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Evolution Solutions of Perturbed Khokhlov-Zabolotskaya Equation

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Abstract. This paper is concerned with the exact solutions of Khokhlov-Zabolotskaya (KZ) equation with general perturbation. With the help of appropriate transformations and assumptions, the wave theory of Hopf equation is applied to get partial exact solutions. In addition, some examples and numerical simulations are presented to illustrate our analytical results.

AMS Subject Classifications: 35A09, 35E15

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Key Words: Perturbed KZ equation; exact solution; Hopf equation.

1 Introduction

The canonical equation for weakly nonlinear and weakly diffracting waves is the Khokhlov-Zabolotskaya (KZ) equation [1], which can be written in the general form

$$(u_t+\alpha uu_x)_x+\frac{1}{2}u_{yy}=0, \tag{1.1}$$

where $u=u(x,y,t)$ is typically a measure of the wave disturbance, x is a spatial variable measured in a frame moving with the wave, y is a transverse spatial variable, and t is a time-like variable. Then constant α is a quadratic nonlinearity parameter. It is well known that (1.1) provides an accurate description of the evolution of many systems including those corresponding to acoustic waves in air and water, shallow water waves, acoustic and magnetosonic waves in in nonlinear medium without dispersion or absorption. The study of numerous approximations to the KZ equation in (1.1) has a prominent history

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concerning the symbiotic interaction of mathematical model and scientific computing to gain insight in the topic.

Actually, for nonlinear optics, Alfvén waves in magnetohydrodynamics, and shear waves in an isotropic nonlinear solid, the quadratic nonlinearity coefficient α is small or vanishing altogether, (1.1) reduces to

$$u_t+\frac{1}{2}u_{yy}\approx 0.$$

However, if the initial wavefront is curved, near the focal region nonlinear effects will become noticeable. Consequently, Zabolotskaya [2] derived the explicit form of the extension of (1.1) for the case of shear waves propagating in a nonlinear solid in the undisturbed state. Due to the net effect of perturbation analysis, the uu_x term in (1.1) is replaced by the cubic term. (1.1) enjoys a new version

$$(u_t+\alpha u^2u_x)_x+\frac{1}{2}u_{yy}=0.$$

which was investigated by Kluwick-Cox [3] and Cramer-Webb [4]. Afterwards, a series of extension systems with weakly relaxing, weakly dissipative, and weakly dispersive were developed, one can refer to [5–9] and the latest result [10]. In this paper, we consider a perturbed acoustic wave equation with general nonlinear term and mixed derivative,

$$\big(u_t+f(u)(u_x+\gamma u_y)\big)_x+\theta u_{yy}+\frac{1}{2}\triangle_\perp u=0, \tag{1.2}$$

where $f(u)=1+u^n$, which is an arbitrary function of u. γ and θ are real constants, γ decides the propagation direction on xOy plane, and $\triangle_\perp$ denotes the transverse Laplacian in Cartesian coordinates. Our goal is to present a simple and direct method of finding partial exact solutions of KZ type equations with the help of the Hopf equation, which is available to classical KZ equation (1.1).

The rest of this paper is organized as follows. In Section 2, some elementary definitions of Hopf equation have been presented and the implicit solutions of generalized KZ equation are given by mathematical analysis. In Section 3, some examples and relevant numerical simulations are shown at the end of the paper.

2 Outline of the derivation

Here we first recall the solution of Hopf equation. As the special case of Burgers model [11], the general Hopf equation enjoys the following form:

$$u_t+\tilde{f}(u)u_x=0. \tag{2.1}$$

By the hodograph transformation $x=x(u,t)$, we have $u_x=x_u^{-1}$ and $u_t=-x_t x_u^{-1}$. Hopf equation can be rewritten to $x_t-\tilde{f}(u)=0$, which implies the solution

$$x-\tilde{f}(u)t=F(u),$$

where $F(u)$ is an arbitrary function.

