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An Averaging Principle for Caputo Fractional Stochastic Differential Equations with Compensated Poisson Random Measure

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Abstract. This article deals with an averaging principle for Caputo fractional stochastic differential equations with compensated Poisson random measure. The main contribution of this article is impose some new averaging conditions to deal with the averaging principle for Caputo fractional stochastic differential equations. Under these conditions, the solution to a Caputo fractional stochastic differential system can be approximated by that of a corresponding averaging equation in the sense of mean square.

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1 Introduction

Most systems in science and industry are perturbed by some random environmental effects, described by stochastic differential equations with (fractional) Brownian motion, Lévy process, Poisson process and etc. A series of useful theories and methods have been proposed to explore stochastic differential equations, such as invariant manifolds , averaging principle , homogenization principle. All of these theories and methods develop to extract an effective dynamics from these stochastic differential equations, which

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is more effective for analysis and simulation. For averaging principle, its often used to approximate dynamical systems with random fluctuations, and provides a powerful tool for simplifying nonlinear dynamical systems. The essence of averaging principle is to establish an approximation theorem that a simplified stochastic differential equation is presented to replace the original one in some senses.

The averaging principle for stochastic differential equations was first introduced by Khasminskii in paper [1], which extending the deterministic result of [2]. Since then, the theory of averaging principle for stochastic differential equations driven by different noise are considered by many authors, see [3–7] .

Because the non-local property of time derivatives, the model of Caputo fractional stochastic differential equations applied in many areas, such as biology, physics and chemistry and etc.. Existence and uniqueness of solution for Caputo fractional stochastic differential have been discussed by many papers. Quite recently, some types of Caputo fractional stochastic (partial) differential equations problem are considered from the dynamic viewpoint. For example, in paper [8], existence of stable manifolds is established. In [9], the existence of global forward attracting set for stochastic lattice systems with a Caputo fractional time derivative in the weak mean-square topology is considered. The asymptotic distance between two distinct solutions under a temporally weighted norm is discussed by [10]. To the best of our knowledge, averaging principle for Captuo fractional differential equations only considered by two articles in present, see [11] and [12], it is worth noting that the average conditions given in these two papers are different, under this conditions, the corresponding averaging conclusions are drawn respectively. In this paper, we also impose a new averaging condition for our framework, which let us to derive the averaging principle for our consider problem from the theoretical derivation.

Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ be a complete stochastic base. Let $(Z,\mathcal{B}(Z))$ be a measurable space and $\nu(dz)$ a $\sigma-$finite measure on it. Let $p=(p(t)),t\in D_p$, be a stationary $\mathcal{F}_t-$Poisson point process on Z with characteristic measure ν. The counting measure associated with $p(t)$ is given by, for $A\in\mathcal{B}(Z)$, $N((0,t],A):=\sharp\{t\in D_p:0<s\le t,p(s)\in A\}$. Assume that $b:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d,\sigma:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^m$ and $F:\mathbb{R}_+\times\mathbb{R}^d\times Z\to\mathbb{R}^d$ are measurable functions. For $\alpha\in(\frac{1}{2},1)$, we consider a stochastic fractional differential equation with compensated Poisson random measure of the form:

$$D_t^\alpha X_\epsilon(t)=\epsilon b(t,X_\epsilon(t-))\mathrm{d}t+\sqrt{\epsilon}\sigma(t,X_\epsilon(t-))\mathrm{d}B_s+\sqrt{\epsilon}\int_Z F(t,X_\epsilon(t-),z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \tag{1.1}$$

where ϵ is a small positive parameter, $\tilde{N}(\mathrm{d}t,\mathrm{d}z):=N(\mathrm{d}t,\mathrm{d}z)-\mathrm{d}t v(\mathrm{d}z)$ is the compensated Poisson martingale measures corresponding to $N(\mathrm{d}t,\mathrm{d}z)$ and $\{B_t\}_{t\ge0}$ is an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion. We note that the above equation is a classical equation if $\alpha=1$, which has been studied by many authors. Under some conditions, which can be compared with the classic case as in [12], we derive an averaging principle for the stochastic fractional differential system (1.1).

