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Nonlinear Degenerate Anisotropic Elliptic Equations with Variable Exponents and L^1 Data

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Abstract. This paper is devoted to the study of a nonlinear anisotropic elliptic equation with degenerate coercivity, lower order term and L^1 datum in appropriate anisotropic variable exponents Sobolev spaces. We obtain the existence of distributional solutions.

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1 Introduction

In this paper we prove the existence of solutions to the nonlinear anisotropic degenerate elliptic equations with variable exponents, of the type

$$\begin{cases} -\sum_{i=1}^{N} D_i a_i(x,u,\nabla u)+g(x,u,\nabla u)=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and the right-hand side f in $L^1(\Omega)$, $D_i u = \frac{\partial u}{\partial x_i}$. We suppose that $a_i : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, $i=1,\ldots,N$ are

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Carathéodory functions such that for almost every x in Ω and for every $(\sigma,\xi)\in\mathbb{R}\times\mathbb{R}^N$ the following assumptions are satisfied for all $i=1,\ldots,N$

$$|a_i(x,\sigma,\xi)|\le\beta\left(|k(x)|+|\sigma|^{\overline{p}(x)}+\sum_{j=1}^{N}|\xi_j|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}, \tag{1.2}$$

$$\sum_{i=1}^{N}(a_i(x,\sigma,\xi)-a_i(x,\sigma,\eta))(\xi_i-\eta_i)>0,\quad \forall\xi\neq\eta, \tag{1.3}$$

$$\sum_{i=1}^{N}a_i(x,\sigma,\xi)\xi_i\ge\alpha\sum_{i=1}^{N}\frac{|\xi_i|^{p_i(x)}}{(1+|\sigma|)^{\gamma_i(x)}}, \tag{1.4}$$

where $\beta>0$, $\alpha>0$, and $k\in L^1(\Omega)$, $\gamma_i:\overline{\Omega}\to\mathbb{R}^+$, $p_i:\overline{\Omega}\to(1,+\infty)$ are continuous functions and $\overline{p}$ is such that

$$\frac{1}{\overline{p}(\cdot)}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(\cdot)}.$$

We introduce the function

$$\overline{p}^*(x)=\begin{cases}\frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \overline{p}(x)<N,\\ +\infty, & \text{if } \overline{p}(x)\ge N.\end{cases} \tag{1.5}$$

The nonlinear term $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function such that for a.e. $x\in\Omega$ and all $(\sigma,\xi)\in\mathbb{R}\times\mathbb{R}^N$, we have

$$|g(x,\sigma,\xi)|\le b(|\sigma|)\left(c(x)+\sum_{i=1}^{N}|\xi_i|^{p_i(x)}\right), \tag{1.6}$$

$$g(x,\sigma,\xi)\cdot\sigma\ge0, \tag{1.7}$$

where $b:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function with finite values, $c\in L^1(\Omega)$ and $\exists\rho>0$ such that:

$$|g(x,\sigma,\xi)|\ge\rho\left(\sum_{i=1}^{N}|\xi_i|^{p_i(x)}\right),\quad \forall\sigma \text{ such that } |\sigma|>\rho. \tag{1.8}$$

In [1], the authors obtain the existence of renormalized and entropy solutions for the nonlinear elliptic equation with degenerate coercivity of the type

$$-\operatorname{div}[a(x,u)|\nabla u|^{p(x)-2}\nabla u]+g(x,u)=f\in L^1(\Omega).$$

For $g\equiv0$ and $f\in L^{m(\cdot)}(\Omega)$, with $m(x)\ge m_-\ge1$, equation of the from (1.1) have been widely studied in [2], where the authors obtain some existence and regularity results for the solutions. If $g\equiv|u|^{s(x)-1}u$,

$$a_i(x,u,\nabla u)=\frac{|D_iu|^{p_i(x)-2}D_iu}{(1+|u|)^{\gamma_i(x)}}$$

and $f \in L^m(\Omega)$, with $m \geq 1$, existence and regularity results of distributional solutions have been proved in [3].

As far as the existence results for our problem (1.1) there are three difficulties associated with this kind of problems. Firstly, from hypothesis (1.2), the operator

$$Au = -\sum_{i=1}^{N} D_i a_i(x,u,\nabla u)$$

is well defined between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual space $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$. However, by assumption (1.4), if we take for example

$$\begin{cases} a_i(x,u_n,\nabla u_n) = \dfrac{|D_i u_n|^{p_i(x)-2} D_i u_n}{(1+|u_n|)^{\gamma_i(x)}} \\ u_n(x) = |x|^{\frac{n(p_+^+ - N)}{(n+1)p_+^+}} - 1, \quad |x| \leq 1 \end{cases} \qquad \text{where } p_+^+ \text{ is defined as in (2.3),}$$

the operator A is not coercive. Because, if $\|u_n\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}$ tends to infinity then

$$\frac{\langle Au_n, u_n \rangle}{\|u_n\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}} \to 0.$$

So, the classical methods used in order to prove the existence of a solution for (1.1) cannot be applied. The second difficulty is represented in the fact that $g(x,u,\nabla u)$ can not be defined from $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into its dual, but from $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$. The third difficulty appears when we give a variable exponential growth condition (1.2) for a_i. The operator A possesses more complicated nonlinearities; thus, some techniques used in the constant exponent case cannot be carried out for the variable exponent case. For more recent results for elliptic and parabolic case, see the papers [4–8] and references therein.

