l,m}^{(i,y)}, \quad l, m = 2(1)N, \quad (2.7)
\end{aligned}$$

where  $\bar{T}_{l,m}^{(i,x)} = O(h^4)$  and  $\bar{T}_{l,m}^{(i,y)} = O(h^4)$ . The numerical methods (2.6) and (2.7) are applicable when the fourth-order difference solutions of the  $u^{(i)}$  are known at each internal grid point. Further, the Dirichlet boundary conditions are given by Eq. (1.2). The difference method (2.5) for the determination of the  $u^{(i)}$  can easily be expressed in tri-block-diagonal matrix form, and the methods (2.6) and (2.7) for the determination of  $u_x^{(i)}$  and  $u_y^{(i)}$  can be expressed in diagonal matrices form, and therefore easily solved. The proposed methods (2.5)-(2.7) are directly applicable to singular elliptic problems in the region  $\Omega$ .

### 3. Derivation of the Discretizations

For  $j = 1, 2, \dots, n$ , we denote

$$\alpha^{(i,j)} = \frac{\partial f^{(i)}}{\partial U^{(j)}}, \quad \beta^{(i,j)} = \frac{\partial f^{(i)}}{\partial U_x^{(j)}}, \quad \gamma^{(i,j)} = \frac{\partial f^{(i)}}{\partial U_y^{(j)}}.$$

Simplifying Eqs. (2.1a) to (2.1l) with the help of Taylor series, we obtain

$$\overline{U}_{l\pm\frac{1}{2},m}^{(i)} = U_{l\pm\frac{1}{2},m}^{(i)} + \frac{h^2}{8}U_{xxl,m}^{(i)} \pm O(h^3), \quad (3.1a)$$

$$\overline{U}_{l,m\pm\frac{1}{2}}^{(i)} = U_{l,m\pm\frac{1}{2}}^{(i)} + \frac{h^2}{8}U_{yy l,m}^{(i)} \pm O(h^3), \quad (3.1b)$$

$$\overline{U}_{xl,m}^{(i)} = U_{xl,m}^{(i)} + \frac{h^2}{6}U_{xxxl,m}^{(i)} + O(h^4), \quad (3.1c)$$

$$\overline{U}_{xl\pm\frac{1}{2},m}^{(i)} = U_{xl\pm\frac{1}{2},m}^{(i)} + \frac{h^2}{24}U_{xxxl,m}^{(i)} \pm O(h^3), \quad (3.1d)$$

$$\overline{U}_{xl,m\pm\frac{1}{2}}^{(i)} = U_{xl,m\pm\frac{1}{2}}^{(i)} + \frac{h^2}{6}U_{xxxl,m}^{(i)} + \frac{h^2}{8}U_{xyyl,m}^{(i)} \pm O(h^3), \quad (3.1e)$$

$$\overline{U}_{yl,m}^{(i)} = U_{yl,m}^{(i)} + \frac{h^2}{6}U_{yyyl,m}^{(i)} + O(h^4), \quad (3.1f)$$

$$\overline{U}_{yl\pm\frac{1}{2},m}^{(i)} = U_{yl\pm\frac{1}{2},m}^{(i)} + \frac{h^2}{8}U_{xxyl,m}^{(i)} + \frac{h^2}{6}U_{yyyl,m}^{(i)} \pm O(h^3), \quad (3.1g)$$

$$\overline{U}_{yl,m\pm\frac{1}{2}}^{(i)} = U_{yl,m\pm\frac{1}{2}}^{(i)} + \frac{h^2}{24}U_{yyyl,m}^{(i)} \pm O(h^3), \quad (3.1h)$$

$$\overline{U}_{xxl,m\pm 1}^{(i)} = U_{xxl,m\pm 1}^{(i)} + O(h^2), \quad (3.1i)$$

$$\overline{U}_{xxl,m}^{(i)} = U_{xxl,m}^{(i)} + O(h^2), \quad (3.1j)$$

$$\overline{U}_{yy l\pm 1,m}^{(i)} = U_{yy l\pm 1,m}^{(i)} + O(h^2), \quad (3.1k)$$

$$\overline{U}_{yy l,m}^{(i)} = U_{yy l,m}^{(i)} + O(h^2). \quad (3.1l)$$

Again invoking Taylor series, we first obtain

$$\begin{aligned} L[U^{(i)}] &\equiv [I_1^{(i)}\delta_x^2 + I_2^{(i)}\delta_y^2 + I_3^{(i)}(2\delta_x^2\mu_y\delta_y) + I_4^{(i)}(2\delta_y^2\mu_x\delta_x) + I_5^{(i)}(\delta_x^2\delta_y^2)]U_{l,m}^{(i)} \\ &= h^2 \left[ J_1^{(i)}F_{l+\frac{1}{2},m}^{(i)} + J_2^{(i)}F_{l-\frac{1}{2},m}^{(i)} + J_3^{(i)}F_{l,m+\frac{1}{2}}^{(i)} + J_4^{(i)}F_{l,m-\frac{1}{2}}^{(i)} - 2F_{l,m}^{(i)} \right] + O(h^6); \\ &\quad l, m = 2(1)N. \end{aligned} \quad (3.2)$$

Then with the approximations Eqs. (3.1a)-(3.1h), from Eqs. (2.2a) and (2.2b) and again using Taylor series we obtain

$$\overline{F}_{l\pm\frac{1}{2},m}^{(i)} = F_{l\pm\frac{1}{2},m}^{(i)} + \frac{h^2}{24}T_1^{(i)} + O(\pm h^3 + h^4), \quad (3.3a)$$

$$\overline{F}_{l,m\pm\frac{1}{2}}^{(i)} = F_{l,m\pm\frac{1}{2}}^{(i)} + \frac{h^2}{24}T_2^{(i)} + O(\pm h^3 + h^4), \quad (3.3b)$$

where

$$T_1^{(i)} = \sum_{j=1}^n \left[ 3U_{xxl,m}^{(j)} \alpha_{l,m}^{(i,j)} + U_{xxxl,m}^{(j)} \beta_{l,m}^{(i,j)} + (3U_{xxy,l,m}^{(j)} + 4U_{yyyl,m}^{(j)}) \gamma_{l,m}^{(i,j)} \right],$$

$$T_2^{(i)} = \sum_{j=1}^n \left[ 3U_{yy,l,m}^{(j)} \alpha_{l,m}^{(i,j)} + (4U_{xxxl,m}^{(j)} + 3U_{xyyl,m}^{(j)}) \beta_{l,m}^{(i,j)} + U_{yyyl,m}^{(j)} \gamma_{l,m}^{(i,j)} \right].$$

Now let

$$\begin{aligned} \bar{\bar{U}}_{l,m}^{(i)} &= U_{l,m}^{(i)} + a_{1i} h^2 \left( \bar{F}_{l+\frac{1}{2},m}^{(i)} + \bar{F}_{l-\frac{1}{2},m}^{(i)} \right) + a_{2i} h^2 \left( \bar{F}_{l,m+\frac{1}{2}}^{(i)} + \bar{F}_{l,m-\frac{1}{2}}^{(i)} \right) \\ &\quad + a_{3i} h^2 \bar{U}_{xxl,m}^{(i)} + a_{4i} h^2 \bar{U}_{yy,l,m}^{(i)}, \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \bar{\bar{U}}_{xl,m}^{(i)} &= \bar{U}_{xl,m}^{(i)} + b_{1i} h \left( \bar{F}_{l+\frac{1}{2},m}^{(i)} - \bar{F}_{l-\frac{1}{2},m}^{(i)} \right) + b_{2i} h (\bar{U}_{yy,l+1,m}^{(i)} - \bar{U}_{yy,l-1,m}^{(i)}) \\ &\quad + b_{3i} h^2 \bar{U}_{xxl,m}^{(i)} + b_{4i} h^2 \bar{U}_{yy,l,m}^{(i)}, \end{aligned} \quad (3.4b)$$

$$\begin{aligned} \bar{\bar{U}}_{yl,m}^{(i)} &= \bar{U}_{yl,m}^{(i)} + c_{1i} h \left( \bar{F}_{l,m+\frac{1}{2}}^{(i)} - \bar{F}_{l,m-\frac{1}{2}}^{(i)} \right) + c_{2i} h (\bar{U}_{xxl,m+1}^{(i)} - \bar{U}_{xxl,m-1}^{(i)}) \\ &\quad + c_{3i} h^2 \bar{U}_{xxl,m}^{(i)} + c_{4i} h^2 \bar{U}_{yy,l,m}^{(i)}, \end{aligned} \quad (3.4c)$$

where  $a_{ki}$ s,  $b_{ki}$ s and  $c_{ki}$ s for  $k = 1(1)4$  are suitable parameters yet to be determined.

