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High Order Energy-Preserving Method of the "Good" Boussinesq Equation

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Abstract. The fourth order average vector field (AVF) method is applied to solve the "Good" Boussinesq equation. The semi-discrete system of the "good" Boussinesq equation obtained by the pseudo-spectral method in spatial variable, which is a classical finite dimensional Hamiltonian system, is discretizated by the fourth order average vector field method. Thus, a new high order energy conservation scheme of the "good" Boussinesq equation is obtained. Numerical experiments confirm that the new high order scheme can preserve the discrete energy of the "good" Boussinesq equation exactly and simulate evolution of different solitary waves well.

AMS subject classifications: 65D17
Key words: "Good" Boussinesq equation, average vector field method, solitary waves, conservation law.

1. Introduction

The "good" Boussinesq (GB) equation provides a balance between dispersion and nonlinearity, which leads to the existence of soliton solutions, similar to the Korteweg-de Vries (KdV) equation and cubic nonlinear Schrödinger equation [1,19]. It describes shallow water waves propagating in both directions and possesses a highly complicated mechanism of solitary waves interaction and differs from other nonlinear wave equations. The solitary waves exist only for a finite range of velocities, they can merge into a single soliton, and they interact with each other to give rise to the so-called anti-solitons [6,12,13,15] and the references therein. The general form of the GB equation can be written as

$$u_{tt} - u_{xx} + u_{xxxx} - (u^2)_{xx} = 0, \tag{1.1}$$

in the region D $= \{(x,t) \in \mathbb{R}^2 : -L/2 \leq x \leq L/2, t \geq 0\}$, subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{1.2}$$

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and the boundary conditions

$$u(-L/2,t)=0,\quad u(L/2,t)=0. \tag{1.3}$$

GB equation (1.1) possess the following two global conservation laws with the boundary conditions (1.3), namely global momentum conservation law

$$\mathcal{M}(t)=\int (vu_x)dx=\mathcal{M}(0), \tag{1.4}$$

and energy conservation law

$$\mathcal{E}(t)=\frac{1}{2}\int \Big(v^2+u^2+\frac{2}{3}u^3+u_x^2\Big)dx=\mathcal{E}(0), \tag{1.5}$$

where $v_x=u_t$.

Numerous numerical methods have been proposed to solve the GB equation (1.1). Frutos *et al.* [6]. developed the pseudo-spectral method of the GB equation; Soliton and anti-soliton interactions were studied by Manoranjan using the Galerkin-Petrov method [12, 13]; Ortega and Sanz-Serna [15] investigated the nonlinear stability and convergence behavior of numerical solutions; E-Zoheiry [20] studied the solitary wave interactions of the GB equation using finite-difference schemes; Huang *et al.* [7] constructed the multi-symplectic scheme of the GB equation; Aydin and Karasözen [2] constructed second order symplectic and multi-symplectic integrators for the GB equation using the two-stage Lobatto IIIA-IIIB partitioned Runge-Kutta method; Hu and Deng [8] proposed the implicit multi-symplectic scheme of the generalized Boussinesq equation. Chen [4] investigated the multi-symplectic Fourier pseudo-spectral method of the GB equation. Zeng [21] developed a new fifteen-point difference scheme which is equivalent to the multi-symplectic Preissman integrator.

Hamiltonian system, which has the energy conservation property, is one of the most important dynamical system. It is applied widely in the structural biology, pharmacology, semi-conductor, super-conducting, plasma, celestial mechanics, material, and partial differential equation, and so on. Feng and his research group [11, 16] developed symplectic geometric algorithms of the Hamiltonian system. Bridges and Reich developed the symplectic geometric algorithms to the multi-symplectic geometric algorithms of the partial differential equations [3]. Symplectic and multi-symplectic geometric algorithms [3,9,11,16,18], which have a long accurately computing capability, have been used widely in astronomy, molecular mechanics and quantum mechanics, electromagnetism, optics and so on. However, the symplectic and multisymplectic method only approximately preserve the energy of the Hamiltonian system (they exactly preserve a modified Hamiltonian) [9].

