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Remarks on Blow-Up Phenomena in p-Laplacian Heat Equation with Inhomogeneous Nonlinearity

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Abstract. We investigate the p-Laplace heat equation $u_t-\Delta_p u=\zeta(t)f(u)$ in a bounded smooth domain. Using differential-inequality arguments, we prove blow-up results under suitable conditions on ζ, f, and the initial datum u_0. We also give an upper bound for the blow-up time in each case.

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1 Introduction

In the past decade a strong interest in the phenomenon of blow-up of solutions to various classes of nonlinear parabolic problems has been assiduously investigated. We refer the reader to the books [1, 2] as well as to the survey paper [3]. Problems with constant coefficients were investigated in [4], and problems with time-dependent coefficients under homogeneous Dirichlet boundary conditions were treated in [5]. See also [6] for a related system. The question of blow-up for nonnegative classical solutions of p-Laplacian heat equations with various boundary conditions has attracted considerable attention in the mathematical community in recent years. See for instance [7–10].

There are two effective techniques which have been employed to prove non-existence of global solutions: the concavity method ([11]) and the eigenfunction method ([12]). The

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latter one was first used for bounded domains but it can be adapted to the whole space $\mathbb{R}^N$. The concavity method and its variants were used in the study of many nonlinear evolution partial differential equations (see, e.g., [13–15]).

In the present paper, we investigate the blow-up phenomena of solutions to the following nonlinear p-Laplacian heat equation:

$$\begin{cases} u_t - \Delta_p u = \zeta(t) f(u), & x \in \Omega, \quad t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator, $p \geq 2$, Ω is a bounded sufficiently smooth domain in $\mathbb{R}^N$, $\zeta(t)$ is a nonnegative continuous function. The nonlinearity $f(u)$ is assumed to be continuous with $f(0)=0$. More specific assumptions on f, ζ and u_0 will be made later.

The case of $p=2$ has been studied in [4] for $\zeta(t) \equiv 1$, and in [5] for ζ being a nonconstant function of t. Concerning the case $p>2$, Messaoudi [10] proved the blow-up of solutions with vanishing initial energy when $\zeta(t) \equiv 1$. See also [9] and references therein. Recently, a p-Laplacian heat equations with nonlinear boundary conditions and time-dependent coefficient was investigated in [7]. This note may be regarded as a complement, and in some sense an improvement, of [5, 10].

Let us now precise the assumptions on f and ζ. If $p=2$, we suppose either

$$f \in C^1(\mathbb{R}) \quad \text{is convex with} \quad f(0)=0; \tag{1.2}$$

$$\exists\ \lambda > 0 \quad \text{such that} \quad f(s) > 0 \quad \text{for all} \quad s \geq \lambda; \tag{1.3}$$

$$\int_\lambda^\infty \frac{\mathrm{d}s}{f(s)} < \infty; \tag{1.4}$$

$$\inf_{t \geq 0} \left(\int_0^t (\zeta(s) - 1)\,\mathrm{d}s \right) := m \in (-\infty, 0], \tag{1.5}$$

or

$$s f(s) \geq (2+\epsilon) F(s) \geq C_0 |s|^\alpha, \tag{1.6}$$

for some constants $\epsilon, C_0 > 0$, $\alpha > 2$, and

$$\zeta \in C^1([0,\infty)) \quad \text{with} \quad \zeta(0) > 0 \text{ and } \zeta' \geq 0. \tag{1.7}$$

Here $F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$.

Our first main result concerns the case $p=2$ and reads as follows.

Theorem 1.1. *Suppose that assumptions* (1.2)–(1.5) *are fulfilled. Let* $0 \leq u_0 \in L^\infty(\Omega)$ *such that* $\int_\Omega u_0 \phi_1$ *is large enough. Then the solution* $u(t,x)$ *of problem* (1.1) *blows up in finite time.*

Remark 1.1.

(i) The function ϕ_1 stands for the eigenfunction of the Dirichlet-Laplace operator associated to the first eigenvalue $\lambda_1>0$, that is

$$\Delta\phi_1=-\lambda_1\phi_1,\quad \phi_1>0,\ x\in\Omega;\quad \phi_1=0,\ x\in\partial\Omega,\ \int_\Omega\phi_1=1.$$

(ii) The assumptions (1.2)–(1.5) on f and ζ cover the example

$$f(u)=\mathrm{e}^u-1\quad\text{and}\quad \zeta(t)=\mathrm{e}^{t^2}. \tag{1.8}$$

Note that this example is not studied in [5], and Theorem 1.1 can be seen as an improvement of Theorem 1 of [5].

(iii) As it will be clear in the proof below, an upper bound of the maximal time of existence is given by

$$T^*=-m+2\int_{y_0}^{\infty}\frac{\mathrm{d}s}{f(s)}, \tag{1.9}$$

where m is as in (1.5) and $y_0=\mathrm{e}^{m\lambda_1}\int_\Omega u_0\phi_1$.

