COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
29(1)(2013), 1–13

Existence of Positive Solutions for Higher Order Boundary Value Problem on Time Scales*

XIE DA-PENG[1], LIU YANG[1] AND SUN MING-ZHE[2]
(*1. Department of Mathematics, Hefei Normal University, Hefei, 230061*)
(*2. Department of Mathematics, College of Science, Yanbian University, Yanji, Jilin, 133002*)

Communicated by Li Yong

Abstract: In this paper, we investigate the existence of positive solutions of a class higher order boundary value problems on time scales. The class of boundary value problems educes a four-point (or three-point or two-point) boundary value problems, for which some similar results are established. Our approach relies on the Krasnosel'skii fixed point theorem. The result of this paper is new and extends previously known results.
Key words: higher order boundary value problem, positive solution, semipositone, on time scale, fixed point
2000 MR subject classification: 34B18
Document code: A
Article ID: 1674-5647(2013)01-0001-13

1 Introduction

In this paper, we study the existence of positive solutions of higher order boundary value problem (BVP) on time scales as follows:

$$x^{(n)}(t) + f(t, x(t)) = 0, \qquad a < t < b, \tag{1.1}$$

$$x(a) = h\Big(\int_a^b x(t)\mathrm{d}\varphi(t)\Big), x'(a) = 0, \cdots, x^{(n-2)}(a) = 0, x(b) = g\Big(\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})\Big), \tag{1.2}$$

where $\int_a^b x(t)\mathrm{d}\varphi(\mathrm{t})$, $\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})$ denote the Riemann-Stieltjes integrals.

We assume that

(H_1) φ and ϕ are increasing nonconstant functions defined on $[a, b]$ with $\varphi(a) = \phi(a) = 0$;

*Received date: Jan. 8, 2010.
Foundation item: The NSF (11201109) of China , the NSF (10040606Q50) of Anhui Province, Excellent Talents Foundation (2012SQRL165) of University of Anhui Province and the NSF (2012kj09) of Heifei Normal University.

(H_2) f is continuous and there exists $M \geq 0$ such that

$$f : [a,b] \times \Big[-\frac{(b-a)^n (n-1)^{n-1}}{n^n n!} M, +\infty \Big) \to [-M, +\infty);$$

(H_3) h and g are continuous and there exists $M \geq 0$ such that

$$h,\ g : \Big[-\frac{(b-a)^n (n-1)^{n-1}}{n^n n!} M, +\infty \Big) \to [0, +\infty).$$

Lemma 1.1 [1] *Assume that*

(1) *$x(t)$ is a bounded function on $[a,b]$, i.e., there exist c, $C \in \mathbf{R}$ such that*

$$c \leq x(t) \leq C, \qquad t \in [a,b];$$

(2) *$\varphi(t)$ and $\phi(t)$ are increasing on $[a,b]$;*

(3) *Riemann-Stieltjes integrals $\int_a^b x(t)\mathrm{d}\varphi(\mathrm{t})$ and $\int_a^b x(t)\mathrm{d}\phi(\mathrm{t})$ exist.*

Then there are numbers v_1, $v_2 \in \mathbf{R}$ with $c \leq v_1$, $v_2 \leq C$, such that

$$\int_a^b x(t)\mathrm{d}\varphi(t) = v_1(\varphi(b) - \varphi(a)),$$

$$\int_a^b x(t)\mathrm{d}\phi(t) = v_2(\phi(b) - \phi(a)).$$

Let $\alpha = \varphi(b)$ and $\beta = \phi(b)$. For any continuous solution $x(t)$ of the BVP (1.1)-(1.2), by Lemma 1.1, there exist ξ, $\eta \in (a,b)$ such that

$$\int_a^b x(t)\mathrm{d}\varphi(t) = x(\xi)(\varphi(b) - \varphi(a)) = x(\xi)\varphi(b) = \alpha x(\xi),$$

$$\int_a^b x(t)\mathrm{d}\phi(t) = x(\eta)(\phi(b) - \phi(a)) = x(\eta)\phi(b) = \beta x(\eta).$$

If

$$h(t) = g(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!} M,\ 0 \Big]$$

and

$$h(t) = g(t) = t, \qquad t \in [0, +\infty),$$

then (1.2) reduces to

$$x(a) = \alpha x(\xi), \quad x'(a) = 0, \quad \cdots, \quad x^{(n-2)}(a) = 0, \quad x(b) = \beta x(\eta). \tag{1.3}$$

The existence of positive solutions of the BVP (1.1)-(1.3) has been studied by several authors when $a = 0$, $b = 1$ and $n = 2$ (see [2–4]).