In what follows we return to the KZ equation (1.2). In order to seek for the exact solution, we assume that

$$u(x,t,\mathbf{z})=\alpha(T)U(X,T),\quad X=x+ky+\varphi(t,\mathbf{z}),\quad T=T(t), \tag{2.2}$$

where arbitrary number N of transverse spatial coordinates $\mathbf{z}=(z_1,\ldots,z_N)$, $\triangle_\perp=\frac{\partial^2}{\partial z_1^2}+\ldots+\frac{\partial^2}{\partial z_N^2}$, k is an arbitrary non-zero constant. By a computation with respect to the derivatives, we have

$$\begin{aligned}&u_t=\alpha_T T_t U+\alpha(U_T T_t+U_X\varphi_t),\ u_x=\alpha U_X,\ u_y=\alpha k U_X,\ u_{yy}=\alpha k^2 U_{XX},\\&\triangle_\perp u=\alpha(U_{XX}(\nabla\varphi)^2+U_X\triangle_\perp\varphi).\end{aligned} \tag{2.3}$$

Substituting (2.3) into (1.2), we have

$$\left[U_T T_t+\left(T_t\alpha_T/\alpha+\frac{1}{2}\triangle_\perp\varphi\right)U+(\varphi_t+(1+\alpha^n U^n)(1+\gamma k)+k^2\theta+\frac{1}{2}(\nabla\varphi)^2)U_X\right]_X=0, \tag{2.4}$$

where subscripts denote derivatives with respect to corresponding variables. If we assume that

$$T_t=\alpha^n,\ \varphi_t+\frac{1}{2}(\nabla\varphi)^2+1+\gamma k+k^2\theta=q\alpha^n,\ \triangle_\perp\varphi=2\alpha^n(m-\alpha_T/\alpha), \tag{2.5}$$

where q and m are some constants, then Eq. (2.4) transforms to the equation

$$(U_T+mU+((1+\gamma k)U^n+q)U_X)_X=0. \tag{2.6}$$

The term with q can be excluded by means of the additional replacement

$$\varphi=\psi(t,\mathbf{z})+qT(t), \tag{2.7}$$

then

$$\varphi_t=\psi_t+qT_t. \tag{2.8}$$

Substituting (2.8) into $\varphi_t+\frac{1}{2}(\nabla\varphi)^2+1+\gamma k+\theta k^2=q\alpha^n$, we get

$$\psi_t+\frac{1}{2}(\nabla\psi)^2+1+\gamma k+\theta k^2=0,$$

then

$$\psi(t,\mathbf{z})=\frac{1}{2}\sum_{p=1}^{N}\frac{z_p^2}{t-t_p}-(1+\gamma k+\theta k^2)t, \tag{2.9}$$

where t_p, $p=1,\ldots,N$, are integration constants. Hence Eq. (2.6) transforms to the general Hopf equation

$$(U_T+mU+(1+\gamma k)U^nU_X)_X=0. \tag{2.10}$$

To obtain $T(t)$ in (2.2), substituting (2.7) and (2.9) into $\triangle_\perp\varphi=2\alpha^n(m-\alpha_T/\alpha)$ gives

$$\alpha^{n-1}\alpha_T+\frac{1}{2}\sum_{p=1}^{N}\frac{1}{t-t_p}=m\alpha^n.$$

Introducing $y=\alpha^n$, we arrive at the Bernoulli equation

$$y'+\frac{1}{2}\sum_{p=1}^{N}\frac{1}{t-t_p}ny=nmy^2,$$

which can be solved by a standard method to give

$$y=T_t=\mathcal{C}e^{-nmT(t)}\prod_{p=1}^{N}|t-t_p|^{-n/2}, \tag{2.11}$$

where $\mathcal{C}$ is an integration constant. If $m=0$ then

$$T_t=\mathcal{C}\prod_{p=1}^{N}|t-t_p|^{-n/2}. \tag{2.12}$$

Therefore, we have reduced finding $T(t)$ to integration of the function in the right side of Eqs. (2.11) or (2.12) while $\alpha(t)=(T_t)^{1/n}$.

As a result of the above calculations, the variables in (2.2) can be considered as known and it remains to find the solution of the general Hopf Eq. (2.10). Integration with respect to X gives at once

$$U_T+mU+(1+\gamma k)U^nU_X=h(T),$$

where $h(T)$ is an arbitrary function to be determined from the initial conditions. Consider the linear differential equation

$$U_T+mU=h(T),$$

with the initial $U_0(X)$ at $t=0$. Then

$$U=(U_0(X)+H(T))e^{-mT},$$

where $H(T)=\int_0^T e^{m\tau}h(\tau)\mathrm{d}\tau$. The characteristic curve is determined by the equation

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}T}&=(1+\gamma k)U^n=(1+\gamma k)(U_0+H(T))^n e^{-nmT}\\&=(1+\gamma k)\sum_{j=0}^{n} C_n^j U_0^{n-j}H^j(T)e^{-nmT}.\end{aligned}$$

Integrating this from 0 to T, we have

$$X=X_0+(1+\gamma k)\sum_{j=0}^{n} C_n^j U_0^{n-j}\int_0^T H^j(\tau)e^{-nm\tau}\mathrm{d}\tau. \tag{2.13}$$