This article is organized as follows. In Section 2 we will give some assumptions and basic results for our theory. The solution of convergence in mean square between the stochastic fractional differential equations with Poisson random measure and the corresponding averaged equation are considered in Section 3.

Throughout this paper, the letter C below will denote positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

2 Stochastic differential equations with Poisson random measure

2.1 Basic Hypothesis and some useful results

We impose the following assumptions to guarantee the existence of the solution.

H1 (Lipschitz condition). There exists a bounded function $K_1(t)>0$ such that for all $x,y\in R^d$,

$$\begin{aligned}&|b(t,x)-b(t,y)|^2+|\sigma(t,x)-\sigma(t,y)|^2+\int_Z|F(s,x,z)-F(s,y,z)|^2\nu(\mathrm{d}z)\\ \leq\ & K_1(t)(|x-y|^2).\end{aligned}$$

H2 (Growth condition). There exists a bounded function $K_2(t)>0$ such that for all $x\in R^d$,

$$|b(t,x)|^2+|\sigma(t,x)|^2+\int_Z|F(t,x,z)|^2\nu(\mathrm{d}z)\leq K_2(t)(1+|x|^2).$$

We also assume that the $K_i(t)$ have the same upper bound K, for $i=1,2$.

To deal with fractional differential equation, we need the following generalization of Gronwall's lemma for singular kernels [13,14].

Lemma 2.1. *Suppose that $b\geq 0$ and $\beta>0$. Assume that $a(t)$ and $u(t)$ are nonnegative and locally integrable functions on $0\leq t<T$, satisfying*

$$u(t)\leq a(t)+b\int_0^t(t-s)^{\beta-1}u(s)\mathrm{d}s.$$

Then

$$u(t)\leq a(t)+\int_0^t\left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]\mathrm{d}s,\qquad 0\leq t<T.$$

2.2 Existence and Uniqueness

In this part, we consider the existence and uniqueness for the following equation under conditions **H1** and **H2**:

$$D_t^{\alpha}X(t)=b(t,X(t-))\mathrm{d}t+\sigma(t,X(t-))\mathrm{d}B_t+\int_Z F(t,X(t-),z)\tilde{N}(\mathrm{d}t,\mathrm{d}z).\tag{2.1}$$

Definition 2.1. *A map $X(t):[0,T]\to L^2(\Omega,\mathbb{R}^d)$ is called a solution of the initial value problem* (2.1) *if $X(0)=x_0$ and $X(t)$ is càdlàg and satisfies for $t\in[0,T]$,*

$$\begin{aligned} X(t)=&x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}b(s,X(s-))\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s,X(s-))\mathrm{d}B_s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\int_Z (t-s)^{\alpha-1}F(s,X(s-),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{2.2}$$

Using similarly methods as [15], we can derive the global existence and uniqueness of solutions for Eq.2.1. Similar problem also considered by [13] under different framework. In the following, we just given a priori estimate for the solution $X(t)$, which assure the existence of workspace for our problem.

Theorem 2.1. *Under conditions of* **H1** *and* **H2***, for every $x_0\in L^2(\Omega,\mathbb{R}^d)$ there exists a unique solution to Eq.* (2.2)*, such that*

$$\sup_{0\le t\le T} E|X(t)|^2<\infty.$$

Proof. The existence and uniqueness can be easily proved by the contraction mapping argument and hence omitted here. Next, we estimate the solution $X(t)$ in $L^\infty([0,T],L^2(\Omega;\mathbb{R}^d))$. By employing the simple arithmetic inequality

$$|a+b+c+d|^2\le 4(|a|^2+|b|^2+|c|^2+|d|^2),$$

we have

$$\begin{aligned} E|X(t)|^2\le& 4E|x_0|^2+4E\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}b(s,X(s-))\mathrm{d}s\right|^2 \\ &+4E\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(t,X(s-))\mathrm{d}B_s\right|^2 \\ &+4E\left|\frac{1}{\Gamma(\alpha)}\int_0^t\int_Z (t-s)^{\alpha-1}F(s,X(s-),x)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^2 \\ =:& 4I_1+4I_2+4I_3+4I_4. \end{aligned}$$