Recently, Quispel *et al.* [17] and McLachlan *et al.* [14] proposed the second order averaged vector field (AVF) method, which can preserve the energy of the Hamiltonian

system exactly.

$$\frac{z^{n+1}-z^n}{\tau} = \int_0^1 f((1-\xi)z^n+\xi z^{n+1})d\xi = S\int_0^1 \nabla H((1-\xi)z^n+\xi z^{n+1})d\xi. \tag{1.6}$$

where S is the constant,orthogonal and skew-symmetric matrix and $H(z) : \mathbb{R}^{2d} \to \mathbb{R}$ is usually called as Hamiltonian energy. The second order AVF method (1.6) has been successfully applied to solve the energy conservation partial differential equation [5]. Motivated by the modified vector field of integral-preserving methods which can preserve the original integral, the fourth order AVF scheme [17] has been derived by using the modified vector field for the second order AVF scheme (1.6), as following

$$\frac{z^{n+1}-z^n}{\tau} = \int_0^1 f((1-\xi)z^n+\xi z^{n+1})d\xi - \frac{1}{12}\tau^2 S\mathcal{H}S\mathcal{H}\int_0^1 f((1-\xi)z^n+\xi z^{n+1})d\xi, \tag{1.7}$$

where $\mathcal{H}_{ij} = \frac{\partial^2 H}{\partial z_i \partial z_j}(\frac{z_j^{n+1}+z_j^n}{2})$, z_k is the kth component of the vector z. In this paper, we apply the fourth order AVF method to construct the high order energy preserving scheme of the GB equation.

This paper is organized as follows. In section 2, a new high order energy preserving scheme of the GB equation is obtained by the pseudo-spectral method in spatial variable and the fourth order AVF method in time variable. In section 3, we investigate the new scheme by simulating evolution of different solitary waves. Finally conclusions are given in section 4.

2. High order energy preserving scheme of the GB equation

The GB equation (1.1) can be expressed in the infinite dimensional Hamiltonian system of the form

$$\frac{d\mathbf{z}}{dt} = J\frac{\delta H(\mathbf{z})}{\delta \mathbf{z}}, \quad J = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}. \tag{2.1}$$

where $\mathbf{z} = [u, v]^T$,∂_x is the first order partial derivative and the Hamiltonian function is

$$H(\mathbf{z}) = \frac{1}{2}\int (v^2 + u^2 + \frac{2}{3}u^3 + u_x^2)dx. \tag{2.2}$$

We solve the Hamiltonian system (2.1) by the pseudo-spectral method in spatial variable and fourth order AVF method in time variable.

Supposing the spatial domain $\Omega = [-L/2, L/2]$. The interval Ω is divided into N equal subinterval with grid spacing $h = L/N$, where the integer N is an even number. The spatial collocation nodes are given by $x_j = -L/2 + hj, j = 0, \cdots, N-1$. Denoting u_j and v_j to the approximation to $u(x_j, t)$ and $v(x_j, t)$,respectively. Defining

$$S_N = \Big\{g_j(x); j = 0, \cdots, N-1\Big\}, \tag{2.3}$$

as the interpolation space, where $g_j(x)$ is a trigonometric polynomial of degree $N/2$ given explicitly by

$$g_j(x) = \frac{1}{N}\sum_{l=-N/2}^{N/2}\frac{1}{c_l}e^{il\mu(x-x_j)}, \tag{2.4}$$

where $c_l = 1(|l| \neq N/2), c_{-N/2} = c_{N/2} = 2, \mu = \frac{2\pi}{L}$.

For any function $u(x,t)$ and $v(x,t) \in C^0(\Omega)$, the interpolation operator I_N is defined as follow [4]

$$I_N u(x,t) = \sum_{l=0}^{N-1} u_l g_l(x), \qquad I_N v(x,t) = \sum_{l=0}^{N-1} v_l g_l(x). \tag{2.5}$$

The trigonometric interpolation operator I_N at the collocation points x_j satisfies

$$I_N u(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} u_l g_l(x_j) = u_j, \quad 0 \leq j \leq N-1, \tag{2.6}$$

$$I_N v(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} v_l g_l(x_j) = v_j, \quad 0 \leq j \leq N-1. \tag{2.7}$$

Suppose $\mathbf{u} = (u_0, u_1, \cdots, u_{N-1})^T$, $\mathbf{v} = (v_0, v_1, \cdots, v_{N-1})^T$ and define

$$(D_k)_{j,l} = \frac{d^k g_l(x_j)}{dx^k}, \tag{2.8}$$

where D_k is the k order Fourier differential matrix. The value for the derivatives $\frac{d}{dx}I_N u(x,t)$ and $\frac{d^2}{dx^2}I_N u(x,t)$ at the collocation points x_j are obtained in terms of the value of u_j, i.e

$$\frac{d}{dx}I_N u(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} u_l \frac{dg_l(x_j)}{dx} = (D_1\mathbf{u})_j, \tag{2.9}$$

$$\frac{d^2}{dx^2}I_N u(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} u_l \frac{d^2 g_l(x_j)}{dx^2} = (D_2\mathbf{u})_j. \tag{2.10}$$