Let the initial distribution $U(X)$ at $T=0$ be given by the function $U_0=F^{-1}(X_0)$. Then exclusion of X_0 and U_0 from (2.13) gives the final result:

$$X=F(Ue^{mT}-H(T))+(1+\gamma k)\sum_{j=0}^{n} C_n^j (Ue^{mT}-H(T))^{n-j}\int_0^T H^j(\tau)e^{-nm\tau}\mathrm{d}\tau.$$

This equation determines implicitly u as a function of x, z, t through variables

$$\begin{aligned}&X=x+ky+qT(t)+\frac{1}{2}\sum_{p=1}^{N}\frac{z_p^2}{t-t_p}-(1+\gamma k+\theta k^2)t,\\&T=T(t),\ u=\alpha(T)U(X,T),\end{aligned} \tag{2.14}$$

in terms of two arbitrary functions $F(U)$ and $H(T)$ which have to be found from the initial conditions.

Remark 2.1. In fact (2.14) is a special solution of Eq. (1.2), because the assumptions in (2.5) lead to the loss of partial solutions.

3 Examples

Example 3.1. We consider the spherical generalized KZ equation

$$\big(u_t+(1+u)(u_x+u_y)\big)_x-u_{yy}+\frac{1}{2}u_{zz}=0. \tag{3.1}$$

Choose $N=1$, $k=1$, $\gamma=1$, $q=m=0$, $t_1=t_2=0$. Then we have $T_t=t^{-\frac{1}{2}}$, $T=2\sqrt{t}$, $X=x+y+\frac{z^2}{2t}-t$, $U=\sqrt{t}u$. Hence Eq. (3.1) has a solution

$$x+y+\frac{z^2}{2t}-t-2tu=F^{-1}(\sqrt{t}u),$$

or

$$u=\frac{1}{\sqrt{t}}F\left(x+y+\frac{z^2}{2t}-t-2tu\right). \tag{3.2}$$

Example 3.2. We consider the generalized KZ equation

$$\big(u_t+(1+u^2)(u_x+u_y)\big)_x-u_{yy}+\frac{1}{2}(u_{zz}+u_{ww})=0. \tag{3.3}$$

Choose $N=2,\ k=1,\ \gamma=1,\ q=m=0,\ t_1=t_2=0$. Then we have $T_t=\frac{1}{t^2},\ T=-\frac{1}{t},\ X=x+y+\frac{z^2+w^2}{2t}-t,\ U=tu$. Hence Eq. (3.3) has a solution

$$x+y+\frac{z^2+w^2}{2t}-t+u=F^{-1}(tu),$$

or

$$u=\frac{1}{t}F\left(x+y+\frac{z^2+w^2}{2t}-t+u\right). \tag{3.4}$$

We notice that the t_p means that we consider sound pulse focused in some transverse directions and defocused in the other directions. To the best of our knowledge, the exact solution of the generalized KZ equation have not been consider earlier and we shall apply here our approach of this kind. Therefore, we want to find the solution of Eq. (3.3) propagation of a nonlinear sound pulse that is defocused in z direction and focused in w direction. To this end, we take in the above formulas $N=2,\ k=1,\ q=m=0,\ t_1<0,\ t_2>0,\ h(T)=0$. It yields

$$\alpha(T)=T_t^{1/2}=\left(\frac{1}{(t+|t_1|)(t_2-t)}\right)^{1/2}.$$

We also know

$$T(t)=\frac{1}{t_2+|t_1|}\Big(\ln(t+|t_1|)-\ln(t_2-t)\Big),$$

where the integration constant is chosen in such a way that $T(0)=0$. We assume here that $0\le t\le t_2$. The self-similar variable has now the form

$$X=x+y+\frac{1}{2}\left(\frac{z^2}{t+|t_1|}-\frac{w^2}{t_2-t}\right)-t.$$

The variable $u(x,y,z,w,t)$ is expressed in terms of $U(X,T)$ as

$$u(x,y,z,w,t)=\alpha(T)U(X,T)=\left(\frac{1}{(t+|t_1|)(t_2-t)}\right)^{1/2}U(X,T),$$

where U obeys general Hopf equation

$$U_T + 2U^2 U_X = 0. \tag{3.5}$$

We assume that at $t=0$ the distribution of $u_0(x,y,z,w)$ depends on a single self-similar variable

$$u_0(x,y,z,w) = \frac{1}{\sqrt{|t_1|t_2}} F\left(x+y+\frac{1}{2}\left(\frac{z^2}{|t_1|}-\frac{w^2}{t_2}\right)-t\right). \tag{3.6}$$

