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Shape Analysis and Solution to a Class of Nonlinear Wave Equation with Cubic Term

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Abstract. In this paper, we analyze the relation between the shape of the bounded traveling wave solutions and dissipation coefficient of nonlinear wave equation with cubic term by the theory and method of planar dynamical systems. Two critical values which can characterize the scale of dissipation effect are obtained. If dissipation effect is not less than a certain critical value, the traveling wave solutions appear as kink profile; while if it is less than this critical value, they appear as damped oscillatory. All expressions of bounded traveling wave solutions are presented, including exact expressions of bell and kink profile solitary wave solutions, as well as approximate expressions of damped oscillatory solutions. For approximate damped oscillatory solution, using homogenization principle, we give its error estimate by establishing the integral equation which reflects the relations between the exact and approximate solutions. It can be seen that the error is an infinitesimal decreasing in the exponential form.

AMS subject classifications: 35Q51, 37C29

Key words: Nonlinear wave equation, planar dynamical system, exact solutions, approximate damped oscillatory solutions, error estimate.

1 Introduction

The Klein-Gordon equation with cubic nonlinear term

$$u_{tt} - \beta u_{xx} + a_1 u + a_3 u^3 = 0 \tag{1.1}$$

is the important equation of motion of a quantum scalar or pseudoscalar field. Many researchers have studied Eq. (1.1) in recent years. Wazwaz [1] used the tanh method

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to obtain traveling wave solutions with compactons, solitons, solitary patterns and periodic structures of Eq. (1.1); Jang [2] used the auxiliary equation to study Eq. (1.1) and presented some exact traveling wave solutions; Sirendaoreji gave us new and more traveling wave solutions of Eq. (1.1) in [3]; Ye obtained all explicit expressions of the bounded traveling wave solutions for Eq. (1.1) in [4].

As far as we know, dissipation is inevitable in practical problem. So it is essential to study the following equation

$$u_{tt} - \beta u_{xx} + r u_t + a_1 u + a_3 u^3 = 0. \tag{1.2}$$

Eq. (1.2) is a vital wave equation for nonlinear researchers. It can be regarded as the nonlinear telegraph equation [5], which was proposed when the electric cables were paved at the bottom of the Atlantic. It is very useful in telegraph signal transmission. Additionally, it also can be used to describe the pressure produced by pulsation when the blood flows in arterials.

Comparing with Eq. (1.1), Eq. (1.2) is much more complex. The effect of the dissipation term ru_t makes the shape of the traveling wave solutions for Eq. (1.2) varies a lot. Fan [6] and Shang [7] gave two kink profile solitary wave solutions of Eq. (1.2) by homogeneous balance method and Wu-eliminition method; and a kind of direct combination and ansatze method, respectively. We can prove that the solutions obtained by them are equivalent. Zhang [8] offered us a method to judge the shape of solitary wave and some solitary wave solutions of Eq. (1.2). However, the relation between the shape and the dissipation coefficient or the damped oscillatory solutions was not obtained in [8]. Ma [9] considered the explicit traveling wave solutions of Kolmogorov-Petrovskii-Piskunov equation [10]

$$u_{xx} - u_t + a_1 u + a_2 u^2 + a_3 u^3 = 0, \tag{1.3}$$

and described nonlinear interactions of traveling waves by Cole-Hope transformation.

From the above studies, we may ask the following questions. How many bounded traveling wave solutions does Eq. (1.2) have? Are there any bounded solutions have not been obtained? How to express the solutions which can not be found out easily? We are sure that if these questions were solved, it can help us settle many practical problems, such as helping the engineers to control the telegraphic signals. In this paper, we will answer these questions. We firstly apply the theory and method of planar dynamical systems to make qualitative analysis to the dynamical systems which the traveling wave solutions of Eq. (1.2) corresponds to, present the global phase portraits, and study the number and shapes of the bounded traveling wave solutions. Secondly, we analyze the dissipation effect on the shape of bounded traveling wave solutions, and give two critical values which can characterize the scale of the dissipation effect, i.e., if r^2 is not less than a certain critical value, the traveling wave solutions of Eq. (1.2) appear as kink profile; if r^2 is less than this critical value, they appear as damped oscillatory. Based on above analysis, we give the expressions of the bounded traveling wave solutions for Eq. (1.2), including the exact expressions of bell and kink profile solitary wave solutions, and approximate expressions of damped oscillatory solutions. Finally, we present the error estimate of the approximate damped oscillatory solution. The difficulty of this problem is that we only know the approximate solution but don't know its exact solution. To over come it, we use the homogenization principle to establish the integral equation reflecting the relations between approximate and its exact solutions, and then give the error estimate. The error is an infinitesimal decreasing in the exponential form.

This paper is organized as follows. In Section 2, we present all global phase portraits of the dynamical systems which the traveling wave solutions of Eq. (1.2) corresponds to. In Section 3, we discuss the relation between the shapes of bounded traveling wave solutions for Eq. (1.2) and dissipation coefficient r, and give two critical values. In Section 4, we give all bounded traveling wave solutions, including exact bell and kink profile solitary wave solutions, as well as approximate damped oscillatory solutions. In Section 5, we give the error estimate of approximate damped oscillatory solutions obtained in the previous section. From the expression of the error, we can see it is an infinitesimal decreasing in the exponential form.