If

$$h(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!} M,\ +\infty \Big],$$

$$g(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!} M,\ 0 \Big]$$

and

$$g(t) = t, \qquad t \in [0, +\infty),$$

then (1.2) reduces to

$$x(a) = 0, \quad x'(a) = 0, \quad \cdots, \quad x^{(n-2)}(a) = 0, \quad x(b) = \beta x(\eta). \tag{1.4}$$

The existence of positive solutions of the BVP (1.1)-(1.4) has been studied when $a = 0$, $b = 1$ in [5–6].

If

$$h(t) = g(t) = 0, \qquad t \in \Big[-\frac{(b-a)^{n-1}(n-1)^{n-1}}{n^n n!}M,\ +\infty\Big],$$

then (1.2) reduces to

$$x(a) = 0, \quad x'(a) = 0, \quad \cdots, \quad x^{(n-2)}(a) = 0, \quad x(b) = 0. \tag{1.5}$$

The existence of positive solutions of the BVP (1.1)-(1.5) has been studied when $a = 0$, $b = 1$ in [7–8].

In fact, (H_2) implies that f is not necessarily nonnegative, monotone, superlinear and sublinear. And also this assumption implies that the BVP (1.1)-(1.2) is semipositive. The purpose of this paper is to establish the existence of positive solutions of the BVP (1.1)-(1.2) by using Krasnosel'skii fixed point theorem in cones.

2 Related Lemmas

Lemma 2.1 *If $y(t) \in C[a, b]$, then the boundary value problem*

$$\begin{cases} x^{(n)}(t) + y(t) = 0, \qquad a < t < b, \\ x(a) = h\Big(\displaystyle\int_a^b x(t)\mathrm{d}\varphi(t)\Big),\ x'(a) = 0,\ \cdots,\ x^{(n-2)}(a) = 0, \\ x(b) = g\Big(\displaystyle\int_a^b x(t)\mathrm{d}\phi(t)\Big) \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \int_a^b G(t,s)y(s)\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b x(s)\mathrm{d}\phi(s)\Big) + \Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big] h\Big(\int_a^b x(s)\mathrm{d}\varphi(s)\Big),$$

where

$$G(t,s) = \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1} - (b-a)^{n-1}(t-s)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le s \le t \le b; \\ \dfrac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le t < s \le b. \end{cases} \tag{2.2}$$

Proof. In fact, if $x(t)$ is a solution of the problem (2.1), then we may suppose that

$$x(t) = -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\mathrm{d}s + A(t-a)^{n-1} + \sum_{i=1}^{n-2} A_i (t-a)^i + B.$$

Since

$$x^{(i)}(a) = 0, \qquad i = 1, 2, \cdots, n-2,$$

we get

$$A_i = 0, \qquad i = 1, 2, \cdots, n-2.$$

It follows from

$$x(a) = h\Big(\int_a^b x(t)\mathrm{d}\varphi(t)\Big)$$

that

$$B = h\Big(\int_a^b x(t)\mathrm{d}\varphi(t)\Big).$$

Using this and

$$x(b) = g\Big(\int_a^b x(t)\mathrm{d}\phi(t)\Big),$$

we have

$$A = \frac{1}{(b-a)^{n-1}}\Big[\int_a^b \frac{(b-s)^{n-1}}{(n-1)!}y(s)\mathrm{d}s + g\Big(\int_a^b x(t)\mathrm{d}\phi(t)\Big) - h\Big(\int_a^b x(t)\mathrm{d}\varphi(t)\Big)\Big].$$

Therefore, the problem (2.1) has a unique solution

$$\begin{aligned} x(t) = & -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)\mathrm{d}s + \int_a^b \frac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!}y(s)\mathrm{d}s \\ & + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b x(s)\mathrm{d}\phi(s)\Big) + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b x(s)\mathrm{d}\phi(s)\Big) \\ & + \Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b x(s)\mathrm{d}\varphi(s)\Big) \\ = & \int_a^b G(t,s)y(s)\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b x(s)\mathrm{d}\phi(s)\Big) \\ & + \Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b x(s)\mathrm{d}\varphi(s)\Big), \end{aligned}$$

where $G(t,s)$ is defined by (2.2).

