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HYBRID ITERATION METHOD FOR GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS[*†]

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Abstract

In this paper, weak and strong convergence theorems are established by hybrid iteration method for generalized equilibrium problem and fixed point problems of a finite family of asymptotically nonexpansive mappings in Hilbert spaces. The results presented in this paper partly extend and improve the corresponding results of the previous papers.

Keywords generalized equilibrium problem; a finite family of asymptotically nonexpansive mapping; hybrid iteration method; inverse-strongly monotone mapping; Hilbert space

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, $G : C \times C \to R$ be a bifunction and $A : C \to H$ be a nonlinear mapping. The generalized equilibrium problem (for short, GEP) is to find an $x \in C$ such that

$$G(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \text{for any } y \in C. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $GEP(G)$, that is,

$$GEP(G) = \{x \in C : G(x, y) + \langle Ax, y - x\rangle \geq 0, \text{ for any } y \in C\}.$$

In the case of $A \equiv 0$, GEP is denoted by $EP(G)$. In the case of $G \equiv 0$, GEP is also denoted by $VI(C, A)$. Problem (1.1) covers monotone inclusion problems, saddle point problems, minimization problems, optimization problems, variational inequality problems, and Nash equilibria in noncooperative games. Recently, many authors have studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [1-8] and the references therein.

In 2007, to study the strong and weak convergence of fixed points of nonexpansive mappings, Wang [9] introduced the following hybrid iteration scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_{n+1}} x_n, \quad n \geq 0, \tag{1.2}$$

where $T^\lambda x = Tx - \lambda\mu F(Tx)$ for all $x \in H$ and $x_0 \in H$ is chosen arbitrarily, $F : H \to H$ is an η-strongly monotone and k-Lipschitzian, they obtained that under some suitable conditions, the sequence $\{x_n\}$ converges weakly to a fixed point of T, and under the necessary and sufficient conditions, $\{x_n\}$ converges strongly to a fixed point of T.

In 2009, Ceng and Gao [3] introduced the following iterative scheme for a k-strict pseudo-contraction mapping in Hilbert space: $x_1 = x \in H$,

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for any } y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Tu_n, \end{cases} \tag{1.3}$$

for every $n \in N$, where $\alpha_n \subset [a, b]$ for some $a, b \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf\limits_{n\to\infty} r_n > 0$. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in F(T) \cap EP(G)$, where $z = P_{F(T)\cap EP(G)}x$.

In 2012, Wang and Huang [10] considered hybrid iteration method for a finite asymptotically nonexpansive mappings in Banach space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[T_{i(n)}^{k_(n)} x_n - \lambda_{n+1}\mu f(T_{i(n)}^{k_(n)} x_n)], \quad \text{for any } n \geq 1, \tag{1.4}$$

where $x_1 \in E$ is chosen arbitrarily, $\{T_1, T_2, \cdots, T_N\} : K \to K$ are N asymptotically nonexpansive mappings, $f : K \to K$ is a L-Lipschitzian mapping. And weak and strong convergence theorems are obtained under some suitable conditions.

Motivated by those works of [3] and [10], in this paper we introduce the following hybrid iteration method: $x_1 \in C$

$$\begin{cases} G(u_n, y) + \langle Ax_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for any } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[T_{i(n)}^{k(n)} u_n - \lambda_n \mu f(T_{i(n)}^{k(n)} u_n)], \quad \text{for any } n \geq 1, \end{cases} \tag{1.5}$$

where μ is a positive fixed constant, $\{T_1, T_2, \cdots, T_N\} : C \to C$ are N asymptotically nonexpansive mappings, $f : C \to C$ is a L-Lipschitzian mapping, $A : C \to H$ is an α-inverse-strongly monotone mapping, $\{\lambda_n\} \subset [0, 1]$, $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$.