Then the solution of Eq. (3.5) can be written as $X-2U^2T=F^{-1}(U)$, or returning to the original variables,

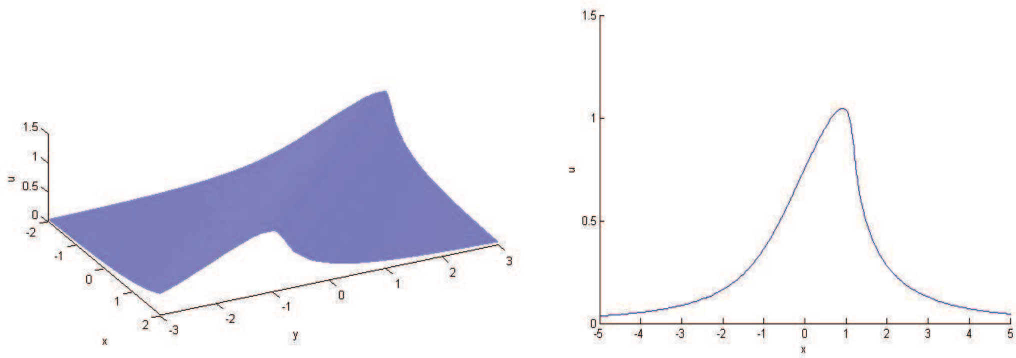
$$\begin{aligned} &x+y+\frac{1}{2}\left(\frac{z^2}{t+|t_1|}-\frac{w^2}{t_2-t}\right)-t-2(t+|t_1|)(t_2-t) \\ &\quad\times\left(\frac{1}{t_2+|t_1|}(\ln(t+|t_1|)-\ln(t_2-t))\right)u^2 = F^{-1}(\sqrt{(t+|t_1|)(t_2-t)}u). \end{aligned} \tag{3.7}$$

The formula determines implicitly u as a function of space coordinates at any moment of time t in the interval $0 \le t \le t_2$. It is worth noticing that this restriction makes it impossible to take the limit $t_1 = t_2 = 0$ and to reproduce the solution (3.4).

In what follows, we present some numerical simulations to demonstrate our analytical results (3.7). Here we suppose the initial distribution is as follows:

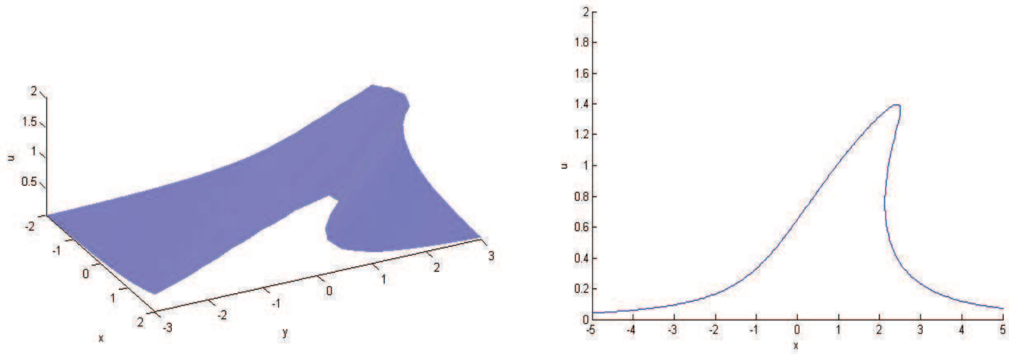
$$u_0 = (x,y,z,w) = \frac{1}{1+\left(x+y+\frac{1}{2}(z^2-w^2)\right)^2}. \tag{3.8}$$

The profiles of the pulse along the xOy plane and the x-axis are shown in Fig. 1 and Fig. 2 for $t=0.3$ and 0.7 , respectively.



(a) (b)

Figure 1: Profile of the pulse $u(x,y,\mathbf{z},t)$ for the moment of time $t=0.3$: (a)along xOy plane; (b) along x-axis.



(a) (b)

Figure 2: Profile of the pulse $u(x,y,\mathbf{z},t)$ for the moment of time $t=0.7$: (a)along xOy plane; (b) along x-axis.

Remark 3.1. In the present numerical simulation, we have drawn the traveling wave 3-D solutions surfaces and corresponding 2-D solution graphs for the obtained exact solutions of Eq. (3.3). We should stress that all these effects are relevant for finite time $t\sim t_p$ important for practical applications. In the study of asymptotical long-time behavior for $t\gg t_p$ we can put all t_p equal to zero, then the wave fronts become asymptotically paraboloidal.

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