Lemma 2.2 *$G(t,s)$ has the following properties*:

(i) $0 \le G(t,s) \le k(s)$ *for* $t, s \in [a,b]$, *where*

$$k(s) = \frac{(s-a)(b-s)^{n-1}}{(b-a)(n-2)!};$$

(ii) $G(t,s) \ge \gamma(t)k(s)$ *for* $t, s \in [a,b]$, *where*

$$\gamma(t) = \begin{cases} \dfrac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}}, & a \le t \le \dfrac{a+b}{2}; \\ \dfrac{(b-t)(t-a)^{n-2}}{(n-1)(b-a)^{n-1}}, & \dfrac{a+b}{2} \le t \le b. \end{cases}$$

Proof. It is obvious that $G(t,s)$ is nonnegative. Moreover,

$$G(t,s) = \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1} - (b-a)^{n-1}(t-s)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le s \le t \le b; \\ \dfrac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le t < s \le b, \end{cases}$$

$$
\begin{aligned}
&= \frac{1}{(b-a)^{n-1}(n-1)!} \cdot \begin{cases} (s-a)(b-t) & \\ \quad \cdot\{[(t-a)(b-s)]^{n-2} & \\ \quad +[(t-a)(b-s)]^{n-3}(b-a)(t-s)+\cdots & \\ \quad +(t-a)(b-s)[(b-a)(t-s)]^{n-3} & \\ \quad +[(b-a)(t-s)]^{n-2}\}, & a \le s \le t \le b; \\ (b-s)^{n-1}(t-a)^{n-1}, & a \le t < s \le b, \end{cases} \\
&\le \frac{1}{(b-a)^{n-1}(n-1)!} \cdot \begin{cases} (n-1)(s-a)(b-s)[(b-a)(b-s)]^{n-2}, & a \le s \le t \le b; \\ (b-s)^{n-1}(s-a)^{n-1}, & a \le t < s \le b, \end{cases} \\
&\le \frac{(s-a)(b-s)^{n-1}}{(b-a)(n-2)!} \\
&= k(s).
\end{aligned}
$$

So, (i) holds.

It is clear that (ii) holds for $s=a$ or $s=b$. For $s \in (a,b)$ and $t \in [a,b]$, we have

$$
\begin{aligned}
\frac{G(t,s)}{k(s)} &= \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1}-(b-a)^{n-1}(t-s)^{n-1}}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}, & a \le s \le t \le b; \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b \end{cases} \\
&= \begin{cases} \dfrac{1}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}(s-a)(b-t) & \\ \quad \cdot\{[(t-a)(b-s)]^{n-2} & \\ \quad\quad +[(t-a)(b-s)]^{n-3}(b-a)(t-s)+\cdots & \\ \quad\quad +(t-a)(b-s)[(b-a)(t-s)]^{n-3} & \\ \quad\quad +[(b-a)(t-s)]^{n-2}\}, & a \le s \le t \le b; \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b \end{cases} \\
&\ge \begin{cases} \dfrac{(s-a)(b-t)(t-a)^{n-2}(b-s)^{n-2}}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}, & a \le s \le t \le b; \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b \end{cases} \\
&\ge \begin{cases} \dfrac{(b-t)(t-a)^{n-2}}{(n-1)(b-a)^{n-1}}, & a \le s \le t \le b; \\ \dfrac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}}, & a \le t < s \le b \end{cases} \\
&\ge \frac{1}{(n-1)(b-a)^{n-1}} \cdot \begin{cases} (b-t)(t-a)^{n-2}, & a \le s \le t \le b; \\ (t-a)^{n-1}, & a \le t < s \le b. \end{cases}
\end{aligned}
$$

Since

$$(b-t)(t-a)^{n-2} \ge (t-a)^{n-1}, \qquad t \in \Big[a,\, \frac{a+b}{2}\Big]$$

and

$$(t-a)^{n-1} \ge (b-t)(t-a)^{n-2}, \qquad t \in \Big[\frac{a+b}{2},\, b\Big],$$

we obtain

$$\frac{G(t,s)}{k(s)} \geq \gamma(t), \qquad s \in (a,b),\ t \in [a,b].$$

Thus, (ii) holds. The proof is completed.