(iv) The conclusion of Theorem 1.1 remains valid for $\Omega=\mathbb{R}^N$ if we replace ϕ_1 by $\varphi(x)=\pi^{-N/2}\mathrm{e}^{-|x|^2}$.

In order to state our next result (again for $p=2$), we introduce the energy functional

$$E(u(t)):=\frac{1}{2}\int_\Omega|\nabla u(t,x)|^2\mathrm{d}x-\zeta(t)\int_\Omega F(u(t,x))\mathrm{d}x. \tag{1.10}$$

Using (1.7), we see that $t\longmapsto E(u(t))$ is nonincreasing along any solution of (1.1). This leads to the following.

Theorem 1.2. *Suppose that assumptions* (1.6)-(1.7) *are fulfilled. Assume that either* $E(u_0)\le 0$ *or* $E(u_0)>0$ *and* $\|u_0\|_2$ *is large enough. Then the corresponding solution* $u(t,x)$ *blows up in finite time.*

Remark 1.2. An upper bound for the blow-up time is given by

$$T^*=\begin{cases}\dfrac{(2+\epsilon)|\Omega|^{\alpha/2-1}\|u_0\|_2^{2-\alpha}}{\epsilon\zeta(0)C_0(\alpha-2)} & \text{if } E(u_0)\le 0,\\[2ex] \displaystyle\int_{\|u_0\|_2^2/2}^{\infty}\frac{\mathrm{d}z}{Az^{\alpha/2}-2E(u_0)} & \text{if } E(u_0)>0,\end{cases} \tag{1.11}$$

where

$$A=\frac{2^{\alpha/2}C_0\epsilon\zeta(0)}{(2+\epsilon)|\Omega|^{\alpha/2-1}}.$$

We turn now to the case $p>2$. In [16], the author studied (1.1) when $\zeta(t)\equiv 1$. He established:

- local existence when $f\in C^1(\mathbb{R})$;
- global existence when $uf(u)\lesssim |u|^q$ for some $q<p$;
- nonglobal existence under the condition

$$\frac{1}{p}\int_\Omega |\nabla u_0)|^p \mathrm{d}x - \int_\Omega F(u_0)\mathrm{d}x < 0. \tag{1.12}$$

Later on Messaoudi [10] improved the condition (1.12) by showing that blow-up can be obtained for vanishing initial energy. Note that by adapting the arguments used in [16], we can show a local existence result as stated below.

Theorem 1.3. *Suppose $\zeta\in C([0,\infty])$ and $f\in C(\mathbb{R})$ satisfy $|f|\le g$ for some C^1-function g. Then for any $u_0\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$, the problem* (1.1) *has a local solution*

$$u\in L^\infty((0,T)\times\Omega)\cap L^p((0,T);W_0^{1,p}(\Omega)),\ u_t\in L^2((0,T)\times\Omega).$$

The energy of a solution u is

$$\mathbf{E}_p(u(t)) = \frac{1}{p}\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x - \zeta(t)\int_\Omega F((u(t,x))\mathrm{d}x. \tag{1.13}$$

We also define the following set of initial data

$$\mathcal{E} = \left\{ u_0\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega);\ u_0\not\equiv 0 \text{ and } \mathbf{E}_p(u_0)\le 0\right\}. \tag{1.14}$$

Our main result concerning $p>2$ ca be stated as follows.

Theorem 1.4. *Suppose that assumption* (1.7) *is fulfilled. Let $f\in C(\mathbb{R})$ satisfy $|f|\le g$ for some C^1-function g and*

$$0\le \kappa F(u)\le uf(u), \quad \kappa>p>2. \tag{1.15}$$

Then for any $u_0\in\mathcal{E}$ the solution $u(t,x)$ of (1.1) *given in Theorem* 1.3 *blows up in finite time.*

Remark 1.3. Theorem 1.4 and its proof are almost the result of [10]. In fact, with ζ satisfying (1.7), it only accelerate the blow-up.

Remark 1.4. Although the proof uses the Poincaré inequality in a crucial way, we believe that a similar result can be obtained for $\Omega=\mathbb{R}^N$. This will be investigated in a forthcoming paper.

We stress that the set $\mathcal{E}$ is non empty as it is shown in the following proposition.

Proposition 1.1. *Suppose that assumption* (1.15) *is fulfilled and $\zeta(0)>0$. Then $\mathcal{E}\neq\emptyset$.*

2 Proofs

This section is devoted to the proof of Theorems 1.1-1.2-1.4 as well as Proposition 1.1.