If $\{T_1, T_2, \cdots, T_N\} : C \to C$ are N nonexpansive mappings, then (1.5) reduces to hybrid iteration method of a finite family of nonexpansive mappings as follows: $x_1 \in C$,

$$\begin{cases} G(u_n, y) + \langle Ax_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for any } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[T_n u_n - \lambda_n \mu f(T_n u_n)], \quad \text{for any } n \geq 1. \end{cases} \tag{1.6}$$

In the case of $A \equiv 0$, (1.5) reduces to hybrid iteration method as follows: $x_1 \in C$,

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for any } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[T_{i(n)}^{k(n)} u_n - \lambda_n \mu f(T_{i(n)}^{k(n)} u_n)], \quad \text{for any } n \geq 1. \end{cases} \tag{1.7}$$

In the case of $A \equiv 0$, $\lambda_n \equiv 0$, (1.5) reduces to hybrid iteration method as follows: $x_1 \in C$,

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for any } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{i(n)}^{k(n)} u_n, \quad \text{for any } n \geq 1. \end{cases} \tag{1.8}$$

The purpose of this paper is to study iteration scheme (1.5) for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a finite family of asymptotically nonexpansive mappings. And we obtain some weak and strong convergence theorems for the iteration method in Hilbert space. The results presented in this paper partly extend and improve the corresponding results of [3,10].

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. $T : C \to C$ is a mapping. $F(T)$ denotes the set of fixed point of mapping T. $G : C \times C \to R$ is a bifunction. For a given sequence $\{x_n\} \subset C$, $\omega_w(x_n)$ denotes the weak accumulation set of $\{x_n\}$, that is, $\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}$.

To solve $GEP(G)$, we assume that G satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;

(A2) G is monotone, that is, $G(x,y)+G(y,x)\le 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$, $\lim\limits_{t\to 0} G(tz+(1-t)x,y)\le G(x,y)$;

(A4) for each $x\in C$, $y\mapsto G(x,y)$ is convex and lower semicontinuous.

Recall that a Banach space E is said to satisfy *Opial's* condition [11], if for each sequence $\{x_n\}$ in E, the sequence $x_n\rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|,$$

for all $y\in E$ with $y\neq x$. It is well known that every Hilbert space satisfies *Opial's* condition.

Definition 2.1 Let C be a closed subset of H, $T:C\to C$, $f:C\to C$ and $A:C\to H$ be mappings.

(1) T is called demiclosed at the origin, if for each sequence $\{x_n\}$ in C, the condition $x_n\to x_0$ weakly and $Tx_0\to 0$ strongly implies $Tx_0=0$.

(2) T is called semicompact, if for any bounded sequence $\{x_n\}$ in C, such that $\|x_n-Tx_n\|\to 0$ $(n\to\infty)$, there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\to x^*\in C$.

(3) T is called asymptotically nonexpansive, if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim\limits_{n\to\infty}k_n=1$ such that

$$\|T^nx-T^ny\|\le k_n\|x-y\|,\quad \text{for any } x,y\in D, n\ge 1.$$

When $k_n\equiv 1$, T is known as nonexpansive mapping.

(4) f is called L-Lipschitzian if there exists a constant $L>0$ such that

$$\|fx-fy\|\le L\|x-y\|,\quad \text{for any } x,y\in C.$$

(5) A is called α-inverse-strongly monotone if there exists an $\alpha>0$ such that

$$\langle x-y,Ax-Ay\rangle\ge\alpha\|Ax-Ay\|^2,\quad \text{for any } x,y\in C.$$

Remark 2.1 If $\{T_i\}_{i=1}^N:C\to C$ be N asymptotically nonexpansive mappings, then there exists a sequence $\{h_n\}\subset[1,\infty)$ with $h_n\to 1$ such that

$$\|T_i^nx-T_i^ny\|\le h_n\|x-y\|,\quad \text{for any } n\ge 1,\ x,y\in K,\ i=1,2,\cdots,N. \tag{2.1}$$