Remark 2.1 By simple computations, we get

$$\int_a^b k(s)\mathrm{d}s = \frac{1}{(b-a)(n-2)!}\int_a^b (s-a)(b-s)^{n-1}\mathrm{d}s = \frac{(b-a)^n}{n(n+1)(n-2)!},$$

$$\begin{aligned}\int_{\frac{(n-1)b+a}{n}}^{\frac{nb+a}{n+1}} k(s)\mathrm{d}s &= \frac{1}{(b-a)(n-2)!}\int_{\frac{(n-1)b+a}{n}}^{\frac{nb+a}{n+1}} (s-a)(b-s)^{n-1}\mathrm{d}s \\ &= \frac{[(n-1)(n+1)^{n+2}-(n+1)n^{n+2}-n^{n+1}+(n+1)^{n+1}](b-a)^n}{n^{n+2}(n+1)^{n+2}(n-2)!},\end{aligned}$$

and

$$\min_{a\leq t\leq b}\gamma(t) = 0,$$

$$\max_{a\leq t\leq b}\gamma(t) = \begin{cases} \dfrac{1}{2}, & n=2; \\ \dfrac{1}{8}, & n=3; \\ \max\left\{\dfrac{1}{(n-1)2^{n-1}}, \dfrac{(n-2)^{n-2}}{(n-1)^n}\right\}, & n\geq 4. \end{cases}$$

For $n \geq 4$, by simple computations, we have

$$\gamma'(t) = \frac{(t-a)^{n-3}[-(t-a)+(n-2)(b-t)]}{(b-a)^{n-1}(n-1)}, \qquad \frac{b+a}{2} \leq t \leq b.$$

From this, we obtain

$$\min_{\frac{(n-1)b+a}{n}\leq t\leq\frac{nb+a}{n+1}}\gamma(t) = \gamma\Big(\frac{nb+a}{n+1}\Big) = \frac{n^{n-2}}{(n-1)(n+1)^{n-1}}, \qquad n\geq 4.$$

Thus,

$$\begin{aligned}\min_{\frac{(n-1)b+a}{n}\leq t\leq\frac{nb+a}{n+1}}\gamma(t) &= \begin{cases} \dfrac{1}{3}, & n=2; \\ \dfrac{3}{32}, & n=3; \\ \dfrac{n^{n-2}}{(n-1)(n+1)^{n-1}}, & n\geq 4 \end{cases} \\ &= \frac{n^{n-2}}{(n-1)(n+1)^{n-1}}, \qquad n\geq 2.\end{aligned}$$

Lemma 2.3 *Let*

$$p(t) := \int_a^b G(t,s)\mathrm{d}s = \frac{(b-t)(t-a)^{n-1}}{n!}, \qquad a\leq t\leq b.$$

Then

(i) $\displaystyle\max_{a\leq t\leq b} p(t) = \frac{(b-a)^n(n-1)^{n-1}}{n^n n!}$, $\displaystyle\min_{a\leq t\leq b} p(t) = 0$;

(ii) $\displaystyle\max_{\frac{(n-1)b+a}{n}\leq t\leq\frac{nb+a}{n+1}} p(t) = \frac{(b-a)^n(n-1)^{n-1}}{n^n n!}$, $\displaystyle\min_{\frac{(n-1)b+a}{n}\leq t\leq\frac{nb+a}{n+1}} p(t) = \frac{n^{n-1}(b-a)^n}{(n+1)^n n!}$;

(iii) $\dfrac{(n-1)(b-a)^n}{2\cdot n!}\gamma(t) \le p(t) \le \dfrac{(n-1)(b-a)^n}{n!}\gamma(t),\ a\le t\le b.$

Proof. Since

$$p'(t) = \frac{(t-a)^{n-2}[(b-t)n-(b-a)]}{n!}, \qquad a \le t \le b,$$

we have

$$\max_{a\le t\le b} p(t) = p\Big(\frac{(n-1)b+a}{n}\Big) = \frac{(b-a)^n(n-1)^{n-1}}{n^n n!},$$
$$\min_{a\le t\le b} p(t) = p(a) = p(b) = 0,$$

and

$$\max_{\frac{(n-1)b+a}{n}\le t\le \frac{nb+a}{n+1}} p(t) = p\Big(\frac{(n-1)b+a}{n}\Big) = \frac{(b-a)^n(n-1)^{n-1}}{n^n n!},$$
$$\min_{\frac{(n-1)b+a}{n}\le t\le \frac{nb+a}{n+1}} p(t) = p\Big(\frac{nb+a}{n+1}\Big) = \frac{n^{n-1}(b-a)^n}{(n+1)^n n!}.$$

Hence, (i) and (ii) hold.