2 Shape analysis to bounded traveling wave solutions

Assume that Eq. (1.2) has the traveling wave solution in the form of $u(x,t) = u(\xi) = u(x-ct)$, where c is wave speed. Consequently, Eq. (1.2) satisfies

$$u_{\xi\xi} + \tilde{r}u_{\xi} + \tilde{b}(u^3 + pu) = 0, \tag{2.1}$$

where

$$\tilde{r} = -\frac{rc}{c^2-\beta}, \qquad \tilde{b} = \frac{a_3}{c^2-\beta} \ (c^2-\beta \neq 0), \qquad p = \frac{a_1}{a_3}.$$

Let $x = u(\xi)$ and $y = u_{\xi}(\xi)$. Then Eq. (1.2) can be converted into the following planar dynamical system

$$\begin{cases} \dfrac{dx}{d\xi} = y \equiv P(x,y), \\ \dfrac{dy}{d\xi} = -\tilde{r}y - \tilde{b}(x^3+px) \equiv Q(x,y). \end{cases} \tag{2.2}$$

Since the number of singular points for system (2.2) is determined by the number of real roots of

$$f(x) \equiv x^3 + px = x(x^2+p),$$

and as we only discuss the bounded traveling wave solutions of Eq. (1.2), we assume $p < 0$ throughout this paper. Thus, system (2.2) has three singular points

$$P_1(x_1,0) = (-\sqrt{-p},0), \quad P_2(x_2,0) = (0,0), \quad P_3(x_3,0) = (\sqrt{-p},0). \tag{2.3}$$

Obviously, x_i, $i = 1,2,3$, are the real roots of $f(x) = 0$, satisfying $x_1 \leq x_2 \leq x_3$ and $x_1 = -x_3$.

Denote the Jacobian matrix of system (2.2) at $P_i(x_i,0)$, $i=1,2,3$, be

$$J(x_i,0)=\begin{pmatrix} 0 & 1 \\ -\tilde{b}f'(x_i) & -\tilde{r} \end{pmatrix}, \qquad i=1,2,3, \tag{2.4}$$

and denote $D_i=\tilde{r}^2-4\tilde{b}f'(x_i)$, $i=1,2,3$.

We apply the theory and method of planar dynamical system [11,12], and to make qualitative analysis for system (2.2) from three aspects (finite singular points, infinite singular points,and existent of limit cycle). Not loss generality, we assume $\tilde{r}<0$, and we can obtain the following propositions.

Proposition 2.1. *The types of singular points of system* (2.2) *are listed as follows:*

(1) If $\tilde{b}>0$, since $\det(J(x_2,0))=p\tilde{b}<0$, $\det(J(x_i,0))=-2p\tilde{b}>0$, $i=1,3$, and $\tilde{r}<0$, P_2 is a saddle point. P_1 and P_3 are unstable node points if $D_1=D_3>0$, while P_1 and P_3 are unstable focus points if $D_1=D_3<0$.

(2) If $\tilde{b}<0$, since $\det(J(x_2,0))=p\tilde{b}>0$, $\det(J(x_i,0))=-2p\tilde{b}<0$, $i=1,3$, and $\tilde{r}<0$, P_1 and P_3 are saddle points. P_2 is an unstable node point if $D_2>0$, while P_2 is an unstable focus point if $D_2<0$.

Proposition 2.2. *We use Poincaré transformation to analyze the singular points at infinity of system* (2.2)*. It is easy to know that there exists a couple of singular points at infinity on the y-axis which are denoted by A_1 and A_2, respectively. If $\tilde{b}>0$, there exists a hyperbolic type region around A_i, respectively; while if $\tilde{b}<0$, there exists a elliptical type region around A_i, respectively. Moreover, the circumference of Poincaré disk is orbits.*

Proposition 2.3. *Since*

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=-\tilde{r}, \tag{2.5}$$

from Bendixson-Dulac criterion we know if $\tilde{r}\neq 0$, system (2.2) *does not have any closed orbit or singular closed orbit on (x,y) phase plane. This shows Eq.* (1.2) *has neither periodic traveling wave solution nor bell profile solitary wave solution as $\tilde{r}\neq 0$.*

Furthermore, based on the above qualitative analysis, we present all possible global phase portraits for (2.2) in case of $\tilde{r}=0$ and $\tilde{r}<0$ (see Figs. 1 and 2).

It is well known that a homoclinic orbit of planar dynamical system corresponds to a bell profile solitary wave solution of its corresponding nonlinear evolution equation; the heteroclinic orbit corresponds to a kink profile solitary or an oscillatory wave solution; the closed orbit corresponds to a periodic traveling wave solution. So we can derive the following theorem from Figs. 1 and 2.

Theorem 2.1. *Eq.* (1.2) *has and only has two bounded traveling wave solutions. The two homoclinic orbits $L(P_2,P_2)$ in Fig. 1(a) correspond to two bell profile solitary wave solutions, respectively; the heteroclinic orbits $L(P_1,P_3)$ and $L(P_3,P_1)$ in Fig. 1(b), $L(P_i,P_2)$, $i=1,3$, in Fig. 2(a) and $L(P_2,P_i)$, $i=1,3$, in Fig. 2(c) correspond to two kink profile solitary wave solutions, respectively; $L(P_i,P_2)$, $i=1,3$, in Fig. 2(b) and $L(P_2,P_i)$, $i=1,3$, in Fig. 2(d) correspond to two oscillatory wave solutions, respectively.*

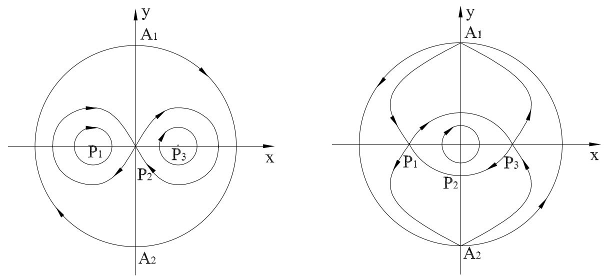
3 Relation between the solution shapes and dissipation coefficients

Below we present three main results of this paper.