The paper is organized as follows. In Section 2, we present results on the Lebesgue and Sobolev spaces with variable exponents both for the isotropic and the anisotropic cases, and state the main results. The proof of the main result will be presented in Section 3. We start by giving an existence result for an approximate problem associated with (1.1). The second part of Section 3 is devoted to proving the main existence result by using a priori estimates and then passing to the limit in the approximate problem.

2 Preliminaries and statement of the main result

2.1 Preliminaries

In this sub-section, we recall some facts on anisotropic spaces with variable exponents and we give some of their properties. For further details on the Lebesgue-Sobolev spaces with variable exponents, we refer to [9–11] and references therein. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$, we denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad p^- = \min_{x\in\overline{\Omega}} p(x) \tag{2.1}$$

and

$$C_+(\overline{\Omega}) = \{p\in C(\overline{\Omega}) \mid p^- > 1\}.$$

Let $p(\cdot)\in C_+(\overline{\Omega})$. We define the space

$$L^{p(\cdot)}(\Omega) = \left\{ u:\Omega\to\mathbb{R}^N \text{ measurable} \mid \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < \infty \right\}$$

then the expression

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda>0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $\left(L^{p(\cdot)}(\Omega),\ \|u\|_{p(\cdot)}\right)$ is a separable Banach space. If $0<\text{meas}(\Omega)<+\infty$ and $p_1,p_2\in C_+(\overline{\Omega})$ with $p_1\leq p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega)\hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous. Moreover, if $1<p^-<p^+<+\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$. For all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\mathrm{d}x\right| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds true. We define the variable exponents Sobolev spaces by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u\in L^{p(\cdot)}(\Omega) \quad \mid |\nabla u|\in L^{p(\cdot)}(\Omega) \right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Finally, we introduce a natural generalization of the variable exponents Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study with sufficient accuracy problem (1.1). Let $\overrightarrow{p}(\cdot)=(p_1(\cdot),\ldots,p_N(\cdot))$, where $p_i:\overline{\Omega}\to(1,+\infty)$ are continuous functions. We introduce the anisotropic variable exponents Sobolev spaces

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u\in L^{p_i(\cdot)}(\Omega) \mid D_iu\in L^{p_i(\cdot)}(\Omega), i=1,\ldots,N\},$$

with respect to the norm

$$\|v\|_{1,\overrightarrow{p}(\cdot)}=\sum_{i=1}^{N}\Big(\|u\|_{L^{p_i(\cdot)}(\Omega)}+\|D_iu\|_{L^{p_i(\cdot)}(\Omega)}\Big). \tag{2.2}$$

We introduce the following notation $p_+^+,p_-^-\in\mathbb{R}^+$ as

$$p_+^+=\max\{p_1^+,...,p_N^+\},\quad p_-^-=\min\{p_1^-,...,p_N^-\}. \tag{2.3}$$

We denote $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)=W_0^{1,1}(\Omega)\cap W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with respect to the norm (2.2).
According to [10], $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

Theorem 2.1 ([10]). *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot)=(p_1(\cdot),p_2(\cdot),...,p_N(\cdot))\in(C_+(\overline{\Omega}))^N$. Suppose that*

$$p^+(x)<\overline{p}^*(x)\quad \textit{for all}\quad x\in\overline{\Omega}. \tag{2.4}$$

Then

$$\|u\|_{L^{p^+(\cdot)}(\Omega)}\le C\sum_{i=1}^{N}\|D_iu\|_{L^{p_i(\cdot)}(\Omega)},\quad \forall u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),$$

where p^+ is defined as in (2.1), *$\overline{p}^*$ as in* (1.5), *and C is a positive constant independent of u. Thus $\sum_{i=1}^N\|D_iu\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

Proposition 2.1. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Then, for all $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we have*

$$\frac{1}{N^{p_-^- -1}}\|u\|_{1,\overrightarrow{p}(\cdot)}^{p_-^-}-N\le\sum_{i=1}^{N}\int_\Omega|D_iu|^{p_i(x)}\mathrm{d}x\le N+\|u\|_{1,\overrightarrow{p}(\cdot)}^{p_+^+}. \tag{2.5}$$