Similarly, we can get

$$\frac{d}{dx}I_N v(x,t)|_{x=x_j} = (D_1\mathbf{v})_j, \qquad \frac{d^2}{dx^2}I_N v(x,t)|_{x=x_j} = (D_2\mathbf{v})_j, \tag{2.11}$$

where D_1 and D_2 represent the first-order and second-order Fourier differential matrices with the elements, respectively

$$(D_1)_{i,j} = \begin{cases} \frac{1}{2}\mu(-1)^{i+j}\cot(\mu\frac{x_i - x_j}{2}), & i \neq j \\ 0. & i = j \end{cases} \tag{2.12}$$

$$(D_2)_{i,j} = \begin{cases} \frac{1}{2}\mu^2(-1)^{i+j+1}\frac{1}{\sin^2(\mu\frac{x_i-x_j}{2})}, & i \neq j \\ -\mu^2\frac{N^2+2}{12}. & i = j \end{cases} \tag{2.13}$$

The semi-discrete Fourier pseud-spectral approximation for system (2.1) is constructed as follow

$$((I_N u(x,t))_t - \partial_x(I_N v(x,t)))|_{x=x_j} = 0, \tag{2.14}$$

$$((I_N v(x,t))_t + \partial_x(I_N u(x,t))_{xx} - \partial_x(I_N u(x,t) + I_N(u(x,t))^2))|_{x=x_j} = 0. \tag{2.15}$$

Using the Fourier differential matrix (2.12)-(2.13) to Eqs. (2.14)-(2.15) and approximating ∂_x by first-order Fourier differential matrix D_1, we can get

$$\frac{du_j}{dt} = (D_1\mathbf{v})_j, \tag{2.16}$$

$$\frac{dv_j}{dt} = -(\mathcal{A}\mathbf{u})_j + \sum_{l=1}^{N} d_{j,l}\left(u_l + u_l^2\right), \tag{2.17}$$

where $\mathcal{A} = D_1D_2$ and $d_{l,k}$ is the lth row and kth column element of the matrix D_1.

The semi-discrete Fourier pseud-spectral approximation of Eqs. (2.16,2.17) is equivalent to

$$\frac{d\mathbf{Z}}{dt} = f(\mathbf{Z}) = \mathcal{J}\nabla H(\mathbf{Z}), \quad \mathcal{J} = \begin{bmatrix} O & D_1 \\ D_1 & O \end{bmatrix}. \tag{2.18}$$

where $\mathbf{Z} = [\mathbf{u}^T, \mathbf{v}^T]^T$ and the corresponding discrete Hamiltonian energy function is

$$H(\mathbf{Z}) = \frac{1}{2}\sum_{j=0}^{N-1}\left(v_j^2 + u_j^2 + \frac{2}{3}u_j^3\right) - \frac{1}{2}\mathbf{u}^T D_2\mathbf{u}. \tag{2.19}$$

The semi-discrete Hamiltonian system (2.21) is solved by the fourth order AVF method (1.7), we can get

$$\begin{aligned} &\frac{\mathbf{Z}^{n+1} - \mathbf{Z}^n}{\tau} \\ &= \int_0^1 f\big((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1}\big)d\xi - \frac{\tau^2}{12}\hat{\mathcal{J}}^2\int_0^1 f\big((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1}\big)d\xi, \end{aligned} \tag{2.20}$$

where

$$\hat{\mathcal{J}}^2 = \begin{bmatrix} -D_1(\mathcal{A}-\mathcal{B}) & O \\ O & -(\mathcal{A}-\mathcal{B})D_1 \end{bmatrix}.$$

In the above matrix, $\mathcal{B} = D_1\mathcal{D}$ and $\mathcal{D}$ can been expressed as

$$\mathcal{D} = \begin{bmatrix} u_1^{n+1}+u_1^n & 0 & \cdots & 0 \\ 0 & u_2^{n+1}+u_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_N^{n+1}+u_N^n \end{bmatrix} + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2.21}$$