For $a \le t \le \dfrac{a+b}{2}$, we have

$$\begin{aligned} p(t) &= \frac{(b-t)(t-a)^{n-1}}{n!} \\ &\le \frac{(b-a)(t-a)^{n-1}}{n!} \\ &= \frac{(b-a)(n-1)(b-a)^{n-1}}{n!}\cdot\frac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}} \\ &= \frac{(n-1)(b-a)^n}{n!}\gamma(t) \end{aligned}$$

and

$$\begin{aligned} p(t) &= \frac{(b-t)(t-a)^{n-1}}{n!} \\ &\ge \frac{b-a}{2}\cdot\frac{(t-a)^{n-1}}{n!} \\ &= \frac{(b-a)(n-1)(b-a)^{n-1}}{2\cdot n!}\cdot\frac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}} \\ &= \frac{(n-1)(b-a)^n}{2\cdot n!}\gamma(t). \end{aligned}$$

Thus,

$$\frac{(n-1)(b-a)^n}{2\cdot n!}\gamma(t) \le p(t) \le \frac{(n-1)(b-a)^n}{n!}\gamma(t), \qquad a\le t\le \frac{a+b}{2}. \tag{2.3}$$

For $\dfrac{a+b}{2} \le t \le b$, we obtain

$$\begin{aligned} p(t) &= \frac{(t-a)^{n-1}(b-t)}{n!} \\ &= \frac{(t-a)(t-a)^{n-2}(b-t)}{n!} \\ &\le \frac{(b-a)(t-a)^{n-2}(b-t)}{n!} \end{aligned}$$

$$\begin{aligned}
&= \frac{(b-a)(n-1)(b-a)^{n-1}}{n!} \cdot \frac{(t-a)^{n-2}(b-t)}{(n-1)(b-a)^{n-1}} \\
&= \frac{(n-1)(b-a)^n}{n!}\gamma(t)
\end{aligned}$$

and

$$\begin{aligned}
p(t) &= \frac{(t-a)^{n-1}(b-t)}{n!} \\
&= \frac{(t-a)(t-a)^{n-2}(b-t)}{n!} \\
&\geq \frac{b-a}{2} \cdot \frac{(t-a)^{n-2}(b-t)}{n!} \\
&= \frac{(b-a)(n-1)(b-a)^{n-1}}{2 \cdot n!} \cdot \frac{(t-a)^{n-2}(b-t)}{(n-1)(b-a)^{n-1}} \\
&= \frac{(n-1)(b-a)^n}{2 \cdot n!}\gamma(t).
\end{aligned}$$

Hence,

$$\frac{(n-1)(b-a)^n}{2 \cdot n!}\gamma(t) \leq p(t) \leq \frac{(n-1)(b-a)^n}{n!}\gamma(t), \qquad \frac{a+b}{2} \leq t \leq b. \tag{2.4}$$

It follows from (2.3) and (2.4) that (iii) holds.

Lemma 2.4 [9] *Let $E = (E, \|\cdot\|)$ be a Banach space and $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be continuous and completely continuous. In addition, suppose either*

(1) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$, and $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$,

or

(2) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, and $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main Result

Let $E = C[a, b]$ be a real Banach space with the maximum norm, and define the cone $P \subset E$ by

$$P = \{x \in E : x(t) \geq \gamma(t)\|x\|, \ t \in [a, b]\},$$

where $\gamma(t)$ is as in Lemma 2.2. For brevity, we denote

$$\Omega_i = \{x \in E : \|x\| < r_i\}, \qquad i = 1, 2,$$

$$\alpha(r) = \max\Big\{f(t, x) : \ a \leq t \leq b, \ -\frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M \leq x \leq r\Big\}, \tag{3.1}$$

$$\begin{aligned}
\beta(r) = \min\Big\{f(t, x) : \ &\frac{(n-1)b+a}{n} \leq t \leq \frac{nb+a}{n+1}, \ \frac{n^{n-2}}{(n+1)^{n-1}(n-1)}r \\
&- \frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M \leq x \leq r - \frac{n^{n-1}(b-a)^n}{(n+1)^n n!}M\Big\},
\end{aligned} \tag{3.2}$$

$$C_1 = \int_a^b k(s)\mathrm{d}s = \frac{(b-a)^n}{n(n+1)(n-2)!}, \tag{3.3}$$

$$C_2 = \left\{ \max_{a \le t \le b} \gamma(t) \int_{\frac{(n-1)b+a}{n}}^{\frac{nb+a}{n+1}} k(s)\mathrm{d}s \right\}^{-1}, \tag{3.4}$$

where $\max\limits_{a \le t \le b} \gamma(t)$ and $\int_{\frac{(n-1)b+a}{n}}^{\frac{nb+a}{n+1}} k(s)\mathrm{d}s$ are as in Remark 2.1.