For I_2, by Cauchy-Schwarz inequality and the condition **H2**, we have

$$\begin{aligned} I_2\le&\frac{TK}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}(1+E|X(s)|^2)\mathrm{d}s \\ \le&\frac{TK}{\Gamma(\alpha)^2}\left[\frac{t^{2\alpha-1}}{2\alpha-1}+\int_0^t (t-s)^{2(\alpha-1)}E|X(s)|^2\mathrm{d}s\right] \\ \le&\frac{K}{\Gamma(\alpha)^2}\frac{T^{2\alpha}}{2\alpha-1}+\frac{TK}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X(s)|^2\mathrm{d}s, \end{aligned}$$

where we have used the fact that $\{s:X(s)\neq X(s-)\}$ is countable.

For I_3, by Itô isometry and the condition **H2**, we have

$$\begin{aligned}I_3 \le& \frac{K}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}(1+E|X(s)|^2)\mathrm{d}s\\ \le& \frac{K}{\Gamma(\alpha)^2}\frac{T^{2\alpha-1}}{2\alpha-1}+\frac{K}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X(s)|^2\mathrm{d}s.\end{aligned}$$

In a similar way, for I_4 we have

$$\begin{aligned}I_4 =& \frac{1}{\Gamma(\alpha)^2}\int_0^t\int_{|x|<c}(t-s)^{2(\alpha-1)}E|F(t,X(s-),z)|^2\nu(\mathrm{d}x)\mathrm{d}s\\ \le& \frac{K}{\Gamma(\alpha)^2}\frac{T^{2\alpha-1}}{2\alpha-1}+\frac{K}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X(s)|^2\mathrm{d}s.\end{aligned}$$

Therefore, we get

$$E|X(t)|^2 \le \left(4E|x_0|^2+\frac{4K}{\Gamma(\alpha)^2}\frac{T^{2\alpha-1}}{2\alpha-1}(2+T)\right)+\frac{4TK+8K}{\Gamma(\alpha)^2}\int_0^t (t-s)^{(2\alpha-1)-1}E|X(s)|^2\mathrm{d}s.$$

By setting $r=4E|x_0|^2+\frac{4K}{\Gamma(\alpha)^2}\frac{T^{2\alpha-1}}{2\alpha-1}(2+T)$ and applying Lemma 2.1, we have

$$\begin{aligned}E|X(t)|^2 \le& r^2\left(1+\int_0^t\sum_{n=1}^{\infty}\frac{(\frac{4TK+8K}{\Gamma(\alpha)^2}\Gamma(2\alpha-1))^n}{\Gamma(2n\alpha-n)}(t-s)^{n(2\alpha-1)-1}\mathrm{d}s\right)\\ \le& r^2\left(1+\sum_{n=1}^{\infty}\frac{(\frac{4TK+8K}{\Gamma(\alpha)^2}\Gamma(2\alpha-1)T^{2\alpha-1})^n}{\Gamma(2n\alpha-n+1)}\right)\\ =& r^2\left(1+E_{2\alpha-1,1}\left(\frac{4TK+8K}{\Gamma(\alpha)^2}\Gamma(2\alpha-1)T^{2\alpha-1}\right)\right)<\infty\end{aligned}$$

for all $t\in[0,T]$, where $E_{2\alpha-1,1}(\cdot)$ is a two-parameter function of the Mittag-Leffler type [13]. Then we have

$$\sup_{0\le t\le T} E|X(t)|^2<\infty.$$

This completes the proof of the theorem. □

3 An averaging principle

We now study an averaging principle for a standard stochastic integral equation in R^d:

$$\begin{aligned}X_\epsilon(t)=& x_0+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}b(s,X_\epsilon(s-))\mathrm{d}s+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s,X_\epsilon(s-))\mathrm{d}B_s\\ &+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t\int_Z (t-s)^{\alpha-1}F(t,X_\epsilon(s-),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \qquad (3.1)\end{aligned}$$

where x_0 is a random vector satisfy $E|x_0|^2<\infty$, and $\epsilon\in(0,\epsilon_0]$ is a positive small parameter with ϵ_0 a fixed number.