Invoking Eqs. (3.3a), (3.3b), (3.1i)-(3.1l) and simplifying Eqs. (3.4a)-(3.4c), we obtain

$$\bar{\bar{U}}_{l,m}^{(i)} = U_{l,m}^{(i)} + \frac{h^2}{6} T_3^{(i)} + O(h^4) \quad (3.5a)$$

$$\bar{\bar{U}}_{xl,m}^{(i)} = U_{xl,m}^{(i)} + \frac{h^2}{6} T_4^{(i)} + O(h^4) \quad (3.5b)$$

$$\bar{\bar{U}}_{yl,m}^{(i)} = U_{yl,m}^{(i)} + \frac{h^2}{6} T_5^{(i)} + O(h^4), \quad (3.5c)$$

where

$$\begin{aligned} T_3^{(i)} &= [12(a_{1i} + a_{2i})A_{l,m}^{(i)} + 6a_{3i}]U_{xxl,m}^{(i)} + [12(a_{1i} + a_{2i})B_{l,m}^{(i)} + 6a_{4i}]U_{yy,l,m}^{(i)} \\ T_4^{(i)} &= (1 + 6b_{1i}A_{l,m}^{(i)})U_{xxxl,m}^{(i)} + 6(b_{1i}A_{xl,m}^{(i)} + b_{3i})U_{xxl,m}^{(i)} + 6(b_{1i}B_{l,m}^{(i)} + 2b_{2i})U_{xyyl,m}^{(i)} \\ &\quad + 6(b_{1i}B_{xl,m}^{(i)} + b_{4i})U_{yy,l,m}^{(i)} \\ T_5^{(i)} &= (1 + 6c_{1i}B_{l,m}^{(i)})U_{yyyl,m}^{(i)} + 6(c_{1i}A_{yl,m}^{(i)} + c_{3i})U_{xxl,m}^{(i)} + 6(c_{1i}A_{l,m}^{(i)} + 2c_{2i})U_{xxy,l,m}^{(i)} \\ &\quad + 6(c_{1i}B_{yl,m}^{(i)} + c_{4i})U_{yy,l,m}^{(i)}. \end{aligned}$$

Finally, from Eq. (2.4) we obtain

$$\bar{\bar{F}}_{l,m}^{(i)} = F_{l,m}^{(i)} + \frac{h^2}{6} T_6^{(i)} + O(h^4), \quad (3.6)$$



where

$$T_6^{(i)} = \sum_{j=1}^n [T_3^{(j)} \alpha_{l,m}^{(i,j)} + T_4^{(j)} \beta_{l,m}^{(i,j)} + T_5^{(j)} \gamma_{l,m}^{(i,j)}] .$$

Substituting the approximations (3.3a), (3.3b) and (3.6) into Eq. (2.5), and invoking Eq. (3.2), we obtain

$$\overline{T}_{l,m}^{(i)} = \frac{h^4}{6} (2T_6^{(i)} - T_1^{(i)} - T_2^{(i)}) + O(h^6) . \quad (3.7)$$

Thus the proposed difference method (2.5) is fourth-order if the coefficient of  $h^4$  in Eq. (3.7) is zero, so we have

$$T_1^{(i)} + T_2^{(i)} - 2T_6^{(i)} = 0 ,$$

which gives

$$\begin{aligned} \sum_{j=1}^n \left\{ \left[ 24(a_{1j} + a_{2j})A_{l,m}^{(j)} + 12a_{3j} - 3 \right] U_{xxl,m}^{(j)} + \left[ 24(a_{1j} + a_{2j})B_{l,m}^{(j)} + 12a_{4j} - 3 \right] U_{yy l,m}^{(j)} \right\} \alpha_{l,m}^{(i,j)} \\ + \left\{ \left[ 12b_{1j}A_{l,m}^{(j)} - 3 \right] U_{xxxl,m}^{(j)} + 12(b_{1j}A_{xl,m}^{(j)} + b_{3j})U_{xxl,m}^{(j)} \right. \\ + \left. \left[ 12(b_{1j}B_{l,m}^{(j)} + 2b_{2j}) - 3 \right] U_{xyyl,m}^{(j)} + 12(b_{1j}B_{xl,m}^{(j)} + b_{4j})U_{yy l,m}^{(j)} \right\} \beta_{l,m}^{(i,j)} \\ + \left\{ \left[ 12c_{1j}B_{l,m}^{(j)} - 3 \right] U_{yyyl,m}^{(j)} + \left[ 12(c_{1j}A_{l,m}^{(j)} + 2c_{2j}) - 3 \right] U_{xxyl,m}^{(j)} \right. \\ + \left. 12(c_{1j}A_{yl,m}^{(j)} + c_{3j})U_{xxl,m}^{(j)} + 12(c_{1j}B_{yl,m}^{(j)} + c_{4j})U_{yy l,m}^{(j)} \right\} \gamma_{l,m}^{(i,j)} \Big\} = 0 . \end{aligned} \quad (3.8)$$

Equating the coefficients of each of  $\alpha_{l,m}^{(i,j)}$ ,  $\beta_{l,m}^{(i,j)}$  and  $\gamma_{l,m}^{(i,j)}$  for  $j = 1, 2, \dots, n$  to zero, and likewise the coefficients of  $U_{xxl,m}^{(j)}$ ,  $U_{yy l,m}^{(j)}$ ,  $U_{xyyl,m}^{(j)}$ ,  $U_{xxyl,m}^{(j)}$ ,  $U_{xxxl,m}^{(j)}$  and  $U_{yyyl,m}^{(j)}$ , for each  $j = 1, 2, \dots, n$  we have

$$\begin{aligned} a_{1j} &= \frac{1}{16A_{l,m}^{(j)}}, & a_{2j} &= \frac{1}{16B_{l,m}^{(j)}}, & a_{3j} &= \frac{1}{8} \left( 1 - \frac{A_{l,m}^{(j)}}{B_{l,m}^{(j)}} \right), & a_{4j} &= \frac{1}{8} \left( 1 - \frac{B_{l,m}^{(j)}}{A_{l,m}^{(j)}} \right), \\ b_{1j} &= \frac{1}{4A_{l,m}^{(j)}}, & b_{2j} &= \frac{1}{8} \left( 1 - \frac{B_{l,m}^{(j)}}{A_{l,m}^{(j)}} \right), & b_{3j} &= -\frac{A_{xl,m}^{(j)}}{4A_{l,m}^{(j)}}, & b_{4j} &= -\frac{B_{xl,m}^{(j)}}{4A_{l,m}^{(j)}}, \\ c_{1j} &= \frac{1}{4B_{l,m}^{(j)}}, & c_{2j} &= \frac{1}{8} \left( 1 - \frac{A_{l,m}^{(j)}}{B_{l,m}^{(j)}} \right), & c_{3j} &= -\frac{A_{yl,m}^{(j)}}{4B_{l,m}^{(j)}}, & c_{4j} &= -\frac{B_{yl,m}^{(j)}}{4B_{l,m}^{(j)}}, \end{aligned}$$

where  $\overline{T}_{l,m}^{(i)} = O(h^6)$  and thus the difference method of  $O(h^4)$  for the system (1.1).

With the fourth-order accurate solution values for the  $u^{(i)}$ , approximate values of  $(\partial u^{(i)}/\partial x)$  and  $(\partial u^{(i)}/\partial y)$  can be obtained by using the standard central differences

$$u_{xl,m}^{(i)} = \frac{1}{2h}(u_{l+1,m}^{(i)} - u_{l-1,m}^{(i)}), \quad (3.9a)$$

$$u_{yl,m}^{(i)} = \frac{1}{2h}(u_{l,m+1}^{(i)} - u_{l,m-1}^{(i)}), \quad (3.9b)$$

which yield second-order accurate results irrespective of whether the fourth-order difference method (2.5) or a standard difference scheme is used to solve the system (1.1). However, our new difference methods for computing the numerical values of  $u_x$  and  $u_y$  are found to yield  $O(h^4)$  accurate results, when used in conjunction with the 9-point formula (2.5). From Taylor series, we obtain

$$\begin{aligned} U_{xl,m}^{(i)} = & \frac{1}{2h}(2\mu_x \delta_x)U_{l,m}^{(i)} + \frac{1}{6A_{l,m}^{(i)}}(A_{xl,m}^{(i)} \delta_x^2 + B_{xl,m}^{(i)} \delta_y^2)U_{l,m}^{(i)} + \frac{B_{l,m}^{(i)}}{12hA_{l,m}^{(i)}}(2\delta_y^2 \mu_x \delta_x)U_{l,m}^{(i)} \\ & - \frac{h}{6A_{l,m}^{(i)}} \left( F_{l+\frac{1}{2},m}^{(i)} - F_{l-\frac{1}{2},m}^{(i)} \right) + O(h^4), \quad l, m = 2(1)N, \end{aligned} \quad (3.10)$$

$$\begin{aligned} U_{yl,m}^{(i)} = & \frac{1}{2h}(2\mu_y \delta_y)U_{l,m}^{(i)} + \frac{1}{6B_{l,m}^{(i)}}(A_{yl,m}^{(i)} \delta_x^2 + B_{yl,m}^{(i)} \delta_y^2)U_{l,m}^{(i)} + \frac{A_{l,m}^{(i)}}{12hB_{l,m}^{(i)}}(2\delta_x^2 \mu_y \delta_y)U_{l,m}^{(i)} \\ & - \frac{h}{6B_{l,m}^{(i)}} \left( F_{l,m+\frac{1}{2}}^{(i)} - F_{l,m-\frac{1}{2}}^{(i)} \right) + O(h^4) \quad l, m = 2(1)N; \end{aligned} \quad (3.11)$$

and then using Eq. (3.3a) in Eq. (2.6) we have

$$\begin{aligned} U_{xl,m}^{(i)} = & \frac{1}{2h}(2\mu_x \delta_x)U_{l,m}^{(i)} + \frac{1}{6A_{l,m}^{(i)}}(A_{xl,m}^{(i)} \delta_x^2 + B_{xl,m}^{(i)} \delta_y^2)U_{l,m}^{(i)} + \frac{B_{l,m}^{(i)}}{12hA_{l,m}^{(i)}}(2\delta_y^2 \mu_x \delta_x)U_{l,m}^{(i)} \\ & - \frac{h}{6A_{l,m}^{(i)}} \left( F_{l+\frac{1}{2},m}^{(i)} - F_{l-\frac{1}{2},m}^{(i)} \right) + \bar{T}_{l,m}^{(i,x)} + O(h^4), \quad l, m = 2(1)N \end{aligned}$$

where  $\bar{T}_{l,m}^{(i,x)} = O(h^4)$  from Eq. (3.10), and the similar result for  $U_{yl,m}$  with  $\bar{T}_{l,m}^{(i,y)} = O(h^4)$ .