Proof. Put

$$\mathcal{I}=\Big\{i\in\{1,...,N\}\mid\|D_iu\|_{p_i(\cdot)}\le1\Big\}\quad\text{and}\quad\mathcal{J}=\Big\{i\in\{1,...,N\}\mid\|D_iu\|_{p_i(\cdot)}>1\Big\}.$$

Thanks to (Proposition 2.1 in [3]), we have

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega|D_iu|^{p_i(x)}\mathrm{d}x&=\sum_{i\in\mathcal{I}}\int_\Omega|D_iu|^{p_i(x)}\mathrm{d}x+\sum_{i\in\mathcal{J}}\int_\Omega|D_iu|^{p_i(x)}\mathrm{d}x\\&\ge\sum_{i\in\mathcal{I}}\|D_iu\|_{p_i(\cdot)}^{p_i^+}+\sum_{i\in\mathcal{J}}\|D_iu\|_{p_i(\cdot)}^{p_i^-}\\&\ge\sum_{i=1}^{N}\|D_iu\|_{p_i(\cdot)}^{p_-^-}-\sum_{i\in\mathcal{I}}\|D_iu\|_{p_i(\cdot)}^{p_-^-}\ge\sum_{i=1}^{N}\|D_iu\|_{p_i(\cdot)}^{p_-^-}-N.\end{aligned}$$

Using the convexity of the application $t\in\mathbb{R}^+\mapsto t^{p_-^-}, p_-^->1$, we obtain

$$\sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\mathrm{d}x\geq \frac{1}{N^{p_-^- -1}}\|u\|_{1,\overrightarrow{p}(\cdot)}^{p_-^-}-N,$$
$$\sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\mathrm{d}x\leq \sum_{i\in\mathcal{I}}\|D_iu\|_{p_i(\cdot)}^{p_i^-}+\sum_{i\in\mathcal{J}}\|D_iu\|_{p_i(\cdot)}^{p_i^+}\leq N+\sum_{i=1}^{N}\|D_iu\|_{p_i(\cdot)}^{p_+^+}. \qquad \square$$

We will use through the paper, the truncation function T_k at height k ($k>0$), that is $T_k(s):=\max\{-k,\min\{k,s\}\}$.

Lemma 2.1 ([12]). *Let $g\in L^{p(\cdot)}(\Omega)$ and $g_n\in L^{p(\cdot)}(\Omega)$ with $\|g_n\|_{p(\cdot)}\leq C$. If $g_n(x)\to g(x)$ almost everywhere in Ω, then $g_n\rightharpoonup g$ in $L^{p(\cdot)}(\Omega)$.*

Lemma 2.2 ([13]). *Let $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, then $T_k(u)\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for all $k>0$. Moreover, we have $T_k(u)\to u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as $k\to\infty$.*

Lemma 2.3 ([13]). *Let $(u_n)_n$ be a bounded sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. If $u_n\rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, then $T_k(u_n)\rightharpoonup T_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

Lemma 2.4 ([13]). *Assume that* (1.2)-(1.4) *hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $u_n\rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and*

$$\sum_{i=1}^{N}\int_\Omega (a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D_iu_n-D_iu)\mathrm{d}x\to 0.$$

Then, $u_n\to u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for a subsequence.

2.2 Statement of main result

We will extend the notion of distributional solution, see [12, 13], to problem (1.1) as follows:

Definition 2.1. *Let $f\in L^1(\Omega)$ a measurable function u is said to be solution in the sense of distributions to the problem* (1.1), *if*

$$u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\ g(x,u,\nabla u)\in L^1(\Omega),\ \text{ and } \forall v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega) \tag{2.6a}$$
$$\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)D_iv\mathrm{d}x+\int_\Omega g(x,u,\nabla u)v\mathrm{d}x=\int_\Omega fv\mathrm{d}x. \tag{2.6b}$$

Our main result is as follows

Theorem 2.2. *Let $f\in L^1(\Omega)$. Assume* (1.2)-(1.8) *and* (2.4). *Then problem* (1.1) *has at least one solution in the sense of distributions.*

3 Proof of the main result

3.1 Approximate solution

Let $(f_n)_n$ be a sequence in $L^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ with $|f_n| \le |f|$ (for example $f_n = T_n(f)$) and we consider the approximate problem

$$\begin{cases} -\sum\limits_{i=1}^{N} D_i a_i(x,T_n(u_n),\nabla u_n) + g(x,u_n,\nabla u_n) = T_n(f) \quad \text{in} \quad \Omega, \\ u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $f \in L^1(\Omega)$. Assume* (1.2)-(1.8) *and* (2.4)*. Then, problem* (3.1) *has at least one solution in the sense of distributions.*

Proof. Let us define the operator A_n from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into its dual $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$, by

$$A_n u = -\sum_{i=1}^{N} D_i a_i(x,T_n(u),\nabla u) \quad \text{and} \quad g^k(x,s,\xi) = \frac{g(x,s,\xi)}{1+|g(x,s,\xi)|/k}.$$