Theorem 3.1. *Suppose that*

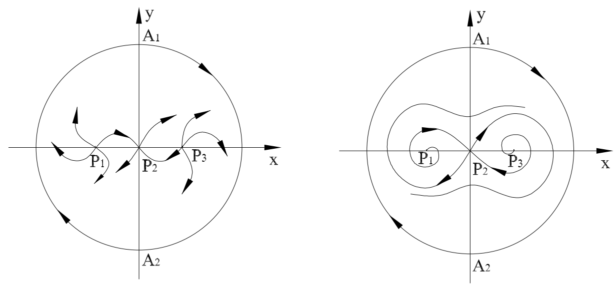
$$a_1 a_3 < 0 \quad \textit{and} \quad \frac{a_3}{c^2 - \beta} > 0.$$

We have

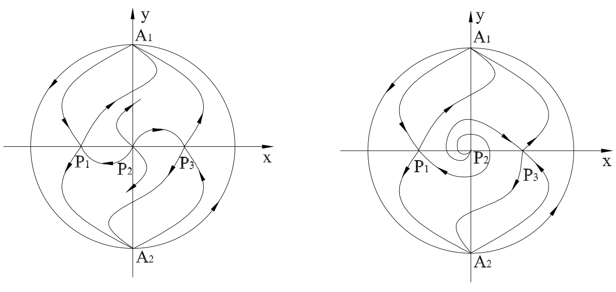


(a) $\tilde{b} > 0$ (b) $\tilde{b} < 0$

Figure 1: The global phase portrait in the case of $\tilde{r} = 0$.



(a) $\tilde{b} > 0$, $D_1 = D_3 > 0$ (b) $\tilde{b} > 0$, $D_1 = D_3 < 0$



(c) $\tilde{b} < 0$, $D_2 > 0$ (d) $\tilde{b} < 0$, $D_2 < 0$

Figure 2: The global phase portrait in the case of $\tilde{r} < 0$.

(1) If

$$r^2 \geq -\frac{8a_1(c^2-\beta)}{c^2},$$

there exists a monotone decreasing kink profile solitary wave solution $u(\xi)$ in Eq. (1.2), *satisfying $u(-\infty) = x_3$ and $u(+\infty) = 0$; meanwhile, there exists a monotone increasing kink profile solitary wave solution $u(\xi)$ in Eq.* (1.2), *satisfying $u(-\infty) = x_1$ and $u(+\infty) = 0$. Here, $u(\xi)$ correspond to the orbits $L(P_3, P_2)$ and $L(P_1, P_2)$ in Fig. 2(a), respectively.*

(2) If

$$r^2 < -\frac{8a_1(c^2-\beta)}{c^2},$$

there exists oscillatory traveling wave solutions $u(\xi)$ in Eq. (1.2), *satisfying $u(-\infty) = x_3$, $u(+\infty) = 0$, and $u(-\infty) = x_1$, $u(+\infty) = 0$. Here, $u(\xi)$ correspond to the orbits $L(P_3, P_2)$ and $L(P_1, P_2)$ in Fig. 2(b), respectively.*

Proof. By using the transformation

$$V(\xi) = \frac{u(\xi) - x_1}{x_3 - x_1},$$

Eq. (2.1) becomes

$$V''(\xi) + \tilde{r}V'(\xi) + \tilde{b}'V(\xi)(1 - V(\xi))(V(\xi) - w) = 0, \tag{3.1}$$

where $w = 1/2$, $\tilde{b}' = -\tilde{b}(x_3 - x_1)^2$. Evidently, the singular points of Eq. (3.1) are $P_1'(0,0)$, $P_2'(1/2,0)$, $P_3'(1,0)$, which correspond to singular points $P_1(x_1,0)$, $P_2(x_2,0)$, $P_3(x_3,0)$ of system (2.2), respectively. Since the linear transformation keeps the properties of singular points, the results on $P_1(x_1,0)$, $P_2(x_2,0)$, $P_3(x_3,0)$ given in Section 2 also hold for $P_1'(0,0)$, $P_2'(w,0)$, $P_3'(1,0)$ under the corresponding conditions.

From the previous qualitative analysis, we have the fact that system (2.2) has three singular points P_i, $i = 1,2,3$, where P_2 is a saddle point and P_i, $i = 1,3$, are unstable singular points, and there exits orbits $L(P_i, P_2)$, $i = 1,3$ (see Fig. 2(a) and Fig. 2(b)), on (x,y) phase plane. Since orbits $L(P_i, P_2)$, $i = 1,3$, tend to P_i, $i = 1,3$, as $\xi \to -\infty$ and to P_2 as $\xi \to +\infty$, Eq. (2.1) can not have the bounded solution satisfying $u(-\infty) = u(+\infty)$, only has the bounded solutions satisfying the following alternative

(i) $u(-\infty) = x_1$, $u(+\infty) = 0$;

(ii) $u(-\infty) = x_3$, $u(+\infty) = 0$.

Namely, a bounded solution of Eq. (3.1) satisfies one of the following two cases

(iii) $V(-\infty) = 0$, $V(+\infty) = \dfrac{1}{2}$;

(iv) $V(-\infty) = 1$, $V(+\infty) = \dfrac{1}{2}$.