Theorem 3.1 *Suppose that* (H_1)–(H_3) *hold. Assume that there exist three positive numbers* r_1, r_2 *and* C_3 *with* $r_1 > r_2 > \max\left\{ \frac{(n-1)(b-a)^n}{n!} M, \ \frac{g^* + h^*}{1 - C_3 C_1} \right\}$ *and* $C_3 < C_1^{-1}$ *such that*

$$\alpha(r_1) \le r_1 C_3 - M, \qquad \beta(r_2) \ge r_2 C_2 - M,$$

where α, β, C_1, C_2 *are as in* (3.1)–(3.4), *and*

$$g^* = \max\left\{ g(t) : -\frac{(b-a)^n (n-1)^{(n-1)}}{n^n n!} M\phi(b) \le t \le r_1 \phi(b) \right\}, \tag{3.5}$$

$$h^* = \max\left\{ h(t) : -\frac{(b-a)^n (n-1)^{(n-1)}}{n^n n!} M\varphi(b) \le t \le r_1 \varphi(b) \right\}. \tag{3.6}$$

Then the BVP (1.1)-(1.2) *has at least one positive solution.*

Proof. Let

$$x_0(t) = Mp(t), \qquad a \le t \le b.$$

Then by Lemma 2.3 (i) and (ii), we have

$$0 \le x_0(t) \le \frac{(b-a)^n (n-1)^{n-1}}{n^n n!} M, \qquad a \le t \le b, \tag{3.7}$$

and

$$\frac{n^{n-1}(b-a)^n}{n!(n+1)^n} M \le x_0(t) \le \frac{(b-a)^n (n-1)^{n-1}}{n^n n!} M, \qquad \frac{(n-1)b+a}{n} \le t \le \frac{nb+a}{n+1}. \tag{3.8}$$

Consider the following boundary value problem:

$$\begin{cases} x^{(n)}(t) + f(t, x(t) - x_0(t)) + M = 0, \qquad a < t < b, \\ x(a) = h\Big(\displaystyle\int_a^b (x(t) - x_0(t))\mathrm{d}\varphi(t) \Big), \ x'(a) = 0, \ \cdots, \ x^{(n-2)}(a) = 0, \\ x(b) = g\Big(\displaystyle\int_a^b (x(t) - x_0(t))\mathrm{d}\phi(t) \Big). \end{cases} \tag{3.9}$$

This problem is equivalent to the integral equation

$$\begin{aligned} x(t) = &\int_a^b G(t,s)[f(s,\, x(s) - x_0(s)) + M]\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\phi(s) \Big) \\ &+ \Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}} \Big] h\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\varphi(s) \Big). \end{aligned}$$

We define the operator A as follows:

$$\begin{aligned} (Ax)(t) = &\int_a^b G(t,s)[f(s,\, x(s) - x_0(s)) + M]\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\phi(s) \Big) \\ &+ \Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}} \Big] h\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\varphi(s) \Big). \end{aligned} \tag{3.10}$$

We claim that $A(P) \subset P$. In fact, for each $x \in P$ and $t \in [a,b]$, by Lemma 2.2(i) we have

$$\begin{aligned}(Ax)(t) = &\int_a^b G(t,s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big] h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ \le &\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+ h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\|Ax\| \le &\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+ h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big).\end{aligned} \tag{3.11}$$

On the other hand, for any $t \in [a,b]$, by (3.11) and Lemma 2.2(ii), we have

$$\begin{aligned}(Ax)(t) = &\int_a^b G(t,s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big] h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ \ge &\,\gamma(t)\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s\\ &+\frac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}\}} g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\Big(1-\frac{t-a}{b-a}\Big)\Big[1+\frac{t-a}{b-a}+\cdots+\frac{(t-a)^{n-3}}{(b-a)^{n-3}}+\frac{(t-a)^{n-2}}{(b-a)^{n-2}}\Big]\\ &\cdot h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ \ge &\,\gamma(t)\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + \gamma(t) g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\Big(1-\frac{t-a}{b-a}\Big)\frac{(t-a)^{n-2}}{(b-a)^{n-2}} h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ = &\,\gamma(t)\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + \gamma(t) g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\frac{(b-t)(t-a)^{n-2}}{(b-a)^{n-1}} h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ \ge &\,\gamma(t)\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + \gamma(t) g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\\ &+\frac{(b-t)(t-a)^{n-2}}{(n-1)(b-a)^{n-1}} h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\\ \ge &\,\gamma(t)\Big\{\int_a^b k(s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s + g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big)\end{aligned}$$

$$+ h\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\varphi(s)\Big)\Big\}$$
$$\geq \gamma(t)\|Ax\|.$$

This implies that $A(P) \subset P$. Similar to the proof of Remark 2.1 in [2], it is easy to prove that the operator $A : P \cap (\bar{\Omega}_1 \setminus \Omega_2) \to P$ is continuous and compact.