Note that $g^k(x,s,\xi)s \ge 0$, $|g^k(x,s,\xi)| \le |g(x,s,\xi)|$ and $|g^k(x,s,\xi)| \le k$ for all $k \in \mathbb{N}-\{0\}$. We define $G_k : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$, by

$$\langle G_k u, v\rangle = \int_\Omega g^k(x,u,\nabla u) v \,\mathrm{d}x.$$

Consider the following problem

$$A_n u_{n_k} + g^k(x,u_{n_k},\nabla u_{n_k}) = T_n(f) \ \text{ in } \Omega, \quad u_{n_k} \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{3.2}$$

Lemma 3.2. *Let $f \in L^1(\Omega)$. Assume that* (1.2)-(1.8) *and* (2.4) *hold, then the problem* (3.2) *has at least one solution u_{n_k} in the sense of distributions.*

Lemma 3.3. *The operator $B_k^n = A_n + G_k$ from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$ is pseudo-monotone, moreover, B_k^n is coercive in the following sense*

$$\frac{\langle B_k^n v, v\rangle}{\|v\|_{1,\overrightarrow{p}(\cdot)}} \to +\infty \quad \text{if } \|v\|_{1,\overrightarrow{p}(\cdot)} \to +\infty \quad \text{for } v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Proof of the Lemma 3.3. Thanks to the Hölder inequality, we have for all $u,v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$|\langle G_k u, v\rangle| \le \left(\frac{1}{p_i^-} + \frac{1}{(p_i^-)'}\right) \|g^k(x,u,\nabla u)\|_{p_i'(\cdot)} \|v\|_{p_i(\cdot)}$$

$$
\begin{aligned}
&\le C_0\left(\frac{1}{(p_-^-)}+\frac{1}{(p_-^-)'}\right)\left(k^{(p_+^+)'}\text{meas}(\Omega)+1\right)^{\frac{1}{(p_-^-)'}}\|v\|_{1,\overrightarrow{p}(\cdot)}\\
&\le C_1\|v\|_{1,\overrightarrow{p}(\cdot)}.
\end{aligned}
\tag{3.3}
$$

Thanks to the Hölder inequality and (1.2), we have for all $u,v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$
\begin{aligned}
|\langle A_n u,v\rangle| &\le 2\sum_{i=1}^{N}\left\|\left(k(x)+|T_n(u)|^{\overline{p}(x)}+\sum_{j=1}^{N}|D_i u|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\right\|_{p_i'(\cdot)}\|D_i v\|_{p_i(\cdot)}\\
&\le C_2\|v\|_{1,\overrightarrow{p}(\cdot)}.
\end{aligned}
\tag{3.4}
$$

Then by using (3.3) and (3.4) we conclude that $B_k^n=A_n+G_k$ is bounded. For the coercivity, by using (1.4), (1.7), and (2.5), we get

$$
\begin{aligned}
\langle B_k^n u,u\rangle &\ge \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D_i u\mathrm{d}x \ge \sum_{i=1}^{N}\int_\Omega \frac{\alpha}{(1+n)^{\gamma_i(x)}}|D_i u|^{p_i(x)}\mathrm{d}x\\
&\ge \frac{C_3}{N^{p_-^- -1}}\|u\|_{1,\overrightarrow{p}(\cdot)}^{p_-^-}-C_3 N,
\end{aligned}
$$

then

$$
\frac{\langle B_k^n u,u\rangle}{\|u\|_{1,\overrightarrow{p}(\cdot)}}\ge \frac{C_3}{N^{p_-^- -1}}\|u\|_{1,\overrightarrow{p}(\cdot)}^{p_-^- -1}-\frac{C_3 N}{\|u\|_{1,\overrightarrow{p}(\cdot)}}\to+\infty \quad \text{as } \|u\|_{1,\overrightarrow{p}(\cdot)}\to+\infty.
$$

It remains to show that B_k^n is pseudo-monotone. Let $(u_m)_m$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$
\begin{cases}
u_m \rightharpoonup u, & \text{in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\\
B_k^n u_m \rightharpoonup \chi_k^n, & \text{in } \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)',\\
\limsup\limits_{m\to\infty}\langle B_k^n u_m,u_m\rangle \le \langle \chi_k^n,u\rangle.
\end{cases}
\tag{3.5}
$$

We will prove that

$$
\chi_k^n=B_k^n u \quad \text{and} \quad \langle B_k^n u_m,u_m\rangle\to\langle\chi_k^n,u\rangle \quad \text{as } m\to+\infty.
$$