It can be verified that Eq. (2.20) is equivalent to

$$\frac{u_j^{n+1}-u_j^n}{\tau}=\sum_{l=1}^{N}d_{j,l}\int_0^1((1-\xi)v_l^n+\xi v_l^{n+1})d\xi+\frac{\tau^2}{12}\sum_{l=1}^{N}d_{j,l}\left(\sum_{p=1}^{N}(a_{l,p}-b_{l,p})\left(\sum_{s=1}^{N}d_{p,s}\int_0^1\left((1-\xi)v_s^n+\xi v_s^{n+1}\right)d\xi\right)\right), \tag{2.22}$$

$$\begin{aligned}\frac{v_j^{n+1}-v_j^n}{\tau}=&-\sum_{l=1}^{N}a_{j,l}\int_0^1\left((1-\xi)u_l^n+\xi u_l^{n+1}\right)d\xi\\&+\sum_{l=1}^{N}d_{j,l}\int_0^1\left(((1-\xi)u_l^n+\xi u_l^{n+1})+((1-\xi)u_l^n+\xi u_l^{n+1})^2\right)d\xi\\&+\frac{\tau^2}{12}\sum_{l=1}^{N}(a_{j,l}-b_{j,l})+\left(\sum_{p=1}^{N}d_{l,p}\left(-\sum_{s=1}^{N}a_{p,s}\int_0^1\left((1-\xi)u_s^n+\xi u_s^{n+1}\right)d\xi\right)\right)\\&+\frac{\tau^2}{12}\sum_{l=1}^{N}(a_{j,l}-b_{j,l})+\sum_{p=1}^{N}d_{l,p}\left(\sum_{s=1}^{N}d_{p,s}\int_0^1((1-\xi)u_s^n+\xi u_s^{n+1}\right.\\&\left.+((1-\xi)u_s^n+\xi u_s^{n+1})^2)d\xi\right).\end{aligned} \tag{2.23}$$

The high order energy-preserving scheme (2.22)-(2.23) is equivalent to

$$\begin{aligned}&\frac{u_j^{n+1}-u_j^n}{\tau}\\&=\left(D_1(\frac{\mathbf{v}^{n+1}+\mathbf{v}^n}{2})\right)_j+\frac{\tau^2}{12}\sum_{l=1}^{N}d_{j,l}\left(\sum_{p=1}^{N}(a_{l,p}-b_{l,p})\left(\sum_{s=1}^{N}d_{p,s}\left(\frac{v_s^{n+1}+v_s^n}{2}\right)\right)\right),\end{aligned} \tag{2.24}$$

$$\begin{aligned}&\frac{v_j^{n+1}-v_j^n}{\tau}\\&=-\left(\mathcal{A}(\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2})\right)_j+\sum_{l=1}^{N}d_{j,l}\left(\frac{u_l^n+u_l^{n+1}}{2}+\frac{(u_l^n)^2+u_l^nu_l^{n+1}+(u_l^{n+1})^2}{3}\right)\\&\quad+\frac{\tau^2}{12}\sum_{l=1}^{N}(a_{j,l}-b_{j,l})\left(\sum_{p=1}^{N}d_{l,p}\left(-\sum_{s=1}^{N}a_{p,s}(\frac{u_s^{n+1}+u_s^n}{2})\right)\right)\\&\quad+\frac{\tau^2}{12}\sum_{l=1}^{N}(a_{j,l}-b_{j,l})\sum_{p=1}^{N}d_{l,p}\left(\sum_{s=1}^{N}d_{p,s}\left(\frac{u_s^n+u_s^{n+1}}{2}+\frac{(u_s^n)^2+u_s^nu_s^{n+1}+(u_s^{n+1})^2}{3}\right)\right).\end{aligned} \tag{2.25}$$

where $a_{l,k}$ and $b_{l,k}$ is the lth row and kth column element of the matrices $\mathcal{A}$ and $\mathcal{B}$ respectively. Eqs. (2.24)-(2.25) can been written as matrix-vector form

$$\begin{bmatrix} \frac{u_j^{n+1}-u_j^n}{\tau} \\ \frac{v_j^{n+1}-v_j^n}{\tau} \end{bmatrix} = \begin{bmatrix} D_1\frac{v^n+v^{n+1}}{2} \\ -(\mathcal{A}\frac{u^n+u^{n+1}}{2} + D_1\frac{(u^n).^2+u^n.*u^{n+1}+(u^{n+1}).^2}{3} \end{bmatrix} + \frac{\tau^2}{12}\begin{bmatrix} D_1(\mathcal{A}-\mathcal{B}) & O \\ O & (\mathcal{A}-\mathcal{B})D_1 \end{bmatrix}\begin{bmatrix} D_1\frac{v^n+v^{n+1}}{2} \\ -(\mathcal{A}\frac{u^n+u^{n+1}}{2} + D_1\frac{(u^n).^2+u^n.*u^{n+1}+(u^{n+1}).^2}{3} \end{bmatrix}.$$