If $x \in P \cap \partial\Omega_1$, we obtain

$$0 = \min_{a\leq t\leq b} \gamma(t)r_1 \leq \min_{a\leq t\leq b} x(t) \leq r_1. \tag{3.12}$$

Then, by (3.7) and (3.12), we have

$$-\frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M \leq x(t) - x_0(t) \leq r_1, \qquad a \leq t \leq b. \tag{3.13}$$

Thus,

$$f(t, x(t) - x_0(t)) \leq \alpha(r_1) \leq r_1 C_3 - M, \qquad a \leq t \leq b, \tag{3.14}$$

$$-\frac{(b-a)^n(n-1)^{(n-1)}}{n^n n!}M\phi(b) \leq \int_a^b (x(t) - x_0(t))\mathrm{d}\phi(t) \leq r_1\phi(b), \tag{3.15}$$

$$-\frac{(b-a)^n(n-1)^{(n-1)}}{n^n n!}M\varphi(b) \leq \int_a^b (x(t) - x_0(t))\mathrm{d}\varphi(t) \leq r_1\varphi(b), \tag{3.16}$$

which imply

$$g\Big(\int_a^b (x(t) - x_0(t))\mathrm{d}\phi(t)\Big) \leq g^*, \qquad h\Big(\int_a^b (x(t) - x_0(t))\mathrm{d}\varphi(t)\Big) \leq h^*. \tag{3.17}$$

Then by (3.14), (3.17) and Lemma 2.2(i), we have

$$\begin{aligned}
\|Ax\| = &\max_{a\leq t\leq b}\Big\{\int_a^b G(t,s)[f(s,\, x(s) - x_0(s)) + M]\mathrm{d}s \\
&+\frac{(t-a)^{n-1}}{(b-a)^{n-1}} g\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\phi(s)\Big) \\
&+\Big[1 - \frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big] h\Big(\int_a^b (x(s) - x_0(s))\mathrm{d}\varphi(s)\Big)\Big\} \\
\leq &\, r_1 C_3 \int_a^b k(s)\mathrm{d}s + g^* + h^* \\
< &\, r_1 \\
= &\, \|x\|.
\end{aligned}$$

Therefore,

$$\|Ax\| < \|x\|, \qquad x \in P \cap \partial\Omega_1. \tag{3.18}$$

For $x \in P \cap \partial\Omega_2$, we have

$$\frac{n^{n-2}}{(n-1)(n+1)^{n-1}}r_2 = \min_{\frac{(n-1)b+a}{n}\leq t\leq \frac{nb+a}{n+1}} \gamma(t)r_2 \leq \min_{\frac{(n-1)b+a}{n}\leq t\leq \frac{nb+a}{n+1}} x(t) \leq r_2. \tag{3.19}$$

In view of (3.8) and (3.19), for

$$\frac{(n-1)b+a}{n} \leq t \leq \frac{nb+a}{n+1},$$

we have

$$\frac{n^{n-2}}{(n-1)(n+1)^{n-1}}r_2 - \frac{(b-a)^n(n-1)^{n-1}}{n^n n!}M \leq x(t) - x_0(t) \leq r_2 - \frac{n^{n-1}(b-a)^a}{n!(n+1)^n}M, \tag{3.20}$$

which implies

$$f(t,x(t)-x_0(t)) \geq \beta(r_2) \geq r_2C_2 - M, \qquad \frac{(n-1)b+a}{n} \leq t \leq \frac{nb+a}{n+1}. \tag{3.21}$$

Using this and Lemma 2.2(ii), for $x \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|Ax\| = {} & \max_{a\leq t\leq b}\bigg\{\int_a^b G(t,s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s \\
& +\frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\phi(s)\Big) \\
& +\Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b (x(s)-x_0(s))\mathrm{d}\varphi(s)\Big)\bigg\} \\
\geq {} & \max_{a\leq t\leq b}\int_a^b G(t,s)[f(s,\, x(s)-x_0(s))+M]\mathrm{d}s \\
\geq {} & r_2C_2\max_{a\leq t\leq b}\gamma(t)\int_{\frac{(n-1)b+a}{n}}^{\frac{nb+a}{n+1}} k(s)\mathrm{d}s = r_2 = \|x\|.
\end{aligned}$$

Thus,

$$\|Ax\| \geq \|x\|, \qquad x \in P \cap \partial\Omega_2. \tag{3.22}$$

According to Lemma 2.4 and using the inequalities (3.18) and (3.22), we assert that the operator A has at least one fixed point $x^* \in P \cap (\bar{\Omega}_1 \setminus \Omega_2)$. This implies that the BVP (3.9) has at least one solution $x^* \in P$ with $r_2 \leq \|x^*\| \leq r_1$.