#### 4. Convergence Analysis

Let us consider the 2D nonlinear elliptic PDE

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \quad (4.1)$$

defined in the region  $\Omega$ , subject to  $u(x, y) = u_0(x, y) \forall (x, y) \in \partial\Omega$ . The difference method (2.5) for the scalar equation (4.1) reduces to

$$\begin{aligned} & \left[ 6\delta_x^2 + 6\delta_y^2 + \delta_x^2 \delta_y^2 \right] u_{l,m} \\ &= 2h^2 \left[ \bar{F}_{l+\frac{1}{2},m} + \bar{F}_{l-\frac{1}{2},m} + \bar{F}_{l,m+\frac{1}{2}} + \bar{F}_{l,m-\frac{1}{2}} - \bar{\bar{F}}_{l,m} \right], \quad l, m = 2(1)N. \end{aligned} \quad (4.2)$$

We now show that, under appropriate conditions, the difference method (4.2) for the PDE (4.1) is  $O(h^4)$  convergent.

For each  $l, m = 2(1)N$ , let

$$\phi_{l,m} = 2h^2 \left[ \bar{F}_{l+\frac{1}{2},m} + \bar{F}_{l-\frac{1}{2},m} + \bar{F}_{l,m+\frac{1}{2}} + \bar{F}_{l,m-\frac{1}{2}} - \bar{\bar{F}}_{l,m} \right] + \text{Boundary Values}.$$

Let  $\mathbf{E} = \mathbf{u} - \mathbf{U}$ ; and for  $S = \phi, u, U, T$  and  $E$  let

$$\mathbf{S} = \left[ S_{2,2}, S_{3,2}, \dots, S_{N,2}, S_{2,3}, S_{3,3}, \dots, S_{N,3}, \dots, S_{2,N}, S_{3,N}, \dots, S_{N,N} \right]_{(N-1)^2 \times 1}^t.$$

Then for  $l, m = 2(1)N$  in Eq. (4.2),

$$\mathbf{D}\mathbf{u} + \phi(\mathbf{u}) = \mathbf{0} \quad (4.3)$$

where  $\mathbf{D} = [\mathbf{B} \mathbf{A} \mathbf{B}]_{(N-1)^2 \times (N-1)^2}$  is a tri-block diagonal matrix, involving the tri-diagonal matrices  $\mathbf{A} = [-4 \ 20 \ -4]_{(N-1) \times (N-1)}$  and  $\mathbf{B} = [-1 \ -4 \ -1]_{(N-1) \times (N-1)}$ . Since  $\mathbf{U}$  is the exact solution vector, it follows that

$$\mathbf{D}\mathbf{U} + \phi(\mathbf{U}) + \mathbf{T} = \mathbf{0} \quad (4.4)$$

where  $\bar{T}_{l,m} = O(h^6)$  for each  $l, m = 2(1)N$ . Denoting

$$\begin{aligned} \bar{f}_{l\pm\frac{1}{2},m} &= f \left( x_{l\pm\frac{1}{2}}, y_m, \bar{u}_{l\pm\frac{1}{2},m}, \bar{u}_{xl\pm\frac{1}{2},m}, \bar{u}_{yl\pm\frac{1}{2},m} \right) \simeq \bar{F}_{l\pm\frac{1}{2},m}, \\ \bar{f}_{l,m\pm\frac{1}{2}} &= f \left( x_l, y_{m\pm\frac{1}{2}}, \bar{u}_{l,m\pm\frac{1}{2}}, \bar{u}_{xl,m\pm\frac{1}{2}}, \bar{u}_{yl,m\pm\frac{1}{2}} \right) \simeq \bar{F}_{l,m\pm\frac{1}{2}}, \\ \bar{\bar{f}}_{l,m} &= f \left( x_l, y_m, \bar{\bar{u}}_{l,m}, \bar{\bar{u}}_{xl,m}, \bar{\bar{u}}_{yl,m} \right) \simeq \bar{\bar{F}}_{l,m}, \end{aligned}$$

we write

$$\begin{aligned} \bar{f}_{l\pm\frac{1}{2},m} - \bar{F}_{l\pm\frac{1}{2},m} &= \left( \bar{u}_{l\pm\frac{1}{2},m} - \bar{U}_{l\pm\frac{1}{2},m} \right) G_{l\pm\frac{1}{2},m}^{(1)} + \left( \bar{u}_{xl\pm\frac{1}{2},m} - \bar{U}_{xl\pm\frac{1}{2},m} \right) H_{l\pm\frac{1}{2},m}^{(1)} \\ &\quad + \left( \bar{u}_{yl\pm\frac{1}{2},m} - \bar{U}_{yl\pm\frac{1}{2},m} \right) I_{l\pm\frac{1}{2},m}^{(1)}, \end{aligned} \quad (4.5a)$$

$$\begin{aligned} \bar{f}_{l,m\pm\frac{1}{2}} - \bar{F}_{l,m\pm\frac{1}{2}} &= \left( \bar{u}_{l,m\pm\frac{1}{2}} - \bar{U}_{l,m\pm\frac{1}{2}} \right) G_{l,m\pm\frac{1}{2}}^{(2)} + \left( \bar{u}_{xl,m\pm\frac{1}{2}} - \bar{U}_{xl,m\pm\frac{1}{2}} \right) H_{l,m\pm\frac{1}{2}}^{(2)} \\ &\quad + \left( \bar{u}_{yl,m\pm\frac{1}{2}} - \bar{U}_{yl,m\pm\frac{1}{2}} \right) I_{l,m\pm\frac{1}{2}}^{(2)}, \end{aligned} \quad (4.5b)$$

and

$$\begin{aligned} \bar{\bar{f}}_{l,m} - \bar{\bar{F}}_{l,m} &= \left( \bar{\bar{u}}_{l,m} - \bar{\bar{U}}_{l,m} \right) G_{l,m}^{(3)} + \left( \bar{\bar{u}}_{xl,m} - \bar{\bar{U}}_{xl,m} \right) H_{l,m}^{(3)} \\ &\quad + \left( \bar{\bar{u}}_{yl,m} - \bar{\bar{U}}_{yl,m} \right) I_{l,m}^{(3)} \end{aligned} \quad (4.5c)$$

for suitable  $Q_{l\pm 1/2,m}^{(1)}$ ,  $Q_{l,m\pm 1/2}^{(2)}$  and  $Q_{l,m}^{(3)}$  where  $Q = G, H$  and  $I$ . Also, for  $Q = H$  and  $I$  we may write

$$Q_{l\pm \frac{1}{2},m}^{(1)} = Q_{l,m}^{(1)} \pm \frac{h}{2} Q_{xl,m}^{(1)} + O(h^2), \quad (4.6a)$$

$$Q_{l,m\pm \frac{1}{2}}^{(2)} = Q_{l,m}^{(2)} \pm \frac{h}{2} Q_{yl,m}^{(2)} + O(h^2), \quad (4.6b)$$

$$G_{l\pm \frac{1}{2},m}^{(1)} = G_{l,m}^{(1)} \pm O(h), \quad (4.6c)$$

$$G_{l,m\pm \frac{1}{2}}^{(2)} = G_{l,m}^{(2)} \pm O(h). \quad (4.6d)$$

From Eqs. (4.5a)-(4.5c) and Eqs. (4.6a)-(4.6d) we obtain

$$\phi(\mathbf{u}) - \phi(\mathbf{U}) = \mathbf{P}\mathbf{E}, \quad (4.7)$$

with  $\mathbf{P} = (P_{r,s})$ ,  $[r = 1(1)(N-1)^2, s = 1(1)(N-1)^2]$  the tri-block diagonal matrix where

$$\begin{aligned} &P_{(m-2)(N-1)+l-1, (m-2)(N-1)+l-1} \\ &= h^2 \left[ 2G_{l,m}^{(1)} + 2G_{l,m}^{(2)} - 2H_{xl,m}^{(1)} - 2I_{yl,m}^{(2)} - 2G_{l,m}^{(3)} + H_{l,m}^{(1)}H_{l,m}^{(3)} + I_{l,m}^{(2)}I_{l,m}^{(3)} \right] + O(h^4), \\ &\quad [l = 2(1)N, \quad m = 2(1)N], \end{aligned}$$

$$\begin{aligned} &P_{(m-2)(N-1)+l-1, (m-2)(N-1)+l-1\pm 1} \\ &= h \left[ \pm 2H_{l,m}^{(1)} \pm H_{l,m}^{(2)} \mp H_{l,m}^{(3)} \right] + \frac{h^2}{2} \left[ 2G_{l,m}^{(1)} + 2H_{xl,m}^{(1)} - H_{l,m}^{(1)}H_{l,m}^{(3)} \right] + O(h^3), \\ &\quad [l = 2(1)N - 1, 3(1)N, \quad m = 2(1)N], \end{aligned}$$

$$\begin{aligned} &P_{(m-2)(N-1)+l-1, (m-2\pm 1)(N-1)+l-1} \\ &= h \left[ \pm 2I_{l,m}^{(2)} \pm I_{l,m}^{(1)} \mp I_{l,m}^{(3)} \right] + \frac{h^2}{2} \left[ 2G_{l,m}^{(2)} + 2I_{yl,m}^{(2)} - I_{l,m}^{(2)}I_{l,m}^{(3)} \right] + O(h^3), \\ &\quad [l = 2(1)N, \quad m = 2(1)N - 1, 3(1)N], \end{aligned}$$

$$\begin{aligned} &P_{(m-2)(N-1)+l-1, (m-1)(N-1)+l-1\pm 1} \\ &= \frac{h}{2} \left[ \pm H_{l,m}^{(2)} + I_{l,m}^{(1)} \right] + \frac{h^2}{8} \left[ \pm 2H_{yl,m}^{(2)} \pm 2I_{xl,m}^{(1)} \mp I_{l,m}^{(1)}H_{l,m}^{(3)} \mp H_{l,m}^{(2)}I_{l,m}^{(3)} \right] + O(h^3), \\ &\quad [l = 2(1)N - 1, 3(1)N, \quad m = 2(1)N - 1], \end{aligned}$$

$$\begin{aligned}
& P_{(m-2)(N-1)+l-1, (m-3)(N-1)+l-1 \pm 1} \\
&= \frac{h}{2} \left[ \pm H_{l,m}^{(2)} - I_{l,m}^{(1)} \right] + \frac{h^2}{8} \left[ \mp 2H_{yl,m}^{(2)} \mp 2I_{xl,m}^{(2)} \pm I_{l,m}^{(1)} H_{l,m}^{(3)} \pm H_{l,m}^{(2)} I_{l,m}^{(3)} \right] + O(h^3), \\
& \quad [l = 2(1)N - 1, 3(1)N, \quad m = 3(1)N].
\end{aligned}$$