3. Numerical experiment

In this section, we investigate the high order energy-preserving scheme (2.24)-(2.25) by simulating evolution of different solitary waves under different initial conditions and comparing the relative discrete energy errors.The energy function of the GB equation is

$$\mathcal{E}^n = \frac{1}{2}\sum_{j=0}^{N-1}\left((v_j^n)^2 + (u_j^n)^2 + \frac{2}{3}(u_j^n)^3 + ((D_1\mathbf{u})_j^n)^2\right). \tag{3.1}$$

We take the relative discrete energy errors as

$$RE(t) = \left|\frac{\mathcal{E}^n - \mathcal{E}^0)}{\mathcal{E}^0}\right|, \tag{3.2}$$

where $\mathcal{E}^0$ is the initial discrete energy and $RE(t)$ is the relative energy errors at $t = n\tau$.

Example 3.1. Firstly, we consider evolution of single solitary wave by choosing the initial condition

$$f(x) = -Asech^2\left(\sqrt{\frac{A}{6}}(x-x^0)\right), \tag{3.3}$$

$$v(x,0) = Ac\ sech^2\left(\sqrt{\frac{A}{6}}(x-x^0)\right), \tag{3.4}$$

where $v(x,0) = \int_{-L/2}^{x} u_t(y,0)dy = -cu(x,0)$, A is the amplitude,c is the velocities. We take $L = 200$, $N = 300$, $\tau = 0.025$, $x^0 = -40$, $c = \sqrt{14/15}$ and boundary condition $u(\pm L/2, t) = 0$.

Fig. 1 shows evolution of single solitary wave at $t \in [0, 80]$. The solitary wave emerges without any changes in their shapes. The solitary wave can propagate well at given velocities. Fig. 2 displays the relative energy errors at $t \in [0, 80]$. The energy errors can be neglected (up to the machine accuracy). From Figs. 1 and 2 we can get that the high order energy-preserving scheme not only has good numerical performance in simulating evolution of single solitary wave, but also preserves the discrete energy of the GB equation exactly.

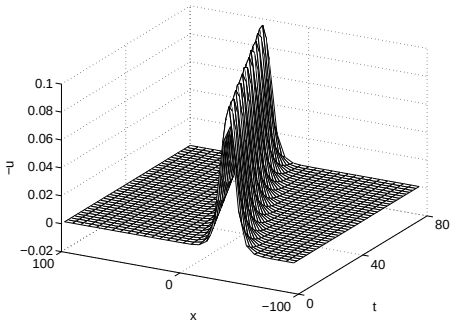

Figure 1: Evolution of single solitary wave at $A = 0.1$,

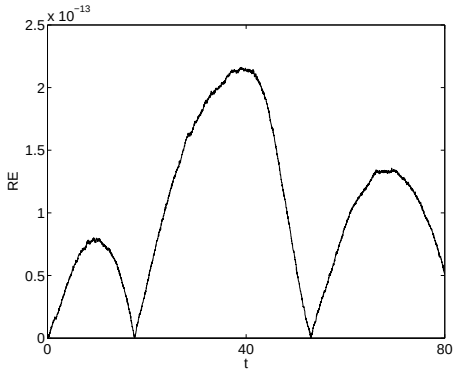

Figure 2: The relative energy errors at $A = 0.1$.

Example 3.2. Then, we consider the birth of solitary waves by choosing the initial condition

$$f(x) = -Asech^2\left(\sqrt{\frac{A}{6}}(x + x^0)\right), \tag{3.5}$$

$$v(x, 0) = 0, \tag{3.6}$$

We take $L = 200$, $N = 400$, $\tau = 0.02$,$x^0 = 0$ and boundary condition $u(\pm L/2, t) = 0$.