Let

$$x_*(t) = x^*(t) - x_0(t), \qquad a \leq t \leq b.$$

We check that x_* is a solution of the BVP (1.1)-(1.2). In fact, since $Ax^* = x^*$, we have

$$\begin{aligned}
x_*(t) + x_0(t) &= x^*(t) \\
&= (Ax^*)(t) \\
&= \int_a^b G(t,s)[f(s,x^*(s)-x_0(s))+M]\mathrm{d}s \\
&\quad + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b (x^*(s)-x_0(s))\mathrm{d}\phi(s)\Big) \\
&\quad + \Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b (x^*(s)-x_0(s))\mathrm{d}\varphi(s)\Big) \\
&= \int_a^b G(t,s)f(s,x_*(s))\mathrm{d}s \\
&\quad + x_0(t) + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b x_*(s)\mathrm{d}\phi(s)\Big) \\
&\quad + \Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b x_*(s)\mathrm{d}\varphi(s)\Big).
\end{aligned}$$

This shows that

$$x_*(t) = \int_a^b G(t,s)f(s,x_*(s))\mathrm{d}s + \frac{(t-a)^{n-1}}{(b-a)^{n-1}}g\Big(\int_a^b x_*(s)\mathrm{d}\phi(s)\Big)$$

$$+\Big[1-\frac{(t-a)^{n-1}}{(b-a)^{n-1}}\Big]h\Big(\int_a^b x_*(s)\mathrm{d}\varphi(s)\Big).$$

In other words, x_* is a solution of the BVP (1.1)-(1.2). Therefore, the BVP (1.1)-(1.2) has at least one solution x_* satisfying $x_*+x_0\in P$ and $r_2\le\|x_*+x_0\|\le r_1$.

Since

$$r_2>\max\Big\{\frac{(n-1)(b-a)^n}{n!}M,\ \frac{g^*+h^*}{1-C_3C_1}\Big\},$$

we have by Lemma 2.4(iii) that

$$\begin{aligned}x_*(t)&=[x_*(t)+x_0(t)]-x_0(t)\\&=[x_*(t)+x_0(t)]-Mp(t)\\&\ge\gamma(t)\|x_*+x_0\|-\frac{(n-1)(b-a)^n}{n!}M\gamma(t)\\&\ge\gamma(t)\Big(r_2-\frac{(n-1)(b-a)^n}{n!}M\Big)>0,\qquad a<t<b,\end{aligned}$$

which implies that x_* is a positive solution of the BVP (1.1)-(1.2).

References

[1] Guo D J, Lakshmikantham V. Nonlinear Problems in Abstract Cones. Notes Rep. Math. Sci. Engrg. vol. 5. Boston-Mass: Academic Press, 1988.

[2] Bai Z B, Du Z J. Positive solutions for some second-order four-point boundary value problems. *J. Math. Anal. Appl.*, 2007, **330**: 34–50.

[3] Bai Z B, Ge W G, Wang Y F. Multiplicity results for some second-order four-point boundary value problems. *Nonlinear Anal.*, 2005, **60**: 491–500.

[4] Bai C Z, Xie D P, Liu Y, Wang C L. Positive solutions for second-order four-point boundary value problems with alternating coefficient. *Nonlinear Anal.*, 2009, **70**: 2014–2023.

[5] Eloe P W, Ahmad B. Positive solutions of a nonlinear nth order boundary value problems with nonlocal conditions. *Appl. Math. Lett.*, 2005, **18**: 521–527.

[6] Hao X A, Liu L S, Wu Y H. Positive solutions of nonlinear nth-order singular nonlocal boundary value problems. *Boundary Value Problems*, 2007, **2007**: 234–244.

[7] Eloe P W, Henderson J. Positive solutions for higher order ordinary differential equations. *Electron. J. Differential Equations*, 1995, **3**: 1–8.

[8] Xie D P, Bai C Z, Liu Y, Wang C L. Existence of positive solutions for nth-order boundary value problem with sign changing nonlinearity. *Electron. J. Qual. Theory Differential Equations*, 2008, **8**: 1–10.

[9] Guo D, Lakshmikantham V. Problem in Abstract Cones. New York: Academic Press, 1988.