Firstly, since $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is compactly embedded in $L^{p_-^-}(\Omega)$, then $u_m\to u$ in $L^{p_-^-}(\Omega)$ and

$$
u_m\to u \quad \text{a.e. in } \Omega,
\tag{3.6}
$$

for a subsequence still denoted $(u_m)_m$. The sequence $(u_m)_m$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Then, by (1.2) we have $a_i(x,T_n(u_m),\nabla u_m)$ is bounded in $L^{p_i'(\cdot)}(\Omega)$. Therefore, there exists a function $\varphi_i^n\in L^{p_i'(\cdot)}(\Omega)$ such that

$$
a_i(x,T_n(u_m),\nabla u_m)\rightharpoonup\varphi_i^n \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{as } m\to\infty.
\tag{3.7}
$$

Similarly, since $(g^k(x,u_m,\nabla u_m))_m$ is bounded in $L^{(p_-^-)'}(\Omega)$ with $(p_-^-)'$ is the conjugate exponent of (p_-^-), there exists a function $\psi^k \in L^{(p_-^-)'}(\Omega)$ such that

$$g^k(x,u_m,\nabla u_m) \rightharpoonup \psi^k \quad \text{in} \quad L^{(p_-^-)'}(\Omega) \quad \text{as } m \to \infty. \tag{3.8}$$

For all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\langle \chi_k^n, v\rangle = \sum_{i=1}^N \int_\Omega \varphi_i^n D_i v \,\mathrm{d}x + \int_\Omega \psi^k v \,\mathrm{d}x. \tag{3.9}$$

Using (3.5), (3.8), (3.9), and that $u_m \to u$ in $L^{p_-^-}(\Omega)$, we have

$$\int_\Omega g^k(x,u_m,\nabla u_m)u_m \,\mathrm{d}x \to \int_\Omega \psi^k u \,\mathrm{d}x, \tag{3.10}$$

therefore, thanks to (3.5), (3.9), and (3.10), we write

$$\limsup_{m\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m) D_i u_m \,\mathrm{d}x \le \sum_{i=1}^N \int_\Omega \varphi_i^n D_i u \,\mathrm{d}x. \tag{3.11}$$

On the other hand, by (1.3), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m) D_i u_m \,\mathrm{d}x \\
&\ge \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m) D_i u \,\mathrm{d}x \\
&\qquad + \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u)(D_i u_m - D_i u) \,\mathrm{d}x.
\end{aligned}$$

In view of Lebesgue dominated convergence theorem and (3.6), we have

$$a_i(x,T_n(u_m),\nabla u) \to a_i(x,T_n(u),\nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega).$$

By (3.7) and (3.5), we get

$$\liminf_{m\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m) D_i u_m \,\mathrm{d}x \ge \sum_{i=1}^N \int_\Omega \varphi_i^n D_i u \,\mathrm{d}x,$$

this implies, thanks to (3.11), that

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m) D_i u_m \,\mathrm{d}x = \sum_{i=1}^N \int_\Omega \varphi_i^n D_i u \,\mathrm{d}x. \tag{3.12}$$

By combining (3.5), (3.10), and (3.12) we deduce that $\langle B_k^n u_m, u_m\rangle \to \langle \chi_k^n, u\rangle$ as $m\to\infty$. Now, by (3.12) we obtain

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u))(D_iu_m-D_iu)\mathrm{d}x=0.$$

In view of Lemma 2.4, we get $u_m\to u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $D_iu_m\to D_iu$ almost everywhere in Ω, then $a_i(x,T_n(u_m),\nabla u_m)\rightharpoonup a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$ and $g^k(x,u_m,\nabla u_m)\rightharpoonup g^k(x,u,\nabla u)$ in $L^{\theta'(\cdot)}(\Omega)$ for all $i=1,...,N$, where $\theta'(x)\geq \max\{p_i(\cdot),\quad i=1,...,N\}$, so we deduce that $\chi_k^n=B_k^n u$, which completes the proof of Lemma 3.3. □

Proof of the Lemma 3.2. In view of Lemma 3.3, there exists at least one weak solution $u_{n_k}\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ of problem (3.2) (see [14]). □

Lemma 3.4. *Let* $f\in L^1(\Omega)$, *assume that* (1.2)-(1.7) *and* (2.4) *hold. Let* $u_{n_k}\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *be a distribution solution of* (3.2). *Then, there exists a constant* $C(n)>0$ *such that*

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\mathrm{d}x\leq C(n).$$

Proof. The proof uses the same technique as in (Lemma 4.1 of [3]) and is omited here. □

Therefore, by Lemma 3.4 the sequence $\{u_{n_k}\}_k$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. As a consequence, there exists a function $u_n\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by u_{n_k}) such that

$$u_{n_k}\rightharpoonup u_n \quad \text{weakly in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \quad \text{and a.e. in} \quad \Omega \text{ as } k\to\infty.$$

Lemma 3.5. *Assume that hypotheses* (1.2), (1.7), *and* (2.4) *hold. Let* $u_{n_k}\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *be a distribution solution of* (3.2). *Then, there exist a subsequence of* (u_{n_k}) *denoted by itself, and a measurable function* $u_n\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, *such that*

$$T_h(u_{n_k})\to T_h(u_n) \quad \text{in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Proof. It is similar to the proof of Theorem 4.2 of [13]. □