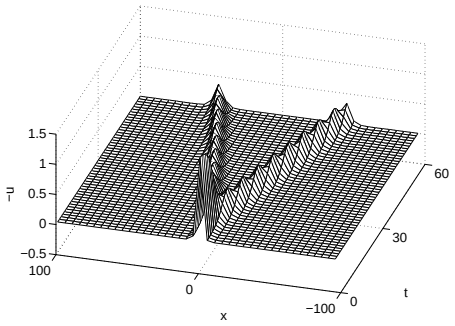

Figure 3: Evolution of solitary wave at $A = 1.48$,

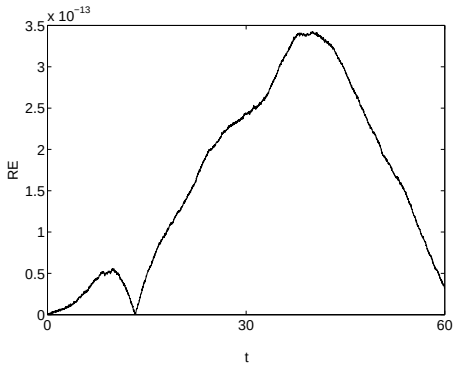

Figure 4: The relative energy errors at $A = 1.48$.

Fig. 3 shows that the numerical solution at $t \in [0, 60]$ with amplitude $A = 1.48$. the initial profile of the wave is splitted into two pulses and moving in different direction. The corresponding relative energy error is investigated in Fig. 4.

Fig. 5 shows the numerical solution at $A = 1.50$. The preservation of the the relative energy error over time is given in Fig. 6. We can see that the energy of the GB equation can been preserved exactly. From Figs. 3 and 6, we have noticed that the high order energy-preserving scheme has good numerical performance in simulating evolution of solitary waves, which is consistent with the results obtained by the symplectic

and multi-symplectic method [2]. Moreover, it can preserve the energy conservation property of the GB equation precisely.

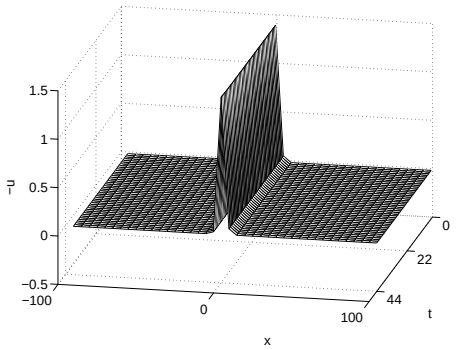

Figure 5: Evolution of solitary wave at $A = 1.50$,

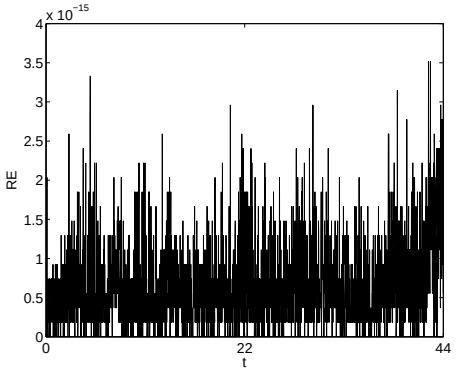

Figure 6: The relative energy errors at $A = 1.50$.

Example 3.3. Finally, we consider interaction of two solitary waves with the initial condition

$$f(x) = -A_1 sech^2\left(\sqrt{\frac{A_1}{6}}(x - x_1^0)\right) - A_2 sech^2\left(\sqrt{\frac{A_2}{6}}(x - x_2^0)\right), \tag{3.7}$$

$$v(x,0) = A_1 c_1\ sech^2\left(\sqrt{\frac{A_1}{6}}(x - x_1^0)\right) + A_2 c_2\ sech^2\left(\sqrt{\frac{A_2}{6}}(x - x_2^0)\right). \tag{3.8}$$

We take $L = 200$ and boundary condition $u(\pm L/2, t) = 0$.