Using the relation (4.7), in the absence of round-off errors we obtain from Eqs. (4.3) and (4.4) the error equation

$$(\mathbf{D} + \mathbf{P})\mathbf{E} = \mathbf{T}. \quad (4.8)$$

Let

$$G_* = \min_{(x,y) \in \bar{\Omega}} \frac{\partial f}{\partial U} \quad \text{and} \quad G^* = \max_{(x,y) \in \bar{\Omega}} \frac{\partial f}{\partial U},$$

where  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Then

$$0 < G_* \leq G_{l \pm \frac{1}{2}, m}^{(1)}, G_{l, m \pm \frac{1}{2}}^{(2)}, G_{l, m}^{(3)} \leq G^* ;$$

and for  $Q = H$  and  $I$  let

$$\begin{aligned}
0 &< |Q_{l \pm \frac{1}{2}, m}^{(1)}|, |Q_{l, m \pm \frac{1}{2}}^{(2)}|, |Q_{l, m}^{(3)}| \leq Q, \\
|Q_{xl, m}^{(1)}| &\leq Q^{(1)}, \quad |Q_{yl, m}^{(2)}| \leq Q^{(2)},
\end{aligned}$$

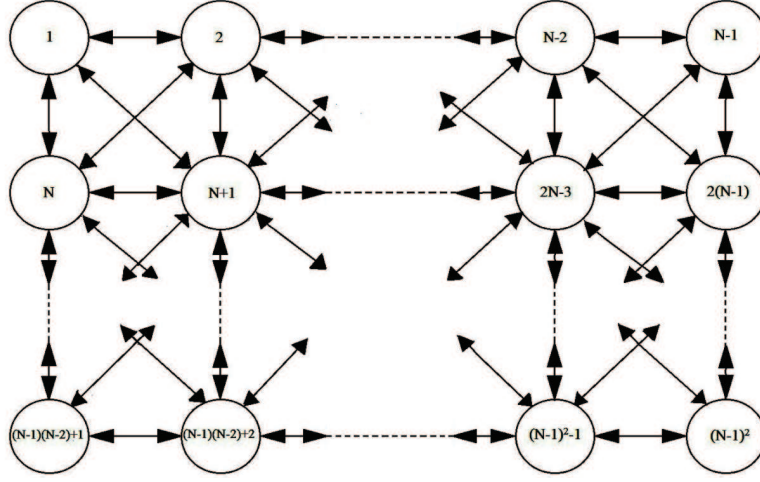
for some positive constants  $Q, Q^{(1)}$  and  $Q^{(2)}$ . It is now easy to verify that for sufficiently small  $h$

$$\begin{aligned}
|P_{(m-2)(N-1)+l-1, (m-2)(N-1)+l-1}| &< 20, & [l = 2(1)N, m = 2(1)N], \\
|P_{(m-2)(N-1)+l-1, (m-2)(N-1)+l-1 \pm 1}| &< 4, & [l = 2(1)N - 1, 3(1)N, m = 2(1)N], \\
|P_{(m-2)(N-1)+l-1, (m-2 \pm 1)(N-1)+l-1}| &< 4, & [l = 2(1)N, m = 2(1)N - 1, 3(1)N], \\
|P_{(m-2)(N-1)+l-1, (m-1)(N-1)+l-1 \pm 1}| &< 1, & [l = 2(1)N - 1, 3(1)N, m = 2(1)N - 1], \\
|P_{(m-2)(N-1)+l-1, (m-3)(N-1)+l-1 \pm 1}| &< 1, & [l = 2(1)N - 1, 3(1)N, m = 3(1)N].
\end{aligned}$$

Further, the Direct graph of  $\mathbf{D} + \mathbf{P}$  shows that  $\mathbf{D} + \mathbf{P}$  is an irreducible matrix — cf. Fig. 2, where the arrows indicate paths  $i \rightarrow j$  for every nonzero entry  $(\mathbf{D} + \mathbf{P})_{(i,j)}$  of the matrix  $\mathbf{D} + \mathbf{P}$ . For any ordered pair of nodes  $i$  and  $j$ , there exists a direct path  $(\overrightarrow{i, l_1}), (\overrightarrow{l_1, l_2}), \dots, (\overrightarrow{l_k, j})$  connecting  $i$  to  $j$ , hence the graph is strongly connected so the matrix  $\mathbf{D} + \mathbf{P}$  is irreducible [23].

Now let  $S_k$  denote the sum of the elements in the  $k$ th row of  $\mathbf{D} + \mathbf{P}$ , so for  $k = 1$  and  $N - 1$  we have

$$S_k = 11 + h^2 \left( 3G_{k+1,2}^{(1)} + 3G_{k+1,2}^{(2)} - 2G_{k+1,2}^{(3)} \right) + \frac{h}{8}(b_k + hc_k) + O(h^3), \quad (4.9a)$$

Figure 2: Directed graph of  $D + P$ .

where

$$b_k = \pm 16H_{k+1,2}^{(1)} \pm 12H_{k+1,2}^{(2)} + 16I_{k+1,2}^{(2)} + 12I_{k+1,2}^{(1)} \mp 8H_{k+1,2}^{(3)} - 8I_{k+1,2}^{(3)},$$

$$c_k = 4H_{k+1,2}^{(1)}H_{k+1,2}^{(3)} + 4I_{k+1,2}^{(2)}I_{k+1,2}^{(3)} - 8H_{xk+1,2}^{(1)} - 8I_{yk+1,2}^{(2)} \pm 2H_{yk+1,2}^{(2)} \pm 2I_{xk+1,2}^{(1)} \\ \mp I_{k+1,2}^{(1)}H_{k+1,2}^{(3)} \mp H_{k+1,2}^{(2)}I_{k+1,2}^{(3)};$$

and also

$$S_{(N-1)(N-2)+k} = 11 + h^2 \left[ 3G_{k+1,N}^{(1)} + 3G_{k+1,N}^{(2)} - 2G_{k+1,N}^{(3)} \right] \\ + \frac{h}{8} \left[ b_{(N-1)(N-2)+k} + hc_{(N-1)(N-2)+k} \right] + O(h^3), \quad (4.9b)$$

where

$$b_{(N-1)(N-2)+k} = \pm 16H_{k+1,N}^{(1)} \pm 12H_{k+1,N}^{(2)} - 16I_{k+1,N}^{(2)} - 12I_{k+1,N}^{(1)} \mp 8H_{k+1,N}^{(3)} + 8I_{k+1,N}^{(3)},$$

$$c_{(N-1)(N-2)+k} = 4H_{k+1,N}^{(1)}H_{k+1,N}^{(3)} + 4I_{k+1,N}^{(2)}I_{k+1,N}^{(3)} - 8H_{xk+1,N}^{(1)} - 8I_{yk+1,N}^{(2)} \mp 2H_{yk+1,N}^{(2)} \\ \mp 2I_{xk+1,N}^{(1)} \pm I_{k+1,N}^{(1)}H_{k+1,N}^{(3)} \pm H_{k+1,N}^{(2)}I_{k+1,N}^{(3)}.$$

For  $q = 2(1)N - 2$ ,

$$S_{(q-1)(N-1)+k} = 6 + h^2 \left[ 3G_{k+1,q+1}^{(1)} + 4G_{k+1,q+1}^{(2)} - 2G_{k+1,q+1}^{(3)} \right] \\ + \frac{h}{2} \left[ b_{(q-1)(N-1)+k} + hc_{(q-1)(N-1)+k} \right] + O(h^3), \quad (4.9c)$$

where

$$b_{(q-1)(N-1)+k} = \pm 4H_{k+1,q+1}^{(1)} \pm 4H_{k+1,q+1}^{(2)} \mp 2H_{k+1,q+1}^{(3)},$$

$$c_{(q-1)(N-1)+k} = -2H_{xk+1,q+1}^{(1)} + H_{k+1,q+1}^{(1)}H_{k+1,q+1}^{(3)}.$$