To ensures an averaging principle, we need to assume the following condition.

H3: There exist measurable functions $\bar{b}:\mathbb{R}^d\to\mathbb{R}^d,\bar{\sigma}:\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^m$ and $\bar{F}:\mathbb{R}^d\times Z\to\mathbb{R}^d$ such that for all $t>0$,

$$\frac{1}{t^{2\alpha-1}}\int_0^t(t-s)^{2(\alpha-1)}|b(s,x)-\bar{b}(x)|^2\mathrm{d}s\le\varphi_1(t)(1+|x|^2), \tag{3.2}$$

$$\frac{1}{t^{2\alpha-1}}\int_0^t(t-s)^{2(\alpha-1)}|\sigma(s,x)-\bar{\sigma}(x)|^2\mathrm{d}s\le\varphi_2(t)(1+|x|^2), \tag{3.3}$$

$$\frac{1}{t^{2\alpha-1}}\int_0^t(t-s)^{2(\alpha-1)}\int_Z|F(s,x,z)-\bar{F}(x,z)|^2\nu(\mathrm{d}z)\mathrm{d}s\le\varphi_3(t)(1+|x|^2), \tag{3.4}$$

where $\varphi_i(t)$ are positive bounded function with $\lim_{t\to\infty}\varphi_i(t)=0$ for $i=1,2,3$.

Remark 3.1. Note that, when we take $\alpha=1$, the above conditions are consistent with classic case.

Our main result, Theorem 3.1, states that the solution of original equation (3.1) is well approximated, in the sense of mean square, by that of following equation

$$\begin{aligned}Z_\epsilon(t)=&x_0+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bar{b}(Z_\epsilon(s-))\mathrm{d}s+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bar{\sigma}(Z_\epsilon(s-))\mathrm{d}B_s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t\int_Z(t-s)^{\alpha-1}\bar{F}(Z_\epsilon(s-),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{3.5}$$

Theorem 3.1. *Assume Hypotheses* **H1**$-$**H3**. *Then, for arbitrary $\alpha\in(\frac{1}{2},1)$, $L>0$ and $\beta\in(0,2\alpha-1)$, there exists a constant $C_{L,\alpha}$ independent of ϵ, such that that*

$$\sup_{0\le t\le L\epsilon^{\frac{-\beta}{2\alpha-1}}}E|X_\epsilon(t)-Z_\epsilon(t)|^2\le C_{L,\alpha}\epsilon^{1-\beta}. \tag{3.6}$$

Proof. Let $[0,T]$ be a fixed time interval. For any $t\in[0,T]$ we have

$$\begin{aligned}X_\epsilon(t)-Z_\epsilon(t)=&\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[b(s,X_\epsilon(s-))-\bar{b}(Z_\epsilon(s-)]\mathrm{d}s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\sigma(s,X_\epsilon(s-))-\bar{\sigma}(Z_\epsilon(s-)]\mathrm{d}B_s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t\int_Z(t-s)^{\alpha-1}[F(s,X_\epsilon(s-),z)-\bar{F}(Z_\epsilon(s-),z)]\tilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

Directly, we get

$$\begin{aligned}
&E|X_\epsilon(t)-Z_\epsilon(t)|^2\\
\leq&3\Big\{E\Big|\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[b(s,X_\epsilon(s-))-\bar{b}(Z_\epsilon(s-))]\mathrm{d}s\Big|^2\\
&+E\Big|\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\sigma(s,X_\epsilon(s-))-\bar{\sigma}(Z_\epsilon(s-))]\mathrm{d}B_s\Big|^2\\
&+E\Big|\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t\int_Z(t-s)^{\alpha-1}[F(s,X_\epsilon(s-),z)-\bar{F}(Z_\epsilon(s-),z)]\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big|^2\Big\}\\
=:&3(J_1+J_2+J_3).
\end{aligned}$$