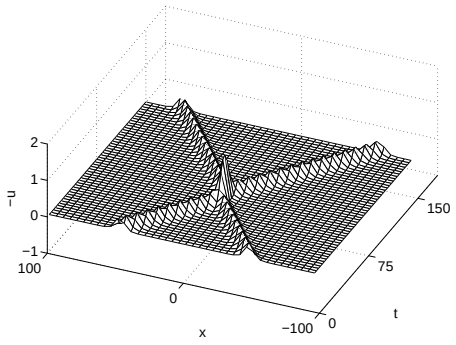

Figure 7: Interaction of two solitary waves at $A_1 = 0.369$ and $A_2 = 0.369$ with $x_1^0 = 50$ and $x_2^0 = -50$,

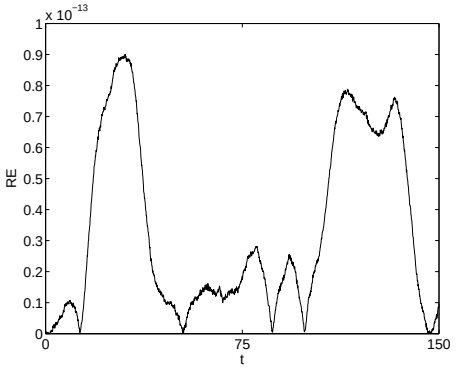

Figure 8: The relative energy errors at $A_1 = 0.369$ and $A_2 = 0.369$ with $x_1^0 = 50$ and $x_2^0 = -50$.

Fig. 7 shows the typical interaction of two solitary waves with equal amplitudes $A_1 = A_2 = 0.369$ and different velocities $c_1 = 0.86833$ and $c_2 = -0.86833$, over

the time interval $0 \leq t \leq 150$ with $N = 320$, $\tau = 0.1$.Two solitary waves initially located at the positions $x_1^0 = 50$ and $x_2^0 = -50$, traveling toward each other, and colliding. During the collision, the amplitude of the pulse doubled and two waves leave each other without changing their shape.The numerical results reveal that the collisions generated no radiation. The relative energy errors of the two solitary waves is shown in Fig. 8. We note that the relative energy errors of the GB equation can been preserved exactly (up to the machine accuracy).

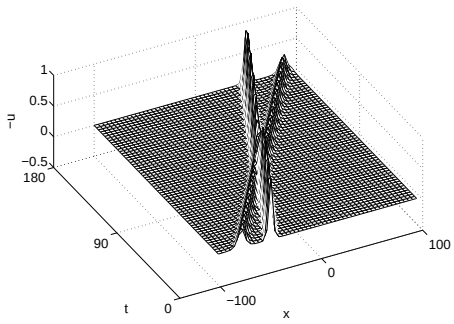

Figure 9: Interaction of two solitary waves at $A_1 = 0.3$ and $A_2 = 1.0$ with $x_1^0 = -80$ and $x_2^0 = -50$,

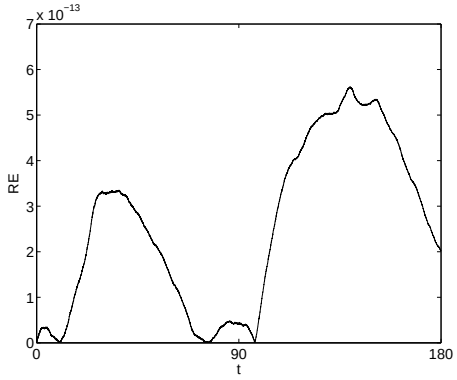

Figure 10: The relative energy errors at $A_1 = 0.3$ and $A_2 = 1.0$ with $x_1^0 = -80$ and $x_2^0 = -50$.

Fig. 9 shows the interaction of two solitary waves over the time interval $0 \leq t \leq 180$ with $N = 300$, $\tau = 0.025$, which is located at $x_1^0 = -80$ and $x_2^0 = -50$ respectively, moving in the same direction with different velocities $c_1 = 0.8944$ and $c_2 = 0.5774$. The interaction has taken place and the faster wave interacted and separated from the slower one and left it behind, which is consistent with the results obtained by the fourth order finite difference method [10]. In Fig. 10 we track the the energy conserved quantity during the interaction scenario. Obviously, the high order energy-preserving scheme can preserve the relative energy error of the GB equation exactly. Moreover,this type of interactions for the GB equation, which can preserve the energy in the long time, seems to not been reported in the literatures as authors know.

4. Concluding remarks

In this paper, a new high order energy-preserving scheme of the GB equation is obtained by the pseudo-spectral method in spatial variable and the fourth order AVF method in time variable. Numerical results show that the new high order energy-preserving scheme can well simulate different solitary wave behaviors of the GB equation in long time and preserve the discrete energy conservation of the GB equation precisely. Obviously, the high order AVF method give the new choice in simulating the energy conservation partial differential equation numerically.

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