For  $r = 2(1)N - 2$ ,

$$S_{(k-1)(N-1)+r} = 6 + h^2[4G_{r+1,k+1}^{(1)} + 3G_{r+1,k+1}^{(2)} - 2G_{r+1,k+1}^{(3)}] + \frac{h}{2}[b_{(k-1)(N-1)+r} + hc_{(k-1)(N-1)+r}] + O(h^3), \quad (4.9d)$$

where

$$b_{(k-1)(N-1)+r} = \pm 4I_{r+1,k+1}^{(1)} \pm 4I_{r+1,k+1}^{(2)} \mp 2I_{r+1,k+1}^{(3)},$$

$$c_{(k-1)(N-1)+r} = -2I_{r+1,k+1}^{(2)} + I_{r+1,k+1}^{(2)} I_{r+1,k+1}^{(3)}.$$

And finally, for  $q = 2(1)N - 2$ ,  $r = 2(1)N - 2$  we have

$$S_{(r-1)(N-1)+q} = h^2 \left[ 4G_{q+1,r+1}^{(1)} + 4G_{q+1,r+1}^{(2)} - 2G_{q+1,r+1}^{(3)} \right] + O(h^4). \quad (4.9e)$$

From Eqs. (4.9a)-(4.9e), since

$$|b_k| \leq 36(H + I),$$

$$|c_k| \leq 4(H^2 + I^2) + 8(H^{(1)} + I^{(2)}) + 2(H^{(2)} + I^{(1)}) + 2IH$$

for  $k = 1, N - 1, (N - 1)(N - 2) + 1$  and  $(N - 1)^2$ ,

$$|b_k| \leq 10H, \quad |c_k| \leq 2H^{(1)} + H^2$$

for  $k = (q - 1)(N - 1) + 1$  and  $q(N - 1)$  where  $q = 2(1)N - 2$ ,

$$|b_k| \leq 10I, \quad |c_k| \leq 2I^{(2)} + I^2$$

for  $k = r$  and  $(N - 2)(N - 1) + r$  where  $r = 2(1)N - 2$ , for sufficiently small  $h$

$$S_k > 6h^2 G_* \quad \text{for } k = 1, N - 1, (N - 1)(N - 2) + 1 \text{ and } (N - 1)^2, \quad (4.10a)$$

$$S_k > 7h^2 G_* \quad \text{for } k = (q - 1)(N - 1) + 1 \text{ and } q(N - 1); q = 2(1)N - 2, \quad (4.10b)$$

$$S_k > 7h^2 G_* \quad \text{for } k = r \text{ and } (N - 2)(N - 1) + r, r = 2(1)N - 2, \quad (4.10c)$$

$$S_{(r-1)(N-1)+q} \geq h^2(8G_* - 2G^*) > 0 \text{ assuming } G^* < 4G_*,$$

$$\text{for } q = 2(1)N - 2 \text{ and } r = 2(1)N - 2. \quad (4.10d)$$

Thus  $\mathbf{D} + \mathbf{P}$  is monotone for sufficiently small  $h$ , so  $(\mathbf{D} + \mathbf{P})^{-1}$  exists and  $(\mathbf{D} + \mathbf{P})^{-1} = \mathbf{J}^{-1} > 0$  [26], where  $\mathbf{J} = (J_{r,s})$  with  $r = 1(1)(N - 1)^2$  and  $s = 1(1)(N - 1)^2$ . Since

$$\sum_{r=1}^{(N-1)^2} J_{p,r} S_r = 1, \quad p = 1(1)(N - 1)^2$$

and  $G_* > 0$ , for  $p = 1(1)(N-1)^2$  it follows from Eqs. (4.10a)-(4.10d) that

$$J_{p,k} \leq \frac{1}{S_k} < \frac{1}{6h^2 G_*} \quad [k = 1, N-1, (N-1)(N-2)+1 \text{ and } (N-1)^2], \quad (4.11a)$$

$$\sum_{q=2}^{N-2} J_{p,k} S_k \leq \frac{1}{\min_{2 \leq q \leq N-2} S_k} < \frac{1}{7h^2 G_*} \quad [k = (q-1)(N-1)+1 \text{ and } q(N-1), q = 2(1)N-2], \quad (4.11b)$$

$$\sum_{r=2}^{N-2} J_{p,k} \leq \frac{1}{\min_{2 \leq r \leq N-2} S_k} < \frac{1}{7h^2 G_*} \quad [k = r \text{ and } (N-2)(N-1)+r, r = 2(1)N-2], \quad (4.11c)$$

$$\sum_{q=2}^{N-2} \sum_{r=2}^{N-2} J_{p,k} \leq \frac{1}{\min_{\substack{2 \leq q \leq N-2 \\ 2 \leq r \leq N-2}} S_k} \leq \frac{1}{h^2(8G_* - 2G^*)} \quad [k = (r-1)(N-1)+q, q = 2(1)N-2 \text{ and } r = 2(1)N-2]. \quad (4.11d)$$

Eq. (4.8) may be written as

$$\|\mathbf{E}\| \leq \|\mathbf{J}\| \|\mathbf{T}\|, \quad (4.12)$$

where

$$\begin{aligned} \|\mathbf{J}\| = \max_{1 \leq p \leq (N-1)^2} & \left[ \left( J_{p,1} + \sum_{q=2}^{(N-2)} J_{p,q} + J_{p,N-1} \right) \right. \\ & + \left( \sum_{q=2}^{N-2} J_{p,(q-1)(N-1)+1} + \sum_{q=2}^{N-2} \sum_{r=2}^{N-2} J_{p,(q-1)(N-1)+r} + \sum_{q=2}^{N-2} J_{p,q(N-1)} \right) \\ & \left. + \left( J_{p,(N-1)(N-2)+1} + \sum_{q=2}^{N-2} J_{p,(N-1)(N-2)+q} + J_{p,(N-1)^2} \right) \right]. \quad (4.13) \end{aligned}$$

Using Eqs. (4.11a)-(4.11d) in Eq. (4.12), for sufficiently small  $h$  we obtain from (4.13)

$$\|\mathbf{E}\| \leq O(h^4). \quad (4.14)$$

This establishes the convergence of the fourth-order difference method (2.5) with  $n = 1$ , for the scalar elliptic equation (4.1).

We may extend the above convergence analysis carried out for a single equation to the system of equations. Thus for each  $i = 1(1)n$ , we denote

$$\mathbf{E}^{(i)} = \mathbf{u}^{(i)} - \mathbf{U}^{(i)}$$

where  $\mathbf{u}^{(i)}$  and  $\mathbf{U}^{(i)}$  are the respective approximate and exact solution vectors for the system of equations. Since the local truncation error is of  $O(h^6)$  for each equation, then in a similar manner to the above we may obtain  $\|\mathbf{E}^{(i)}\| \leq O(h^4)$  for each  $i$ , so the fourth-order convergence of the difference method (2.5) can be established for the system.



## 5. Numerical Illustrations

In this section, we discuss the numerical solution of some linear and nonlinear problems in Cartesian or spherical and cylindrical polar coordinates in specified domains, where the exact solutions are known. (The functions on the right-hand side of the particular PDE and the Dirichlet boundary conditions are obtained from the exact solutions.) The system of linear difference equations is solved using the Block iterative method, and the system of nonlinear difference equations by the Newton-Raphson method — e.g. see Hageman and Young [25]. For all of the problems, the iterations were terminated once the absolute error tolerance  $10^{-12}$  was reached. Computer code was written using the MATLAB programming language.

**Problem 1.** (*Poisson equation in polar coordinates*)

$$(i) \quad u_{rr} + \frac{\alpha}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = G(r, \theta), \quad 0 < r, \theta < 1.$$

For  $\alpha = 1$  and 2, this is the 2D Poisson equation in the  $r$ - $\theta$  plane for cylindrical and spherical coordinates, respectively. The exact solution adopted was  $u = r^2 \cos(\pi\theta)$ .

$$(ii) \quad u_{rr} + \frac{\alpha}{r}u_r + u_{zz} = G(r, z), \quad 0 < r, z < 1.$$

For  $\alpha = 1$ , this is the 2D Poisson equation for cylindrical coordinates in the  $r$ - $z$  plane. The exact solution adopted was  $u = \cosh r \cosh z$ .

The Maximum Absolute Errors (MAE) for  $u$  and its normal derivatives tabulated in Tables 1 and 2 for  $\alpha = 1$  and 2 are for Problems 1(i) and 1(ii), respectively. Figs. 3 and 4 give the plots of the exact and numerical solutions of Problems 1(i) and 1(ii), respectively.

**Problem 2** (*Nonlinear Convection Equation*)

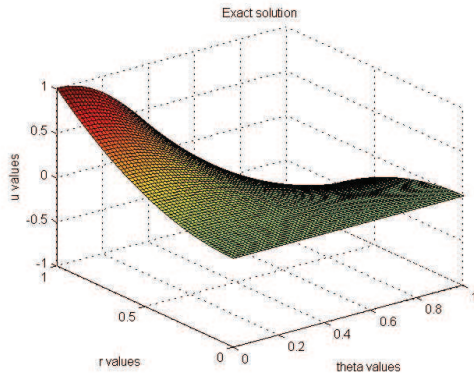
$$\varepsilon(u_{xx} + u_{yy}) = u(u_x + u_y) + g(x, y), \quad 0 < x, y < 1.$$

Table 1: Problem 1(i): The MAE.