3.2 A priori estimates

Lemma 3.6. *Assume* (1.2)-(2.4). *Let* $u_n\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *be a distribution solution of* (1.1). *Then, there exists a constant* $C\geq 0$ *such that*

$$\|u_n\|_{1,\overrightarrow{p}(\cdot)}\leq C.$$

Proof. Let $h>0$. Taking $T_h(u_n)$ as a test function in (3.1), then

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D_iT_h(u_n)\mathrm{d}x+\int_{\Omega} g(x,u_n,\nabla u_n)T_h(u_n)\mathrm{d}x$$
$$=\int_{\Omega} T_n(f)T_h(u_n)\mathrm{d}x. \tag{3.13}$$

By dropping the nonnegative term in (3.13), (1.7), and (1.4) we get

$$\sum_{i=1}^{N}\int_{\Omega}\frac{\alpha}{(1+|T_n(u_n)|)^{\gamma_i(x)}}|D_iT_h(u_n)|^{p_i(x)}\mathrm{d}x\le h\int_{\Omega}|f|\mathrm{d}x,$$

then

$$\sum_{i=1}^{N}\int_{\Omega}\frac{\alpha}{(1+h)^{\gamma_i(x)}}|D_iT_h(u_n)|^{p_i(x)}\mathrm{d}x\le h\|f\|_{L^1(\Omega)}.$$

Consequently,

$$\sum_{i=1}^{N}\int_{\Omega}|D_iT_h(u_n)|^{p_i(x)}\mathrm{d}x\le C_3. \tag{3.14}$$

Taking $T_h(u_n)$ as a test function in (3.1), and dropping the first nonnegative term in the left-hand side, we obtain

$$\int_{\{|u_n|>h\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\le\|f\|_{L^1(\Omega)}. \tag{3.15}$$

By combining (1.8), (3.14) and (3.15), for $h=\rho$, we deduce that

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\mathrm{d}x\le C_4+\frac{1}{h}\int_{\{|u_n|>h\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\le C_4+\frac{\|f\|_{L^1(\Omega)}}{h}=C_5.$$

By (2.5), we get $\|u_n\|_{1,\vec{p}(\cdot)}\le C_6$. Consequently, there exist a subsequence of u_n (denoted by itself) and a measurable function $u\in W_0^{1,\vec{p}(\cdot)}(\Omega)$, such that

$$\begin{cases} u_n\rightharpoonup u, & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ u_n\to u, & \text{in } L^{p_-^-}(\Omega).\end{cases}$$

This ends the proof of Lemma 3.6. □

3.3 The strong convergence of the truncation

Lemma 3.7. *Assume that hypotheses* (1.2)-(1.8) *and* (2.4) *hold, and let* $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *be a distribution solution of* (1.1). *Then, there exist a subsequence of* u_n *denoted by itself, and a measurable function* $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *such that*

$$T_j(u_n) \to T_j(u) \quad \text{strongly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Proof. Let $h \geq j > 0$ and $w_n = T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u))$. We set $\varphi_j(s) = s \cdot \exp(\delta s^2)$, where $\delta = (l(j)/(2\alpha))^2$, $l(j) = b(j)(1+|j|)^{\gamma_+^+}$, and

$$\varphi_j'(s) - \frac{l(j)}{\alpha}|\varphi_j(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R}.$$

Let $M = 4j+h$. Since $D_i w_n = 0$ on $\{|u_n| > M\}$ and $\varphi_j(w_n)$ has the same sign as u_n on the set $\{|u_n| > j\}$ (indeed, if $u_n > j$ then $u_n - T_h(u_n) \geq 0$ and $T_j(u_n) - T_j(u) \geq 0$, it follows that $w_n \geq 0$). Similarly, we show that $w_n \leq 0$ on the set $\{u_n < -j\}$.
By taking $\varphi_j(w_n)$ as a test function in (3.1), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_j'(w_n)D_i w_n \mathrm{d}x\\&\qquad+\int_{|u_n|\leq j} g(x,u_n,\nabla u_n)\varphi_j(w_n)\mathrm{d}x \leq \int_\Omega T_n(f)\varphi_j(w_n)\mathrm{d}x.\end{aligned}$$