For J_1, we have

$$\begin{aligned}
J_1=&E\Big|\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[b(s,X_\epsilon(s-))-\bar{b}(Z_\epsilon(s-))]\mathrm{d}s\Big|^2\\
\leq&\frac{2\epsilon^2}{\Gamma(\alpha)^2}\Big\{E\Big|\int_0^t(t-s)^{\alpha-1}[b(s,X_\epsilon(s-))-b(s,Z_\epsilon(s-))]\mathrm{d}s\Big|^2\\
&+E\Big|\int_0^t(t-s)^{\alpha-1}[b(s,Z_\epsilon(s-))-\bar{b}(Z_\epsilon(s-))]\mathrm{d}s\Big|^2\Big\}.
\end{aligned}$$

Due to the Cauchy-Schwarz inequality, conditions **H1** and (3.2) in **H3**, we get

$$\begin{aligned}
J_1\leq&\frac{2tK\epsilon^2}{\Gamma(\alpha)^2}\int_0^t(t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s\\
&+\frac{2K\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\Big\{\frac{1}{t^{2\alpha-1}}E\int_0^t(t-s)^{2\alpha-2}|b(s,Z(s-))-\bar{b}(Z_\epsilon(s-))|^2\mathrm{d}s\Big\}\\
\leq&\frac{2tK\epsilon^2}{\Gamma(\alpha)^2}\int_0^t(t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s+\frac{2K\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\varphi_1(t)\Big(1+E|Z_\epsilon(s-)|^2\Big).
\end{aligned}\tag{3.7}$$

For J_2, using the Itô isometry, we have

$$\begin{aligned}
J_2=&\frac{\epsilon}{\Gamma(\alpha)^2}\int_0^t(t-s)^{2(\alpha-1)}E|\sigma(s,X_\epsilon(s-))-\bar{\sigma}(Z_\epsilon(s-))|^2\mathrm{d}s\\
\leq&\frac{2\epsilon}{\Gamma(\alpha)^2}\Big\{\int_0^t(t-s)^{2(\alpha-1)}E|\sigma(s,X_\epsilon(s-))-\sigma(s,Z_\epsilon(s-))|^2\mathrm{d}s\\
&+\int_0^t(t-s)^{2(\alpha-1)}E|\sigma(s,Z_\epsilon(s-))-\bar{\sigma}(Z_\epsilon(s-))|^2\mathrm{d}s\Big\}.
\end{aligned}$$

By conditions **H1** and (3.3) in **H3** , we obtain

$$\begin{aligned}J_2 \le&\frac{2\epsilon}{\Gamma(\alpha)^2}\Big\{\int_0^t (t-s)^{2(\alpha-1)}E|\sigma(s,X_\epsilon(s-))-\sigma(s,Z_\epsilon(s-))|^2\mathrm{d}s\\&\quad+\int_0^t (t-s)^{2(\alpha-1)}E|\sigma(s,Z_\epsilon(s-))-\bar{\sigma}(Z_\epsilon(s-))|^2\mathrm{d}s\Big\}\\\le&\frac{2K\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s\\&\quad+\frac{2K\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_2(t)\Big(1+E|Z_\epsilon(s-)|^2\Big).\end{aligned}\tag{3.8}$$