2.1 Proof of Theorem 1.1

The main idea in the proof is to define a suitable auxiliary function $y(t)$ and obtain a differential inequality leading to the blow-up. Define the function $y(t)$ as

$$y(t)=a(t)\int_\Omega u(t,x)\phi_1(x)\,\mathrm{d}x, \tag{2.1}$$

where

$$a(t)=\mathrm{e}^{\lambda_1(m-\Theta(t))}, \tag{2.2}$$

$$\Theta(t)=\int_0^t (\zeta(s)-1)\,\mathrm{d}s. \tag{2.3}$$

We compute

$$\begin{aligned} y'(t)=&\frac{a'(t)}{a(t)}y(t)-\lambda_1 y(t)+a(t)\zeta(t)\int_\Omega f(u(t,x))\phi_1(x)\,\mathrm{d}x \\ =&-\lambda_1\zeta(t)y(t)+a(t)\zeta(t)\int_\Omega f(u(t,x))\phi_1(x)\,\mathrm{d}x, \end{aligned}$$

where we have used $a'/a-\lambda_1=-\lambda_1\zeta$. By using (1.2) and the fact that $0\le a\le 1$, we easily arrive at

$$y'(t)\ge\zeta(t)\Big(-\lambda_1 y(t)+f(y(t))\Big). \tag{2.4}$$

Since f is convex and due to (1.4), there exists a constant $C\ge\lambda$ such that $f(s)\ge 2\lambda_1 s$ for all $s\ge C$. Suppose $y(0)>C$. It follows from (2.4) that, as long as u exists, $y(t)\ge C$. Therefore

$$y(t)\ge\frac{\zeta(t)}{2}f(y(t)).$$

Hence

$$\frac{t+m}{2}\le\frac{1}{2}\int_0^t\zeta(s)\,\mathrm{d}s\le\int_{y(0)}^\infty\frac{\mathrm{d}s}{f(s)}<\infty.$$

This means that the solution u cannot exist globally and leads to the upper bound given by (1.9). $\square$

2.2 Proof of Theorem 1.2

Let $y(t)$ be the auxiliary function defined as follows

$$y(t)=\frac{1}{2}\int_\Omega u^2(t,x)\,\mathrm{d}x.$$

We compute

$$\begin{aligned} y'(t)&=\int_\Omega u\,(\Delta u+\zeta(t)f(u))\,\mathrm{d}x\\ &=-\int_\Omega |\nabla u|^2\mathrm{d}x+\zeta(t)\int_\Omega uf(u)\,\mathrm{d}x\\ &=-2E(u(t))+\zeta(t)\int_\Omega (uf(u)-F(u))\,\mathrm{d}x, \end{aligned}$$

where $E(u(t))$ is given by (1.13). Taking advantage of (1.6), we obtain that

$$y'(t)\geq -2E(u(t))+\frac{\epsilon C_0}{2+\epsilon}\zeta(t)\int_\Omega |u|^\alpha\,\mathrm{d}x. \tag{2.5}$$

Moreover, we compute

$$E'(u(t))=-\int_\Omega u_t^2\,\mathrm{d}x-\zeta'(t)\int_\Omega F(u)\,\mathrm{d}x\leq 0, \tag{2.6}$$

thanks to (1.7). It then follows that $E(u(t))$ is non-decreasing in t so that we have

$$E(u(t))\leq E(u(0))=E(u_0),\quad t\geq 0. \tag{2.7}$$

From (2.5), (2.7), and the Hölder inequality, we find that

$$y'(t)\geq -2E(u_0)+\frac{\epsilon\zeta(0)C_0 2^{\alpha/2}}{(2+\epsilon)|\Omega|^{\alpha/2-1}}y(t)^{\alpha/2}. \tag{2.8}$$

To conclude the proof we use the following result.

Lemma 2.1. *Let $y:[0,T)\to[0,\infty)$ be a C^1-function satisfying*

$$y'(t)\geq -C_1+C_2 y(t)^q, \tag{2.9}$$

for some constants $C_1\in\mathbb{R}, C_2>0, q>1$. Then

$$T\leq\begin{cases} \frac{y^{1-q}(0)}{C_2(q-1)} & \textit{if } C_1\leq 0,\\[2ex] \displaystyle\int_{y(0)}^\infty \frac{dz}{C_2 z^q-C_1} & \textit{if } C_1>0 \textit{ and } y(0)>\left(\frac{C_1}{C_2}\right)^{1/q}. \end{cases} \tag{2.10}$$

Proof of Lemma 2.1. We give the proof here for completeness. If $C_1 \leq 0$ then $y'(t) \geq C_2 y(t)^q$. It follows that

$$y' y^{-q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y^{1-q}}{1-q}\right) \geq C_2.$$

Integrating this differential inequality yields the desired upper bound in this case.