Let

$$r_1 = -2\sqrt{-\frac{1}{2\tilde{b}'}}.$$

From [13], we have

(a) If $\tilde{b}' < 0$, $\tilde{r} \le r_1$, then Eq. (3.1) has a monotone increasing solution $V(\xi)$ satisfying the case (iii) and a monotone decreasing solution $V(\xi)$ satisfying the case (iv);

(b) If $\tilde{b}' < 0$, $r_1 < \tilde{r} < 0$, then Eq. (3.1) has two oscillatory solutions $V(\xi)$ satisfying the case (iii) and case (iv), repectively.

Since there exists a traveling wave transformation and a linear transformation

$$V(\xi) = \frac{u(\xi) - x_1}{x_3 - x_1},$$

between Eq. (3.1) and Eq. (1.2) and we observe that

$$r_1 = -2\sqrt{-\frac{1}{2\tilde{b}'}} = -2\sqrt{-2\tilde{b}p} = -2\sqrt{\frac{-2a_1}{c^2 - \beta}}.$$

Furthermore, we can bring r_1 into Eq. (1.2), Theorem 3.1 holds. $\square$

By the same way, we can give the following theorem without the proof.

Theorem 3.2. *Suppose that*

$$a_1 a_3 < 0 \quad \textit{and} \quad \frac{a_3}{c^2 - \beta} < 0.$$

We have

(1) If

$$r^2 \ge \frac{2a_1(c^2 - \beta)}{c^2},$$

there exists a monotone decreasing kink profile solitary wave solution $u(\xi)$ in Eq. (1.2), satisfying $u(-\infty) = 0$ and $u(+\infty) = x_1$; meanwhile, there exists a monotone increasing kink profile solitary wave solution $u(\xi)$ in Eq. (1.2), satisfying $u(-\infty) = 0$ and $u(+\infty) = x_3$. Here, $u(\xi)$ correspond to the orbits $L(P_2, P_1)$ and $L(P_2, P_3)$ in Fig. 2(c), respectively.

(2) If

$$r^2 < \frac{2a_1(c^2 - \beta)}{c^2},$$

there exists oscillatory traveling wave solutions $u(\xi)$ in Eq. (1.2), satisfying $u(-\infty) = 0$, $u(+\infty) = x_1$, and $u(-\infty) = 0$, $u(+\infty) = x_3$. Here, $u(\xi)$ correspond to the orbits $L(P_2, P_1)$ and $L(P_2, P_3)$ in Fig. 2(d), respectively.

We take the oscillatory traveling wave solutions corresponding to the focus-saddle orbits $L(P_2, P_i)$, $i = 1, 3$, in Fig. 2(b) for example, to discuss their damped properties. Those corresponding to the orbits in Fig. 2(d) can be discussed similarly.

Theorem 3.3. *Suppose that*

$$a_1a_3 < 0, \quad \frac{a_3}{c^2-\beta} < 0 \quad \text{and} \quad r^2 < -\frac{8a_1(c^2-\beta)}{c^2}.$$

We have

(1) The oscillatory traveling wave solution $u(\xi)$ of Eq. (1.2) *corresponding to the focus-saddle orbit $L(P_1,P_2)$ in Fig. 2(b) has minimum at $\check{\xi}_1$. The solution has the property of monotonically increasing at the right of $\check{\xi}_1$ and damped at the left of $\check{\xi}_1$. That is, there exists numerably infinite maximum points $\hat{\xi}_i$ $(i=1,2,\cdots,+\infty)$ and minimum points $\check{\xi}_i$ $(i=1,2,\cdots,+\infty)$ on ξ axis, such that*

$$\begin{cases} -\infty < \cdots < \hat{\xi}_n < \check{\xi}_n < \hat{\xi}_{n-1} < \check{\xi}_{n-1} < \cdots < \hat{\xi}_1 < \check{\xi}_1 < +\infty, \\ \lim\limits_{n\to\infty} \hat{\xi}_n = \lim\limits_{n\to\infty} \check{\xi}_n = -\infty, \end{cases} \tag{3.2a}$$

$$\begin{cases} u(\check{\xi}_1) < \cdots < u(\check{\xi}_n) < \cdots < u(-\infty) < \cdots < u(\hat{\xi}_n) < u(\hat{\xi}_{n-1}) < \cdots < u(\hat{\xi}_1) < u(+\infty), \\ \lim\limits_{n\to\infty} u(\hat{\xi}_n) = \lim\limits_{n\to\infty} u(\check{\xi}_n) = u(-\infty), \end{cases} \tag{3.2b}$$

and

$$\lim_{n\to\infty}(\hat{\xi}_n - \hat{\xi}_{n+1}) = \lim_{n\to\infty}(\check{\xi}_n - \check{\xi}_{n+1}) = \frac{4\pi(c^2-\beta)}{\sqrt{4a_1(c^2-\beta)-c^2r^2}}. \tag{3.3}$$

(2) The oscillatory traveling wave solution $u(\xi)$ of Eq. (1.2) *corresponding to the focus-saddle orbit $L(P_3,P_2)$ in Fig. 2(b) has maximum at $\hat{\xi}_1$. The solution has the property of monotonically decreasing at the right of $\hat{\xi}_1$ and damped at the left of $\hat{\xi}_1$. That is, there exists numerably infinite maximum points $\hat{\xi}_i$ $(i=1,2,\cdots,+\infty)$ and minimum points $\check{\xi}_i$ $(i=1,2,\cdots,+\infty)$ on ξ axis, such that*

$$\begin{cases} -\infty < \cdots < \check{\xi}_n < \hat{\xi}_n < \check{\xi}_{n-1} < \hat{\xi}_{n-1} < \cdots < \check{\xi}_1 < \hat{\xi}_1 < +\infty, \\ \lim\limits_{n\to\infty} \hat{\xi}_n = \lim\limits_{n\to\infty} \check{\xi}_n = -\infty, \end{cases} \tag{3.4a}$$