Taking $y_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(w_n)D_i w_n \mathrm{d}x\\\geq&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(w_n)(D_iT_k(u_n)-D_iT_k(u))\mathrm{d}x\\&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(w_n)D_iT_k(u)\mathrm{d}x\\&\quad-\varphi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|a_i(x,T_M(u_n),\nabla T_M(u_n))||D_iT_k(u)|\mathrm{d}x,\end{aligned}$$

that is equivalent to

$$\sum_{i=1}^{N}\int_\Omega \big(a_i(x,T_j(u_n),\nabla T_j(u_n)) - a_i(x,T_j(u_n),\nabla T_j(u))\big)$$

$$
\begin{aligned}
&\times (D_i T_j(u_n) - D_i T_j(u))\varphi_j'(w_n)\mathrm{d}x\\
&\le \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_j'(w_n) D_i w_n \mathrm{d}x + (\mathbf{A}) + (\mathbf{B}) + (\mathbf{C}),
\end{aligned}
\tag{3.16}
$$

where

$(\mathbf{A})$ $\quad -\sum_{i=1}^{N} \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(w_n) D_i T_k(u)\mathrm{d}x,$

$(\mathbf{B})$ $\quad +\varphi_j'(2j)\sum_{i=1}^{N} \int_{\{|u_n|>j\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))||D_i T_j(u)|\mathrm{d}x,$

$(\mathbf{C})$ $\quad -\sum_{i=1}^{N} \int_\Omega a_i(x, T_j(u_n), \nabla T_j(u))(D_i T_j(u_n) - D_i T_j(u))\varphi_j'(w_n)\mathrm{d}x.$

Arguing as in [13], we can prove that

$$
(\mathbf{A}) = \varepsilon_1(n), \quad (\mathbf{B}) = \varepsilon_2(n) \quad \text{and} \quad (\mathbf{C}) = \varepsilon_3(n). \tag{3.17}
$$

By (3.16) and (3.17) we conclude that

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega (a_i(x, T_j(u_n), \nabla T_j(u_n)) - a_i(x, T_j(u_n), \nabla T_j(u)))\\
&\qquad \times (D_i T_j(u_n) - D_i T_j(u))\varphi_j'(w_n)\mathrm{d}x\\
&\le \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_j'(w_n) D_i w_n \mathrm{d}x + \varepsilon_4(n).
\end{aligned}
\tag{3.18}
$$

Using (3.18) and arguing as in [13], we get

$$
\begin{aligned}
&\frac{l(j)}{\alpha}\sum_{i=1}^{N} \int_\Omega (a_i(x, T_j(u_n), \nabla T_j(u_n)) - a_i(x, T_j(u_n), \nabla T_j(u)))\\
&\qquad \times (D_i T_j(u_n) - D_i T_j(u))|\varphi_j(w_n)|\mathrm{d}x\\
&\ge \left| \int_{\{|u_n|\le j\}} g_i(x, T_j(u_n), \nabla T_j(u_n))\varphi_j(w_n)\mathrm{d}x \right| + \varepsilon_5(n).
\end{aligned}
\tag{3.19}
$$

Thanks to (3.18) and (3.19), we obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N} \int_\Omega (a_i(x, T_j(u_n), \nabla T_j(u_n)) - a_i(x, T_j(u_n), \nabla T_j(u))) \times (D_i T_j(u_n) - D_i T_j(u))\mathrm{d}x\\
&\le \int_\Omega T_n(f)\varphi_j(T_{2j}(u - T_h(u)))\mathrm{d}x + \varepsilon_6(n).
\end{aligned}
$$

Then by letting h tends to infinity in the previous inequality, we get

$$\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_j(u_n),\nabla T_j(u_n))-a_i(x,T_j(u_n),\nabla T_j(u))) \\ \times(D_iT_j(u_n)-D_iT_j(u))\mathrm{d}x\to 0, \qquad \text{as } n\to\infty.$$

Using Lemma 2.4, we deduce that $T_j(u_n)\to T_j(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. □

Thanks to Lemma 2.2, we obtain

$$\begin{cases} u_n\to u, & \text{in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\\ \nabla u_n\to\nabla u, & \text{a.e in } \Omega. \end{cases} \tag{3.20}$$

3.4 The equi-integrability of $g(x,u_n,\nabla u_n)$ and passage to the limit

Thanks to (3.20), we have

$$\begin{aligned} &a_i(x,T_n(u_n),\nabla u_n)\to a_i(x,u,\nabla u) &&\text{a.e. in } \Omega,\\ &g(x,u_n,\nabla u_n)\to g(x,u,\nabla u) &&\text{a.e. in } \Omega. \end{aligned}$$

Using that $(a_i(x,u_n,\nabla u_n))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and Lemma 2.1, we obtain

$$a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega).$$

Now, let E be a measurable subset of Ω. For all $m>0$, we have by using (1.6)

$$\int_{\Omega}|g(x,u_n,\nabla u_n)|\mathrm{d}x \\ \le h(m)\int_{E\cap\{|u_n|\le m\}}\left(c(x)+\sum_{i=1}^{N}|D_iT_m(u_n)|^{p_i(x)}\right)\mathrm{d}x+\int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x.$$