Finally, we take estimate on J_3. Using the Itô isometry yields

$$\begin{aligned}J_3=&E\Big|\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t\int_{|x|<c}(t-s)^{\alpha-1}[F(t,X_\epsilon(s-),z)-\bar{F}(Z_\epsilon(s-),z)]\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big|^2\\\le&\frac{2\epsilon}{\Gamma(\alpha)^2}\Big\{\int_0^t\int_Z (t-s)^{2(\alpha-1)}E|F(s,X_\epsilon(s-),z))-F(s,Z_\epsilon(s-),z))|^2\nu(\mathrm{d}z)\mathrm{d}s\\&\quad+\int_0^t\int_Z (t-s)^{2(\alpha-1)}E|F(s,Z_\epsilon(s-),z)-\bar{F}(Z_\epsilon(s-),z)|^2\nu(\mathrm{d}z)\mathrm{d}s\Big\},\end{aligned}$$

and so, by **H1** and (3.4) in **H3**, we have

$$\begin{aligned}J_3\le&\frac{2K\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s\\&\quad+\frac{2\epsilon}{\Gamma(\alpha)^2}E\int_0^t\int_Z (t-s)^{2(\alpha-1)}|F(s,Z(s-),z)-\bar{F}(Z_\epsilon(s-),z)|^2\nu(\mathrm{d}z)\mathrm{d}s\\\le&\frac{2K\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s\\&\quad+\frac{2K\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_3(t)\Big(1+E|Z_\epsilon(s-)|^2\Big).\end{aligned}\tag{3.9}$$

Therefore, from (3.7)–(3.9), Theorem 2.1, one can get

$$\begin{aligned}&E|X_\epsilon(t)-Z(t)|^2\\\le&\frac{6tK\epsilon^2}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s+\frac{6K\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\varphi_1(t)\Big(1+E|Z_\epsilon(s-)|^2\Big)\\&\quad+\frac{6K\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s+\frac{6K\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_2(t)\Big(1+E|Z_\epsilon(s-)|^2\Big)\\&\quad+\frac{6K\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E|X_\epsilon(s-)-Z_\epsilon(s-)|^2\mathrm{d}s+\frac{6K\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_3(t)\Big(1+E|Z_\epsilon(s-)|^2\Big)\\\le&6\epsilon T^{2\alpha-1}\Big(\frac{C_1K\epsilon}{\Gamma(\alpha)^2}T+\frac{C_2K}{\Gamma(\alpha)^2}\Big)\end{aligned}$$

$$+6\epsilon\Big(\frac{K\epsilon}{\Gamma(\alpha)^2}T+\frac{2K}{\Gamma(\alpha)^2}\Big)\int_0^t(t-s)^{(2\alpha-1)-1}E|X_\epsilon(s)-Z_\epsilon(s)|^2\mathrm{d}s,$$

where we have used the fact that $\{s:X_\epsilon(s)\neq X_\epsilon(s-)\ or\ Z_\epsilon(s-)\neq Z_\epsilon(s)\}$ is countable. By setting $r_1=6(\frac{C_1K\epsilon}{\Gamma(\alpha)^2}T+\frac{C_2K}{\Gamma(\alpha)^2})$ and $r_2=6(\frac{K\epsilon}{\Gamma(\alpha)^2}T+\frac{2K}{\Gamma(\alpha)^2})$, from Lemma 2.1, we have

$$\begin{aligned}E|X_\epsilon(t)-Z_\epsilon(t)|^2&\le\epsilon T^{2\alpha-1}r_1\Big(1+\int_0^t\sum_{n=1}^{\infty}\frac{(r_2\epsilon\Gamma(2\alpha-1))^n}{\Gamma(2n\alpha-n)}(t-s)^{n(2\alpha-1)-1}\mathrm{d}s\Big)\\&\le\epsilon T^{2\alpha-1}r_1\Big(1+\sum_{n=1}^{\infty}\frac{(r_2\epsilon\Gamma(2\alpha-1)T^{2\alpha-1})^n}{\Gamma(2n\alpha-n+1)}\Big)\\&\le\epsilon T^{2\alpha-1}r_1\Big(1+E_{2\alpha-1,1}(r_2\epsilon\Gamma(2\alpha-1)T^{2\alpha-1})\Big).\end{aligned}$$

By selecting $\beta\in(0,2\alpha-1)$ and $L>0$, such that for every $t\in(0,L\epsilon^{\frac{-\beta}{2\alpha-1}})\subseteq[0,T]$, we obtain (3.6). This completes the proof. □

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