$$\begin{cases} u(\check{\xi}_1) < \cdots < u(\check{\xi}_n) < \cdots < u(-\infty) < \cdots < u(\hat{\xi}_n) < u(\hat{\xi}_{n-1}) < \cdots < u(\hat{\xi}_1) < u(+\infty), \\ \lim\limits_{n\to\infty} u(\hat{\xi}_n) = \lim\limits_{n\to\infty} u(\check{\xi}_n) = u(-\infty), \end{cases} \tag{3.4b}$$

and (3.3) *hold.*

Proof. (1) By the theory of planar dynamical systems, we deduce that P_1 is an unstable focus point and P_2 is a saddle point. $L(P_1,P_2)$ tends to P_1 spirally as $\xi\to-\infty$. Moreover, it is easy to see that the interSection points of $L(P_1,P_2)$ and ξ axis at the right of P_1 correspond to maximum points of $u(\xi)$, while at the left of P_1 correspond to minimum points of $u(\xi)$. Hence, both (3.2a) and (3.2b) hold. When $L(P_1,P_2)$ approaches to P_1 sufficiently, its properties tend to the properties of the linear approximate solution of system (2.2) at P_1. The frequency of $L(P_1,P_2)$ rotating around P_1 tends to $\sqrt{4a_1(c^2-\beta)-c^2r^2}/4\pi(c^2-\beta)$. Thus, (3.3) holds.

(2) This part can be proved similarly. $\square$

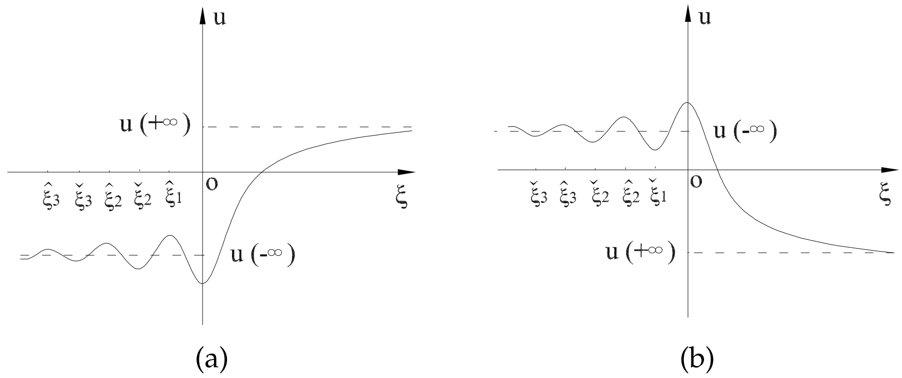


Figure 3: The portraits of the oscillatory traveling waves.

Synthesize above contents, the shape of bounded traveling wave solution for Eq. (1.2) is determined by the scale of dissipation effect. The two critical values are

$$r_1^2 = -\frac{8a_1(c^2-\beta)}{c^2} \quad \text{and} \quad r_2^2 = \frac{2a_1(c^2-\beta)}{c^2}.$$

If dissipation effect is large, namely r^2 is more than a critical value, the traveling wave solutions of Eq. (1.2) appear as kink profile solitary wave solutions; while if dissipation effect is small, namely r^2 less than the critical value, they appears as damped oscillatory wave solutions.

Since a traveling wave solution $u(\xi)$ keeps the shape and speed unchanged when parallel shifting on ξ axis, without loss of generality, we assume that $\hat{\xi}_1 = 0$ and $\check{\xi}_1 = 0$. The portraits of the oscillatory traveling waves described by Theorem 3.3 are shown in Fig. 3

4 Solitary wave solutions and damped oscillatory solutions

4.1 Bell profile solitary solutions of Eq. (1.2)

Inspired by [14], we can bring

$$u(\xi) = \frac{Ae^{\alpha(\xi-\xi_0)}}{(1+e^{\alpha(\xi-\xi_0)})^2 + Be^{\alpha(\xi-\xi_0)}} + W, \tag{4.1a}$$

$$u(\xi) = \frac{Ae^{\alpha(\xi-\xi_0)}}{1+e^{\alpha(\xi-\xi_0)}} + W, \tag{4.1b}$$

into Eq. (1.2), respectively. Computing them, we can have the following theorems.

Theorem 4.1. *When $r=0$, if*

$$a_1a_3 < 0 \quad \textit{and} \quad \frac{a_3}{c^2-\beta} > 0, \tag{4.2}$$

Eq. (1.2) *has bell profile solitary wave solutions in the following form*

$$u^{\pm}(\xi) = \pm\sqrt{-\frac{2a_1}{a_3}}sech\Big(\sqrt{-\frac{a_1}{c^2-\beta}}(\xi-\xi_0)\Big). \tag{4.3}$$

These two solutions correspond to the two homoclinic orbits $L(P_2, P_2)$ in Fig. 1(a).

Theorem 4.2. *When $r = 0$, if*

$$a_1a_3 < 0 \quad and \quad \frac{a_3}{c^2-\beta} < 0, \tag{4.4}$$

Eq. (1.2) *has kink profile solitary wave solutions in the following form*

$$u^{\pm}(\xi) = \pm\sqrt{-\frac{a_1}{a_3}}\tanh\Big(\sqrt{\frac{a_1}{2(c^2-\beta)}}(\xi-\xi_0)\Big). \tag{4.5}$$

These two solutions correspond to the heteroclinic orbits $L(P_1, P_3)$ and $L(P_3, P_1)$ in Fig. 1(b).