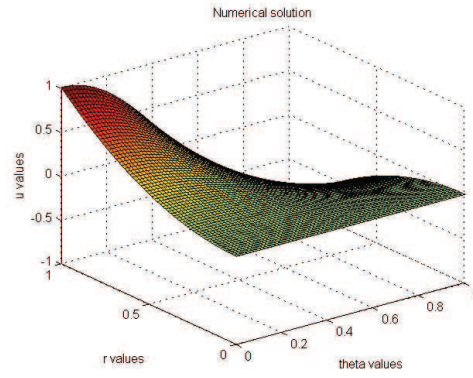
$h$		Proposed $O(h^4)$ - methods		$O(h^4)$ - methods discussed in [12, 13]	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1/8	$u$	2.3294(-06)	4.6091(-06)	4.2944(-06)	6.8672(-06)
	$u_r$	8.1179(-06)	1.6529(-05)	8.8246(-06)	4.2834(-05)
	$u_\theta$	3.9153(-04)	6.2317(-04)	4.4823(-04)	8.6298(-04)
1/16	$u$	1.4731(-07)	2.9153(-07)	2.7278(-07)	4.4629(-06)
	$u_r$	9.7119(-07)	1.8818(-06)	8.1105(-07)	3.0421(-06)
	$u_\theta$	2.8537(-05)	4.5514(-05)	3.2188(-05)	6.3244(-05)
1/32	$u$	9.2898(-09)	1.8373(-08)	1.8642(-08)	3.2187(-07)
	$u_r$	7.9264(-08)	1.5078(-07)	7.0243(-08)	2.2424(-07)
	$u_\theta$	1.9185(-06)	3.0633(-06)	2.3156(-06)	4.4341(-06)
1/64	$u$	5.8207(-10)	1.1480(-09)	1.1022(-09)	1.9068(-08)
	$u_r$	5.5941(-09)	1.0553(-08)	5.8124(-09)	1.4172(-08)
	$u_\theta$	1.2425(-07)	1.9850(-07)	1.5510(-07)	2.7012(-07)

Table 2: Problem 1(ii): The MAE.

$h$		Proposed $O(h^4)$ - methods		$O(h^4)$ - methods discussed in [12, 13]	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1/8	$u$	1.6604(-06)	2.8030(-06)	2.8604(-06)	4.2166(-06)
	$u_r$	1.0721(-05)	1.0390(-05)	2.1444(-05)	1.9833(-05)
	$u_z$	2.3486(-05)	3.6014(-05)	3.6218(-05)	4.8412(-05)
1/16	$u$	1.0530(-07)	1.7649(-07)	1.9884(-07)	2.6261(-07)
	$u_r$	8.1375(-07)	8.1496(-07)	1.1721(-06)	1.0104(-06)
	$u_z$	1.6895(-06)	2.5690(-06)	2.7520(-06)	3.8224(-06)
1/32	$u$	6.6915(-09)	1.1082(-08)	1.1645(-08)	1.6224(-08)
	$u_r$	5.6905(-08)	5.8312(-08)	8.2169(-07)	7.6186(-08)
	$u_z$	1.1430(-07)	1.7298(-07)	1.6644(-07)	2.3242(-07)
1/64	$u$	4.2259(-10)	6.9292(-10)	7.0120(-10)	8.8844(-10)
	$u_r$	4.5970(-09)	7.3664(-09)	5.0210(-08)	4.5458(-09)
	$u_z$	7.4703(-09)	1.1278(-08)	8.9744(-09)	1.4242(-08)



Exact Solution



Numerical Solution

Figure 3: Solution of 2D Poisson equation in polar coordinates.

Table 3: Problem 2: The MAE.

$h$		Proposed $O(h^4)$ - methods			$O(h^4)$ - methods discussed in [12, 13]		
		$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$
1/16	$u$	2.0616(-05)	2.4785(-04)	2.5130(-03)	4.4185(-05)	4.8266(-04)	3.9282(-03)
	$u_x$	1.2837(-04)	2.5155(-03)	1.5332(-02)	2.5208(-04)	2.8427(-03)	2.5834(-02)
	$u_y$	1.6447(-04)	1.3841(-03)	1.5891(-02)	3.0162(-04)	2.7129(-03)	2.4345(-02)
1/32	$u$	1.2695(-06)	1.7462(-05)	1.4391(-04)	2.9287(-06)	3.4324(-05)	2.8411(-04)
	$u_x$	1.6905(-05)	2.0866(-04)	1.6390(-03)	1.9812(-05)	2.2186(-04)	2.6928(-03)
	$u_y$	1.1825(-05)	8.9167(-05)	1.0726(-03)	2.1441(-05)	2.0412(-04)	1.8184(-03)
1/64	$u$	7.9012(-08)	1.1296(-06)	1.0375(-05)	1.8119(-07)	2.1204(-06)	2.1686(-05)
	$u_x$	1.5530(-06)	1.4118(-05)	2.4812(-04)	1.2124(-06)	1.3016(-05)	2.8928(-04)
	$u_y$	8.0908(-07)	6.2655(-06)	5.4500(-05)	1.2888(-06)	1.2162(-05)	1.1421(-04)

The exact solution adopted was  $u = e^x \sin(\pi y/2)$ . The MAE for  $u$ ,  $u_x$  and  $u_y$  are tabulated in Table 3 for  $\varepsilon = 0.1, 0.01$  and  $0.001$ .

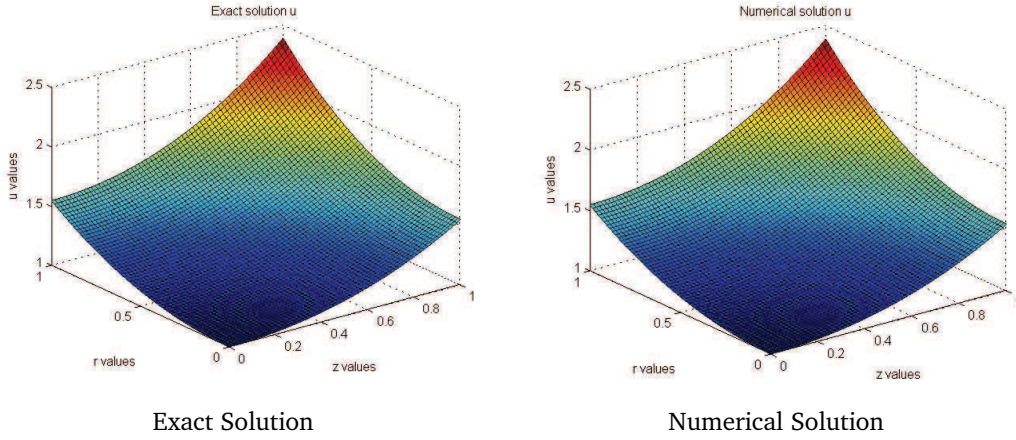


Figure 4: Solution of 2D Poisson equation with cylindrical symmetry.

Table 4: Problem 3: The MAE.

$h$		Proposed $O(h^4)$ - methods			$O(h^4)$ - methods discussed in [12, 13]		
		$\alpha = 1$	$\alpha = 10$	$\alpha = 25$	$\alpha = 1$	$\alpha = 10$	$\alpha = 25$
1/16	$u$	6.6498(-06)	1.4642(-04)	4.8477(-04)	7.6488(-06)	2.4122(-04)	5.6678(-04)
	$u_x$	9.4134(-05)	1.3702(-03)	3.1661(-03)	1.1012(-04)	2.3816(-03)	4.0465(-03)
	$u_y$	3.0028(-04)	2.5306(-03)	5.9721(-03)	3.8149(-04)	3.6296(-03)	6.8764(-03)
1/32	$u$	4.1944(-07)	9.1279(-06)	2.9675(-05)	4.8894(-07)	1.5158(-03)	3.6654(-05)
	$u_x$	6.6876(-06)	1.3007(-04)	4.3732(-04)	7.2724(-06)	2.2244(-04)	3.8975(-04)
	$u_y$	2.0286(-05)	1.7751(-04)	4.1266(-04)	2.6261(-05)	2.4890(-04)	4.8243(-04)
1/64	$u$	2.6367(-08)	5.6914(-07)	1.8430(-06)	3.0189(-08)	9.2284(-07)	2.2056(-06)
	$u_x$	4.4521(-07)	1.0153(-05)	4.3173(-05)	4.8484(-07)	1.8446(-05)	3.6698(-05)
	$u_y$	1.3205(-06)	1.1963(-05)	2.8454(-05)	1.6185(-06)	1.5177(-05)	3.0125(-05)

**Problem 3 (Nonlinear Elliptic Equation)**

$$(1 + x^2)u_{xx} + (1 + y^2)u_{yy} = \alpha u(u_x + u_y) + f(x, y), \quad 0 < x, y < 1$$

with the exact solution  $u = e^x \sin(\pi y)$  adopted. The MAE for  $u$ ,  $u_x$  and  $u_y$  are tabulated in Table 4 for various values of  $\alpha$ .

**Problem 4 (2D steady-state Navier-Stokes equations in Cartesian coordinates)**

$$\begin{aligned} \frac{1}{R_e}(u_{xx} + u_{yy}) &= uu_x + vu_y + f(x, y), & 0 < x, y < 1 \\ \frac{1}{R_e}(v_{xx} + v_{yy}) &= uv_x + vv_y + g(x, y), & 0 < x, y < 1 \end{aligned}$$

where the constant  $R_e > 0$  is the Reynolds number.

The exact solution  $u = \sin(\pi x) \sin(\pi y)$ ,  $v = \cos(\pi x) \cos(\pi y)$  was adopted. The MAE for  $u$  and  $v$  are tabulated in Table 5 for  $R_e = 10, 10^2$  and  $10^3$ . Fig. 5 provides a comparison of the exact and numerical solutions.