Since $(D_iT_m(u_n))_n$ converges strongly in $L^{p_i(\cdot)}(\Omega)$, then for all $\varepsilon>0$, there exists $\delta>0$ such that $meas(E)<\delta$ and

$$h(m)\sum_{i=1}^{N}\int_{E}|D_iT_m(u_n)|^{p_i(x)}\mathrm{d}x<\frac{\varepsilon}{3} \quad \text{and} \quad h(m)\int_{E}c(x)\mathrm{d}x<\frac{\varepsilon}{3}. \tag{3.21}$$

On the other hand, using $T_1(u_n-T_{m-1}(u_n))$ as a test function in (3.1) for $m>1$, we obtain

$$\int_{\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\le\int_{\{|u_n|>m-1\}}|f|\mathrm{d}x,$$

there exists $m_0>0$ such that

$$\int\limits_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\mathrm{d}x<\frac{\varepsilon}{3} \quad \text{for all } m>m_0. \tag{3.22}$$

Using (3.21) and (3.22), we deduce the equi-integrability of $g(x,u_n,\nabla u_n)$. In view of Vitali's theorem, we obtain

$$g(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \quad \text{in } L^1(\Omega).$$

Finally for, $v\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\sum_{i=1}^N \int\limits_\Omega a_i(x,T_n(u_n),\nabla u_n)D_i v\mathrm{d}x+\int\limits_\Omega g(x,u_n,\nabla u_n)v\mathrm{d}x=\int\limits_\Omega T_n(f)v\mathrm{d}x.$$

Letting $n\to+\infty$, we can easily pass to the limit in this equation, to see that this last integral identity is true for u instead of u_n. This proves Theorem (2.2). □

Example 3.1. As a prototype example, we consider the model problem

$$\begin{cases} -\sum\limits_{i=1}^N D_i\left(\dfrac{|u|^{\frac{\overline{p}(x)}{p_i(x)}(p_i(x)-1)}(1+|D_iu|)^{-1}D_iu+|D_iu|^{p_i(x)-2}D_iu}{(1+|u|)^{\gamma_i(x)}}\right) \\ \qquad\qquad +u\sum\limits_{i=1}^N |D_iu|^{p_i(x)}=f, \text{ in } \Omega, \\ u=0, \quad \text{in } \partial\Omega, \end{cases}$$

where $f\in L^1(\Omega)$ and $\gamma_i:\overline{\Omega}\to\mathbb{R}^+$, $p_i:\overline{\Omega}\to(1,+\infty)$ as in Theorem 2.2.

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References

[1] Li Z., Gao W., Existence results to a nonlinear $p(x)-$ Laplace equation with degenerate coercivity and zero-order term : renormalized and entropy solutions. *Appl. Anal.* **95** (2) (2015), 1-17.
[2] Zhang X., Fu Y., Solutions for nonlinear elliptic equations with variable growth and degenerate coercivity. *Annali di Matematica.* **193** (1) (2014), 133-161.
[3] Ayadi H., Mokhtari F., Nonlinear anisotropic elliptic equations with variable exponents and degenerate coercivity. *Electron. J. Differ. Equations.* **45** (2018) (2018), 1-23.

[4] Ayadi H., Mokhtari F., Entropy solutions for nonlinear anisotropic elliptic equations with variable exponents and degenerate coercivity. *Complex Variables and Elliptic Equations.* **DOI: 10.1080/17476933.2019.1615899** (2019), 1-23.

[5] Mokhtari F., Anisotropic elliptic equations in $\mathbb{R}^N$ with variable exponents and locally integrable data. *Nonlinear . Math. Methods Appl.* **40** (2017), 2265-2276.

[6] Mokhtari F., Mecheter R., Anisotropic degenerate parabolic problems in $\mathbb{R}^N$ with variable exponent and locally integrable data. *Mediterr. J. Math.* **16** (3) (2019), 1-21.

[7] Mokhtari F., Regularity of the solution to nonlinear anisotropic elliptic equations with variable exponents and irregular data. *Mediterr. J. Math.* **14** (2017), 1-18.

[8] Naceri M., Mokhtari F., Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity. *Appl. Anal.* **DOI: 10.1080/00036811.2019.1682136** (2019), 1-21.

[9] Fan X., Shen J., Zhao D., Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **262** (2001) (2001), 749-760.

[10] Fan X., Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(x)-$ Laplacian equations. *Complex Var. Elliptic Equations.* **56** (7-9) (2011), 623-642.

[11] Kovacik O., Rakosnik J., On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math. J.* **41** (4) (1991), 592-618.

[12] Benboubker M.B., Azroul E., Barbara A., Quasilinear elliptic problems with nonstandard growths. *Electron. J. Differ. Equations.* **2011** (62) (2011), 1-16.

[13] Benboubker M.B., Chrayteh H., Hjiaj H., Chihab Y., Existence of solutions in the sense of distributions of anisotropic nonlinear elliptic equations with variable exponent. *Topol. Methods Nonlinear Anal.* **46** (2) (2015), 665-693.

[14] Lions J.L., Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, 1969.