Theorem 4.3. *If $a_1a_3 < 0$, when*

$$r^2 = \frac{9(c^2-\beta)}{2c^2}, \tag{4.6}$$

Eq. (1.2) *has kink profile solitary wave solutions in the following form*

$$u_1^{\pm}(\xi) = \mp\frac{1}{2}\sqrt{-\frac{a_1}{a_3}}\tanh\Big(\frac{3a_1}{4rc}(\xi-\xi_0)\Big) \pm \frac{1}{2}\sqrt{-\frac{a_1}{a_3}}, \tag{4.7a}$$

$$u_2^{\pm}(\xi) = \pm\frac{1}{2}\sqrt{-\frac{a_1}{a_3}}\tanh\Big(-\frac{3a_1}{4rc}(\xi-\xi_0)\Big) \pm \frac{1}{2}\sqrt{-\frac{a_1}{a_3}}. \tag{4.7b}$$

$u_1^{\pm}(\xi)$ correspond to the heteroclinic orbits $L(P_1, P_2)$ and $L(P_2, P_3)$ in Fig. 2(a), while $u_2^{\pm}(\xi)$ correspond to the heteroclinic orbits $L(P_1, P_2)$ and $L(P_2, P_3)$ in Fig. 2(c).

Remark 4.1. (1) To obtain above Theorems, we use the following two formulae

$$\frac{Ae^{\alpha(\xi-\xi_0)}}{(1+e^{\alpha(\xi-\xi_0)})^2 + Be^{\alpha(\xi-\xi_0)}} + W = \frac{A\mathrm{sech}^2(\frac{\alpha}{2}(\xi-\xi_0))}{4+B\mathrm{sech}^2(\frac{\alpha}{2}(\xi-\xi_0))} + W, \tag{4.8a}$$

$$\frac{Ae^{\alpha(\xi-\xi_0)}}{1+e^{\alpha(\xi-\xi_0)}} + W = \frac{A}{2}\tanh\Big(\frac{\alpha}{2}(\xi-\xi_0)\Big) + \Big(W+\frac{A}{2}\Big). \tag{4.8b}$$

(2) We can prove that (4.5), (4.7a) and (4.7b) are equivalent to (12), (13) and (14), respectively, in [9] if $\nu(\xi) = \tanh(\xi-\xi_0)$.

4.2 Approximate damped oscillatory solutions of Eq. (1.2)

This Section we take the damped oscillatory solutions of Eq. (1.2) corresponding to the focus-saddle orbits $L(P_1,P_2)$ in Fig. 2(b) for example, to discuss its approximate damped oscillatory solution. Other cases can be discussed similarly. By the theory of planar dynamical systems, it is easy to see that focus-saddle orbit $L(P_1,P_2)$ in Fig. 2(b) comes from the break of left homoclinic orbit $L(P_2,P_2)$ in Fig. 1(b) under the effect of dissipation term ru_t (the dissipation coefficient r satisfies $r^2 < -8a_1(c^2-\beta)/c^2$. Hence, the non-oscillatory part of the damped oscillatory solution corresponding to $L(P_3,P_2)$ can be denoted by the bell profile solitary wave solution of the form

$$u^*(\xi) = -\sqrt{-\frac{2a_1}{a_3}}\operatorname{sech}\Big(\sqrt{-\frac{a_1}{c^2-\beta}}(\xi-\xi_0)\Big), \quad \xi\in[\xi_0,\ +\infty). \tag{4.9}$$

To express the oscillatory part of this damped oscillatory solution approximatively, we use the following solution of the form

$$u(\xi) = e^{\alpha(\xi-\xi_0)}\Big(A_1\cos(B(\xi-\xi_0))-A_2\sin(B(\xi-\xi_0))\Big)+C, \qquad \xi\in(-\infty,\xi_0), \tag{4.10}$$

where A_1, A_2, B, C, α are undetermined constants. The reason why we chose (4.10) is that (4.10) has both damped and oscillatory properties, due to $e^{\alpha(\xi-\xi_0)}$ has damped property while $(A_1\cos(B(\xi-\xi_0))-A_2\sin(B(\xi-\xi_0)))$ has oscillatory property.

Substituting (4.10) into Eq. (2.1), and neglecting the terms including $\mathcal{O}(e^{\alpha(\xi-\xi_0)})$, we have

$$B^2 = \frac{-r^2c^2+12a_3C^2(c^2-\beta)+4a_1(c^2-\beta)}{4(c^2-\beta)^2}, \tag{4.11a}$$

$$\alpha = \frac{rc}{2(c^2-\beta)}, \qquad a_1C-a_3C^3=0. \tag{4.11b}$$

In order to derive approximate damped oscillatory solution of Eq. (1.2), there still requires some conditions to connect (4.9) and (4.10). Since the properties of solutions are unchangeable as translating on ξ axis, we take $\xi_0=0$ as a connective point, and choose

$$\frac{d^i}{d\xi^i}u(0) = \frac{d^i}{d\xi^i}u^*(0), \qquad i=0,1, \tag{4.12}$$

namely,

$$A_1+C=u^*(0), \qquad \alpha A_1-A_2B=0, \tag{4.13}$$

as connective conditions. $\xi_0=0$ is the extremal point of the bell profile solitary wave solutions, thus

$$\frac{d}{d\xi}u^*(0)=0, \tag{4.14}$$

holds. Due to (4.10) tends to x_1 as $\xi \to -\infty$, thus

$$C = x_1 = -\sqrt{-\frac{a_1}{a_3}}. \tag{4.15}$$

Further,

$$B^2 = \frac{-r^2c^2 - 8a_1(c^2-\beta)}{4(c^2-\beta)^2}, \quad A_1 = (-\sqrt{2}+1)\sqrt{-\frac{a_1}{a_3}}, \quad A_2 = \frac{\alpha A_1}{B}. \tag{4.16}$$

According to above analysis, we have the following theorem.