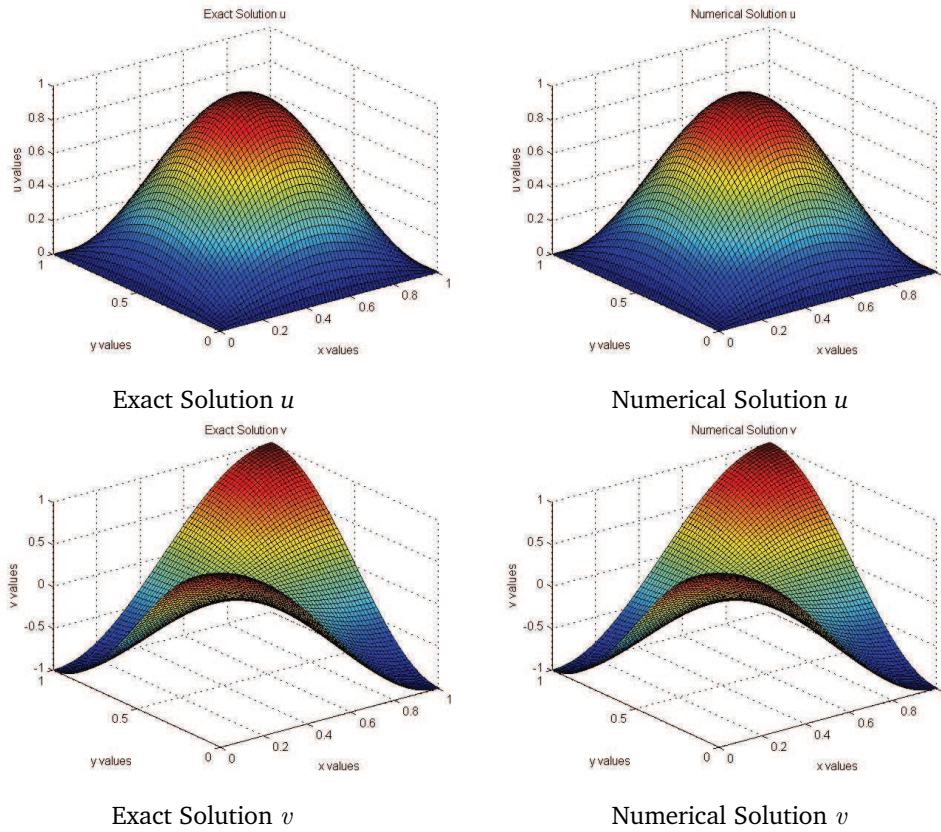


Figure 5: Solution of the 2D Navier-Stokes equations in Cartesian coordinates.

Table 5: Problem 4: The MAE.

$h$		Proposed $O(h^4)$ - methods			$O(h^4)$ - methods discussed in [12]		
		$R_e = 10$	$R_e = 10^2$	$R_e = 10^3$	$R_e = 10$	$R_e = 10^2$	$R_e = 10^3$
1/16	$u$	3.8170(-05)	7.9117(-04)	1.1370(-02)	4.2172(-05)	8.1235(-04)	1.4212(-02)
	$v$	2.0205(-05)	8.1179(-04)	1.6068(-02)	2.3785(-05)	8.4455(-04)	1.9872(-02)
1/32	$u$	2.4148(-06)	4.4070(-05)	6.8308(-04)	2.7868(-06)	4.7602(-05)	7.6880(-04)
	$v$	1.2680(-06)	4.6982(-05)	1.9517(-03)	1.5742(-06)	4.9852(-05)	1.9702(-03)
1/64	$u$	1.5149(-07)	2.6562(-06)	5.3901(-05)	1.7044(-07)	2.9885(-06)	5.8280(-05)
	$v$	7.9504(-08)	3.0175(-06)	1.0199(-04)	8.4324(-08)	3.3046(-06)	1.2692(-04)

**Problem 5** (2D steady-state Navier-Stokes equations in polar coordinates)(i) Spherical polar coordinates ( $r$ - $\theta$  plane):

$$\frac{1}{R_e} \left( u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{2}{r} u_r + \frac{\cot \theta}{r^2} u_{\theta} - \frac{2}{r^2} v_{\theta} - \frac{2}{r^2} u - \frac{2 \cot \theta}{r^2} v \right) \\ = uu_r + \frac{1}{r} vu_{\theta} - \frac{1}{r} v^2 + H(r, \theta), \quad 0 < r, \theta < 1,$$

Table 6: Problem 5(i): The MAE.

$h$		Proposed $O(h^4)$ - methods			$O(h^4)$ - methods discussed in [12]		
		$R_e = 10$	$R_e = 100$	$R_e = 500$	$R_e = 10$	$R_e = 100$	$R_e = 500$
1/4	$u$	4.0350(-03)	4.7668(-02)	1.0430(-01)	4.6880(-03)	5.3210(-02)	2.1678(-01)
	$v$	2.8464(-03)	2.4988(-02)	8.0180(-02)	3.6584(-03)	3.0824(-02)	8.8249(-02)
1/8	$u$	4.5574(-04)	6.9580(-03)	2.4922(-02)	5.0324(-04)	7.3214(-03)	2.9911(-02)
	$v$	1.9624(-04)	2.9897(-03)	1.5191(-02)	2.4454(-04)	3.4789(-03)	1.9923(-02)
1/16	$u$	2.6553(-05)	6.3235(-04)	2.4796(-03)	3.0355(-05)	6.6532(-04)	2.7697(-03)
	$v$	1.2965(-05)	3.0081(-04)	1.0288(-03)	1.5569(-05)	3.3180(-04)	1.3288(-03)

Table 7: Problem 5(ii): The MAE.

$h$		Proposed $O(h^4)$ - methods			$O(h^4)$ - methods discussed in [12]		
		$R_e = 10$	$R_e = 100$	$R_e = 500$	$R_e = 10$	$R_e = 100$	$R_e = 500$
1/8	$u$	4.4475(-04)	6.6056(-03)	7.5996(-03)	5.2574(-04)	7.4650(-03)	8.3699(-03)
	$v$	1.0410(-03)	9.8564(-03)	2.0399(-02)	1.8014(-03)	1.1565(-02)	2.8993(-02)
1/16	$u$	2.9587(-05)	8.6235(-04)	1.9389(-03)	3.2785(-05)	9.1532(-04)	2.4983(-03)
	$v$	6.6121(-05)	8.9744(-04)	3.6708(-03)	1.1060(-04)	9.3447(-04)	4.1807(-03)
1/32	$u$	1.9121(-06)	6.3020(-05)	3.8183(-04)	2.0111(-06)	6.7022(-05)	4.1381(-04)
	$v$	4.1297(-06)	5.7202(-05)	4.9547(-04)	6.8267(-06)	6.0204(-05)	5.2745(-04)

$$\begin{aligned} & \frac{1}{R_e} \left( v_{rr} + \frac{1}{r^2} v_{\theta\theta} + \frac{2}{r} v_r + \frac{\cot \theta}{r^2} v_{\theta} + \frac{2}{r^2} u_{\theta} - \frac{\cos ec^2 \theta}{r^2} v \right) \\ &= u v_r + \frac{1}{r} v v_{\theta} + \frac{1}{r} u v + I(r, \theta), \quad 0 < r, \theta < 1. \end{aligned}$$

The exact solution adopted was  $u = 2r^3 \cos \theta$ ,  $v = -5r^3 \sin \theta$ .

(ii) Cylindrical polar coordinates ( $r$ - $\theta$  plane):

$$\begin{aligned} & \frac{1}{R_e} \left( u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r - \frac{2}{r^2} v_{\theta} - \frac{1}{r^2} u \right) \\ &= u u_r + \frac{1}{r} v u_{\theta} - \frac{1}{r} v^2 + H(r, \theta), \quad 0 < r, \theta < 1, \\ & \frac{1}{R_e} \left( v_{rr} + \frac{1}{r^2} v_{\theta\theta} + \frac{1}{r} v_r + \frac{2}{r^2} u_{\theta} - \frac{1}{r^2} v \right) \\ &= u v_r + \frac{1}{r} v v_{\theta} + \frac{1}{r} u v + I(r, \theta), \quad 0 < r, \theta < 1. \end{aligned}$$

The exact solution adopted was  $u = r^3 \sin \theta$ ,  $v = 4r^3 \cos \theta$ .

The MAE for  $u$  and  $v$  are tabulated in Tables 6 and 7 for various values of the Reynolds number  $R_e > 0$ . Figs. 6 and 7 provide a comparison of the exact and numerical solutions to Problems 5(i) and 5(ii), respectively.

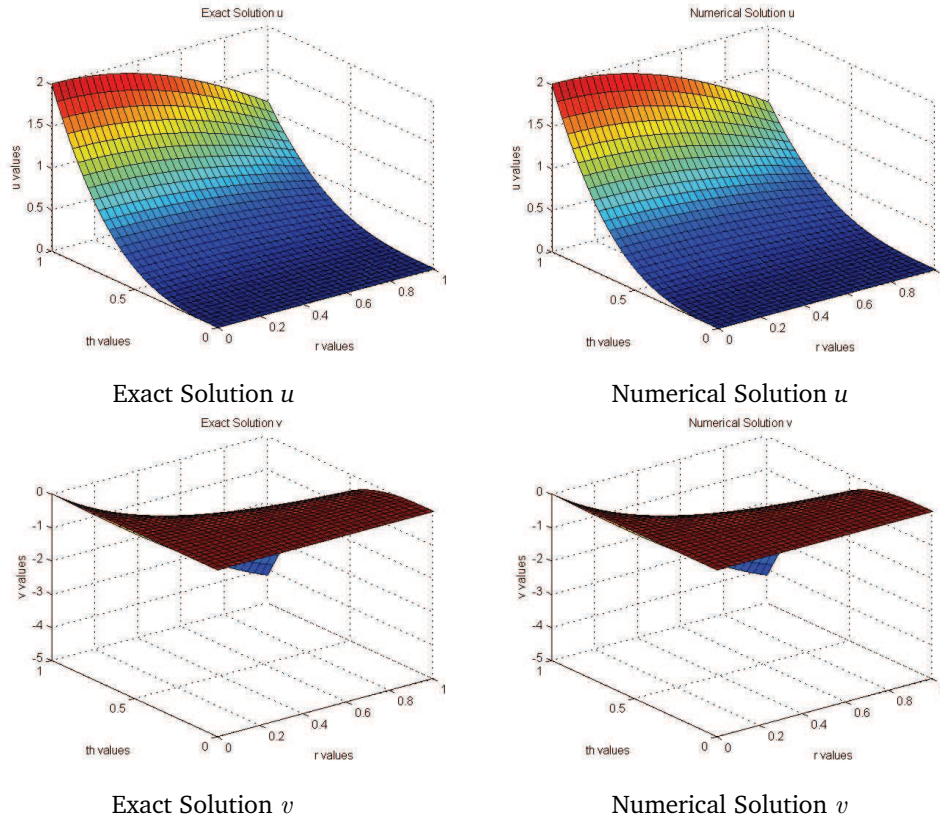


Figure 6: Solution of the Navier-Stokes equations in spherical polar coordinates  $(r, \theta)$ .