Suppose now that $C_1 > 0$ and $y(0) > \left(\frac{C_1}{C_2}\right)^{1/q}$. Then $y(t) > \left(\frac{C_1}{C_2}\right)^{1/q}$ for all $0 \leq t < T$. Therefore

$$\frac{y'(t)}{C_2 y(t)^q - C_1} \geq 1, \quad 0 \leq t < T.$$

Integrating this differential inequality yields

$$t \leq \int_0^t \frac{y'(\tau)\mathrm{d}\tau}{C_2 y(\tau)^q - C_1} \leq \int_{y(0)}^{\infty} \frac{\mathrm{d}z}{C_2 z^q - C_1} < \infty.$$

This finishes the proof of Lemma 2.1. □

2.3 Proof of Theorem 1.4

We define

$$H(t) = \zeta(t) \int_\Omega F((u(t,x))\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x, \tag{2.11}$$

and

$$L(t) = \frac{1}{2}\|u(t)\|_2^2. \tag{2.12}$$

By using (1.1), we obtain that

$$\begin{aligned} H'(t) &= \int_\Omega u_t^2(t,x))\mathrm{d}x + \zeta'(t)\int_\Omega F(u(t,x))\mathrm{d}x \\ &= \frac{\zeta'(t)}{\zeta(t)} H(t) + \int_\Omega u_t^2(t,x))\mathrm{d}x + \frac{\zeta'(t)}{p\zeta(t)}\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x \\ &\geq \frac{\zeta'(t)}{\zeta(t)} H(t). \end{aligned}$$

Hence $H(t) \geq H(0) \geq 0$, by virtue of (1.7).

Recalling (1.1), (2.11), and (1.15), we compute

$$\begin{aligned} L'(t) &= -\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x + \zeta(t)\int_\Omega u(t,x) f(u(t,x))\mathrm{d}x \\ &\geq -\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x + \kappa\zeta(t)\int_\Omega F(u(t,x))\mathrm{d}x \\ &\geq \kappa H(t) + \left(\frac{\kappa}{p} - 1\right)\int_\Omega |\nabla u(t,x)|^p \mathrm{d}x \end{aligned}$$

$$\geq \left(\frac{\kappa}{p}-1\right)\int_\Omega |\nabla u(t,x)|^p\,\mathrm{d}x.$$

Applying Hölder inequality and then Poincaré inequality yields

$$L(t)\leq |\Omega|^{1-2/p}\left(\int_\Omega |u(t,x)|^p\,\mathrm{d}x\right)^{2/p}\leq C\left(\int_\Omega |\nabla u(t,x)|^p\,\mathrm{d}x\right)^{2/p},$$

where $C>0$ is a constant depending only on Ω and p. Hence

$$L'(t)\geq \frac{\kappa-p}{pC^{p/2}}L^{p/2}(t). \tag{2.13}$$

Integrating the differential inequality (2.13) leads to

$$t\leq \frac{2pC^{p/2}L^{1-p/2}(0)}{(p-2)(\kappa-p)}<\infty.$$

Therefore u blows up at a finite time $T^*\leq \frac{2pC^{p/2}L^{1-p/2}(0)}{(p-2)(\kappa-p)}$. □

2.4 Proof of Proposition 1.1

Recalling (1.15), we obtain that

$$F(u)\geq Cu^\kappa \text{ for all } u\geq 1 \tag{2.14}$$

for some constant $C>0$. Let $K\subset\Omega$ be a compact nonempty subset of Ω. Fix a smooth cut-of function $\phi\in C^\infty(\Omega)$ such that

$$\phi(x)=1 \quad \text{for } x\in K.$$

We look for initial data $u_0=\lambda\phi$ where $\lambda>0$ to be chosen later. Clearly $u_0\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$, and for $\lambda\geq 1$ we have using (2.14)

$$\begin{aligned}\mathbf{E}_p(u_0)&=\frac{1}{p}\int_\Omega |\nabla u_0|^p-\zeta(0)\int_\Omega F(u_0),\\ &=\frac{\lambda^p}{p}\int_\Omega |\nabla\phi|^p-\zeta(0)\int_K F(\lambda)-\zeta(0)\int_{\Omega\setminus K}F(u_0),\\ &\leq\frac{\lambda^p}{p}\int_\Omega|\nabla\phi|^p-\tilde{C}\lambda^\kappa,\end{aligned}$$

for some constant $\tilde{C}>0$. Since

$$\frac{\lambda^p}{p}\int_\Omega|\nabla\phi|^p-\tilde{C}\lambda^\kappa\leq 0 \text{ for } \lambda\geq\left(\frac{\|\nabla\phi\|_p^p}{p\tilde{C}}\right)^{1/(\kappa-p)},$$

we deduce that $u_0\in\mathcal{E}$ for λ large enough. This finishes the proof of Proposition 1.1. □

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