Theorem 4.4. *Suppose*

$$\frac{a_3}{c^2-\beta} > 0. \tag{4.17}$$

If

$$r^2 < -\frac{8a_1(c^2-\beta)}{c^2}, \tag{4.18}$$

Eq. (1.2) *has damped oscillatory solutions corresponding to focus-saddle orbit $L(P_1, P_2)$ and $L(P_3, P_2)$ in Fig. 2(b), whose approximate solutions are*

$$u(\xi) \approx \begin{cases} \pm u^*(\xi), & \xi \in [0, +\infty), \\ e^{\frac{rc}{2(c^2-\beta)}\xi}\big(A_1\cos(B\xi) - A_2\sin(B\xi)\big) \mp \sqrt{-\frac{a_1}{a_3}}, & \xi \in (-\infty, 0), \end{cases} \tag{4.19}$$

where $u^*(\xi)$ is given by (4.9),

$$B^2 = \frac{-r^2c^2 - 8a_1(c^2-\beta)}{4(c^2-\beta)^2}, \quad A_1 = \mp(\sqrt{2}-1)\sqrt{-\frac{a_1}{a_3}}, \quad A_2 = \frac{\alpha A_1}{B}. \tag{4.20}$$

Similarly, we can derive the approximate damped oscillatory solutions, corresponding to the heterclinic orbits $L(P_2, P_1)$ and $L(P_2, P_3)$ in Fig. 2(d) which break from the heterclinic orbits $L(P_3, P_1)$ and $L(P_1, P_3)$ in Fig. 1(b) under the effect of dissipation term ru_t (the dissipation coefficient r satisfies $r^2 < 2a_1(c^2-\beta)/c^2$. There is one thing we must point out if we choose $\xi = 0$ as the connective point, the shape of the approximate damped oscillatory will deform seriously. So at this time we choose

$$\xi = \frac{\varphi + \pi/2 - 2k\pi}{B},$$

as the connective point, where $\tan\varphi = A_1/A_2$ and k is any integer.

Then we can imitate the reduction of Theorem 4.4 to obtain the following result.

Theorem 4.5. *Suppose*

$$\frac{a_3}{c^2-\beta}<0. \tag{4.21}$$

If

$$r^2<\frac{2a_1(c^2-\beta)}{c^2}, \tag{4.22}$$

Eq. (1.2) *has damped oscillatory solutions corresponding to focus-saddle orbit* $L(P_2,P_3)$ *and* $L(P_2,P_1)$ *in Fig. 2(d), whose approximate solutions are*

$$u(\xi)\approx\begin{cases}\pm u^*(\xi), & \xi\in[0,+\infty),\\ e^{\frac{rc}{2(c^2-\beta)}\xi}\big(A_1\cos(B\xi)-A_2\sin(B\xi)\big), & \xi\in(-\infty,0),\end{cases} \tag{4.23}$$

where $u^*(\xi)$ *is given by* (4.5)*,*

$$B^2=\frac{-r^2c^2+4a_1(c^2-\beta)}{4(c^2-\beta)^2},\qquad \xi=\frac{\varphi+\pi/2-2k\pi}{B}. \tag{4.24}$$

A_1 *and* A_2 *are determined by*

$$\pm\sqrt{-\frac{a_1}{a_3}}\tanh\Big(\sqrt{\frac{a_1}{2(c^2-\beta)}}\xi\Big)=e^{\frac{rc}{2(c^2-\beta)}\xi}\sin(\varphi-B\xi), \tag{4.25a}$$

$$\begin{aligned}&\pm 4\sqrt{-\frac{a_1^2}{a_3(c^2-\beta)}}\big(e^{\frac{rc}{2(c^2-\beta)}\xi}+e^{-\frac{rc}{2(c^2-\beta)}\xi}\big)^{-2}\\ &\quad=\frac{rc}{2(c^2-\beta)}e^{\frac{rc}{2(c^2-\beta)}\xi}\sin(\varphi-B\xi)+Be^{\frac{rc}{2(c^2-\beta)}\xi}\cos(\varphi+B\xi).\end{aligned} \tag{4.25b}$$

5 Error estimates of damped oscillatory solutions

In this section, we investigate error estimates between approximate damped oscillatory solutions and its exact solutions given in Section 4. We still take the approximate solution (4.19) and its exact solution corresponding to the focus-saddle orbit $L(P_1,P_2)$ in Fig. 2(b) as example. Other error estimates can be discussed similarly.