## 6. Conclusions

Nine-point compact difference methods of order four were developed in Ref. [12] for the numerical solution of the system of second order non-linear 2D elliptic equations (1.1), but these methods fail at singular points and a special treatment was required to solve singular problems. Indeed, it was found that such numerical methods fail if it is difficult to differentiate the singular coefficients twice. In this paper, a new stable method of accuracy four for the solution of the system of non-linear elliptic equations (1.1) is developed, using the same number of grid points. In addition, fourth order compact difference methods are derived to estimate the normal derivatives of the solutions. Our new numerical methods are directly applicable to elliptic equations in polar coordinates, and do not require any fictitious points for computation. No special technique is needed to solve singular elliptic problems, whether linear or nonlinear, and we obtain better results than some existing fourth order numerical methods have produced. Tables 3, 5, 6 and 7 show the results for model nonlinear equations, including the Navier-Stokes equations of motion. Although the order of accuracy drops for high Reynolds number in the Navier-Stokes equations, there is no numerical oscillation arising in the computed solution, in contrast to a second order



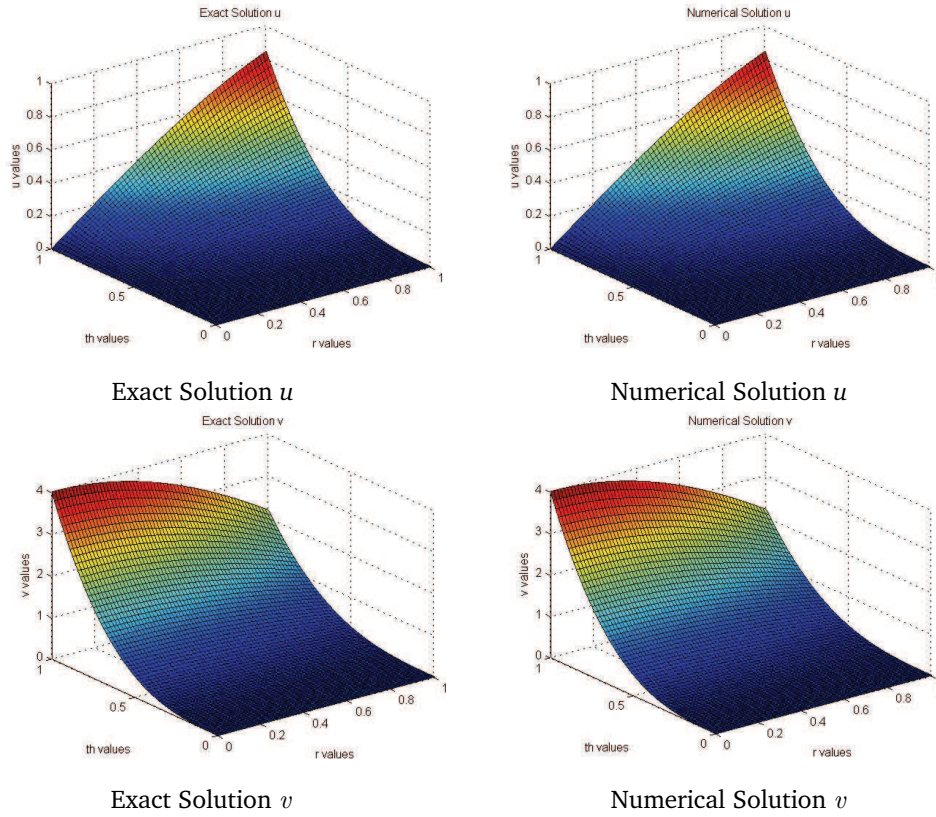


Figure 7: Solution of the Navier-Stokes equations in cylindrical polar coordinates in  $r$ - $\theta$  plane.

method that becomes totally unstable [3,12,13]. We are now extending our new methods to time-dependent problems.

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### References

- [1] M. Li, T. Tang, B. Fornberg, A compact fourth-order finite difference scheme for the steady state incompressible Navier Stokes equations, *Int. J. Numer. Meth. Fluids*, 20 (1995) 1137-1151.
- [2] W.F. Spitz, G.F. Carey, High Order Compact scheme for the steady stream function vorticity equations, *Int. J. Numer. Meth. Eng.*, 38 (1995) 3497-3512.

- [3] M.K. Jain, R.K. Jain, R.K. Mohanty, Fourth order difference methods for the system of 2-D nonlinear elliptic PDEs, *Numer. Meth. Partial Diff. Eqns.*, 7 (1991) 227-244.
- [4] U. Ananthakrishnaiah, G. Saldanha, A fourth-order finite difference scheme for 2D nonlinear elliptic PDEs, *Numer. Meth. Partial Diff. Eqns.*, 11 (1995) 33-40.
- [5] M.M. Gupta, A fourth-order poisson solver, *J. Comp. Phys.*, 55 (1984) 166-172.
- [6] M.M. Gupta, R.M. Manohar, J.H. Stephenson, A Single cell high order scheme for the convection-diffusion equation with variable coefficients *Int. J. Numer. Meth. Fluids*, 4 (1984) 641-651.
- [7] I.V. Yavneh, Analysis of a fourth-order compact scheme for convection diffusion, *J. Comp. Phys.*, 133 (1997) 361-364.
- [8] M.K. Jain, R.K. Jain, M. Krishna, Fourth order difference method for quasi-linear Poisson equation in cylindrical symmetry, *Comm. Numer. Meth. Eng.*, 10 (1994) 291-296.
- [9] J. Zhang, On convergence of iterative methods for a fourth-order discretization scheme, *App. Math. Letters*, 10 (1997) 49-55.
- [10] J. Zhang, On Convergence and performance of iterative methods with Fourth order Compact Schemes, *Numer. Meth. Partial Diff. Eqns.*, 14 (1998) 263-280.
- [11] G.F. Carey, *Computational Grids: Generation, Adaption and Solution Strategies*, Taylor and Francis, Washington, DC, 1997.
- [12] R.K. Mohanty, Order difference methods for a class of singular two-space dimensional elliptic boundary value problems, *Numer. Meth. Partial Diff. Eqns.*, 81 (1997) 229-247.
- [13] R.K. Mohanty, S. Dey, A new finite difference discretization of order four for 2D quasi-linear elliptic boundary value problems, *Int. J. Comp. Math.*, 76 (2001) 505-576.
- [14] R.K. Mohanty, S. Singh, A new fourth-order discretization for singularly perturbed two dimensional nonlinear elliptic boundary value problems, *App. Math. Comp.*, 175 (2006) 1400-1414.
- [15] M.M. Chawla, P.N. Shivakumar, An efficient finite difference method for two-point boundary value problems, *Neur. Par. Sci. Comp.*, 4 (1996) 387-396.
- [16] J.W. Stephenson, Single cell discretization of order two and four for biharmonic problems, *J. Comp. Phys.*, 55 (1984) 65-80.
- [17] R.K. Mohanty, S. Karraa, U. Arora, Fourth order nine-point unequal mesh discretization for the Solution of 2-D nonlinear elliptic PDEs, *Neur. Par. Sci. Comp.*, 14 (2006) 453-470.
- [18] G. Saldanha, Technical Note: A Fourth order finite difference scheme for a system of 2D nonlinear elliptic PDEs, *Numer. Meth. Partial Diff. Eqns.*, 17 (2001) 43-53.
- [19] K. Ito, Z. Qiao, A high order compact MAC finite difference scheme for the Stokes equations: Augmented variable approach, *J. Comp. Phys.*, 227 (2008) 8177-8190.
- [20] E. Erturk, C. Gökçöl, Fourth-order compact formulation of Navier-Stokes equations and driven cavity flow at high Reynolds numbers, *Int. J. Numer. Meth. Fluids*, 50 (2006) 421-436.
- [21] J. Liu, C. Wang, A fourth-order numerical method for the primitive equations formulated in mean vorticity, *Comm. Comp. Phys.*, 4 (2008) 26-55.
- [22] C.T. Kelly, *Iterative Methods for Linear and Non-Linear Equations*, SIAM Publications, Philadelphia, 1995.
- [23] R.S. Varga, *Matrix Iterative Analysis*, Springer, New York, 2000.
- [24] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM Publisher, Philadelphia, 2003.
- [25] L.A. Hageman, D.M. Young, *Applied Iterative Methods*, Dover Publication, New York, 2004.
- [26] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York, 1962.