Substitute

$$V(\xi)=\frac{u(\xi)-x_1}{-2x_1}\quad\text{and}\quad \xi=-\eta\ \ (\eta>0),$$

into Eq. (2.1). Consequently, the problem of finding an exact damped oscillatory solution for Eq. (2.1), which satisfies $u(0)=-\sqrt{-2a_1/a_3}$, $u'(0)=0$ is converted into solving the following initial value problem

$$\begin{cases}\bar{V}_{\eta\eta}(\eta)-\tilde{r}\bar{V}_{\eta}(\eta)-2\tilde{b}x_1^2\bar{V}(\eta)-4\tilde{b}x_1^2V^2(\eta)\Big(\bar{V}(\eta)-\dfrac{3}{2}\Big)=0,\\ \bar{V}(0)=\dfrac{u(0)-x_1}{-2x_1},\quad \bar{V}_{\eta}(0)=0,\end{cases} \tag{5.1}$$

where

$$\bar{V}(\eta) = V(-\eta) = V(\xi).$$

By the principle of homogenization, we can solve initial value problem (5.1) and obtain the implicit expression of damped oscillatory solution of Eq. (1.2)

$$\begin{aligned}\bar{V}(\eta) =& e^{\alpha_1\eta}(c_1\cos(\beta\eta) + c_2\sin(\beta\eta)) \\ &- \frac{4\tilde{b}x_1^2}{\beta}\int_0^\eta e^{\alpha_1(\eta-\tau)}\sin(\beta(\eta-\tau))\bar{V}^2(\tau)\Big(\bar{V}(\tau) - \frac{3}{2}\Big)d\tau, \end{aligned} \tag{5.2}$$

where

$$\alpha_1 = \frac{\tilde{r}}{2}, \quad \beta = \frac{\sqrt{-\tilde{r}^2 - 8\tilde{b}p}}{2}, \quad c_1 = \frac{u(0) - x_1}{-2x_1}, \quad c_2 = \frac{\alpha_1 c_1}{\beta}.$$

Substituting

$$\eta = -\xi \quad \text{and} \quad V(\xi) = \frac{u(\xi) - x_1}{-2x_1},$$

into (5.2), and making the transformation $t = -\tau$. Then we have

$$\begin{aligned}u(\xi) - x_1 =& e^{-\alpha_1\xi}(\bar{c}_1\cos(\beta\xi) + \bar{c}_2\sin(\beta\xi)) \\ &+ \frac{\tilde{b}}{\beta}\int_0^\xi e^{-\alpha_1(\xi-t)}\sin(\beta(\xi-t))(u(t)-x_1)^2(u(t)+2x_1)dt, \end{aligned} \tag{5.3}$$

where

$$\bar{c}_1 = u(0) - x_1, \qquad \bar{c}_2 = \frac{\alpha_1 c_1}{\beta}.$$

It is obvious that β, $\bar{c}_1$ and $\bar{c}_2$ are equal to B, A_1 and $-A_2$ in Theorem 4.4, respectively.

Since damped oscillatory solution $u(\xi)$ is bounded, there exists $M > 0$, such that $|u(\xi)| < M$. Moreover, from (5.2) we have

$$|u(\xi) - x_1| \le C_1 e^{-\alpha_1\xi} + \frac{\tilde{b}T}{\beta}\int_0^\xi e^{-\alpha_1(\xi-t)}|u(t) - x_1|dt, \tag{5.4}$$

where $C_1 = |\bar{c}_1| + |\bar{c}_1|$, $T = (M - x_1)(M - 2x_1)$. By using Gronwall inequality, the above formula becomes

$$|u(\xi) - x_1| \le C_2 e^{-\alpha_1\xi}, \tag{5.5}$$

where

$$C_2 = C_1\exp\Big(\frac{-\tilde{b}T}{\alpha_1\beta}\Big).$$

(5.5) is the amplitude estimate of damped oscillatory solution of Eq. (1.2). From (5.5), it is obvious that $u(\xi)$ rapidly tends to x_1 as $\xi \to -\infty$.

From (5.3) to (5.5), we have

$$\begin{aligned}&|u(\xi)-(e^{-\alpha_1\xi}(\bar{c}_1\cos(\beta_1\xi)+\bar{c}_2\sin(\beta_1\xi))+x_1)|\\ \le&-\frac{\tilde{b}|M-2x_1|C_2^2}{\alpha_1\beta}e^{-2\alpha_1\xi}(1-e^{-\alpha_1\xi}).\end{aligned} \tag{5.6}$$

(5.6) shows that the error estimate between the approximate solution (4.19) and its exact damped oscillatory solution is less than

$$\varepsilon_1(\xi)=-\frac{\tilde{b}|M-2x_1|C_2^2}{\alpha_1\beta}e^{-2\alpha_1\xi}(1-e^{-\alpha_1\xi}).$$

Due to

$$\varepsilon_1(\xi)=\mathcal{O}(e^{-\frac{\tilde{r}}{2}\xi}),\quad \xi\to-\infty,$$

(4.19) is meaningful to be an approximate solution of Eq. (1.2) when the conditions in Theorem 4.4 hold.

By using similar method, we can get error estimates between other approximate damped oscillatory solutions obtained above and their exact solutions. Their errors are all infinitesimals decreasing in the exponential form.

6 Conclusions

In this paper, we discuss the relation between the shape of the bounded traveling wave solutions and dissipation coefficients of nonlinear wave equation with cubic term by the theory and method of planar dynamical systems. Two critical values which can characterize the scale of dissipation effect are obtained. If dissipation effect is not less than a certain critical value, the traveling wave solutions appear as kink profile; while if it less than this critical value, they appear as damped oscillatory. We all know that as the traveling wave, solitary wave's energy is centralized relatively, and it has large damages on affected objects; while for damped oscillatory wave, which has a bell profile head and oscillatory tail, its destructive force is less than solitary wave, and its energy decreases gradually with the time flying; while for decreasing kink wave, its energy losses promptly, so the destructive force is smaller. So these two critical values can help us control the shape of the wave; therefore, we can reduce the hazard to nature and loss in economy. Except for exact expressions of bell and kink profile solitary wave solutions, we still give approximate expression damped oscillatory solutions. For the approximate damped oscillatory solution, we give its error estimate. It can be seen that the error is an infinitesimal decreasing in the exponential form.

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