Lemma 2.1[12] *For a real Hilbert space H, the following identities hold:*

(i) $\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle,$ *for any* $x,y\in H$;

(ii) $\|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2,$ *for any* $t\in[0,1],\ x,y\in H$;

(iii) *If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty}\|x_n-y\|^2=\limsup_{n\to\infty}\|x_n-z\|^2+\|z-y\|^2,\quad \text{for any } y\in H.$$

Lemma 2.2[13] *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be three nonnegative sequences satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad for\ any\ n = 1, 2, \cdots$$

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum\limits_{n=1}^{\infty} b_n < \infty$, *then* $\lim\limits_{n\to\infty} a_n$ *exists.*

Lemma 2.3[5] *Let* C *be a nonempty closed convex subset of* H *and* $G : C \times C \to R$ *be a bifunction satisfying* (A1)-(A4). *Let* $r > 0$ *and* $x \in H$. *Then there exists a* $z \in C$ *such that*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad for\ any\ y \in C.$$

Lemma 2.4[5] *Assume that* $G : C \times C \to R$ *satisfying* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $S_r : H \to C$ *as follows:*

$$S_r(x) = \{z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ for\ any\ y \in C\}, \quad for\ any\ x \in H.$$

Then,

(1) S_r *is single-valued*;

(2) S_r *is firmly nonexpansive, that is,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle, \quad for\ any\ x, y \in H;$$

(3) $F(S_r) = EP(G)$;

(4) $EP(G)$ *is nonempty, closed and convex.*

Lemma 2.5[14] *Let* E *be a uniformly convex Banach space,* b *and* c *be two constants with* $0 < b < c < 1$. *Suppose that* $\{t_n\}$ *is a sequence in* $[b, c]$ *and* $\{x_n\}, \{y_n\}$ *are two sequences in* E. *Then conditions*

$$\begin{cases} \lim\limits_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \\ \limsup\limits_{n\to\infty} \|x_n\| \leq d, \\ \limsup\limits_{n\to\infty} \|y_n\| \leq d \end{cases}$$

imply that $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$ *where* $d \geq 0$ *is some constant.*

Lemma 2.6[15] *Let* K *be a nonempty closed convex subset of uniformly convex Banach space* E *and* $T : K \to K$ *be a asymptotically nonexpansive mapping. If* T *has a fixed point, then* $I - T$ *is demi-closed at zero, where* I *is the identity mapping of* H, *that is, whenever* $\{x_n\}$ *is a sequence in* K *weakly convergent to some* $x \in K$ *and the sequence* $\{(I - T)x_n\}$ *strongly converges to some* y, *it follows that* $(I - T)x = y$.

3 Main Results

Theorem 3.1 *Suppose that* H *is a real Hilbert space,* C *is a nonempty closed convex subset of* H. *Let* $\{T_1, T_2, \cdots, T_N\} : C \to C$ *be* N *asymptotically nonexpansive* *mappings with* $\Omega = \bigcap\limits_{n=1}^{N} F(T_n) \cap GEP(G) \neq \emptyset$*, and* $f : C \to C$ *be a L-Lipschitzian mapping. If hybrid iteration* $\{x_n\}$ *defined by* (1.5)*, where* μ *is a positive fixed constant,* $\{\alpha_n\}$*,* $\{\lambda_n\}$*,* $\{r_n\}$ *and* $\{h_n\}$ *defined by* (2.1) *satisfy the following conditions:*

(i) $a \leq \alpha_n \leq b$ *for some* $a, b \in (0,1)$;

(ii) $\sum\limits_{n=1}^{\infty} (h_n - 1) < \infty$;

(iii) $\{\lambda_n\} \subset [0.1)$, $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$;

(iv) $0 < c \leq r_n \leq d < 2\alpha$;

(v) $1 < h_n + \lambda_n \mu < 2$.

Then,

(1) $\{x_n\}$*,* $\{u_n\}$ *converge weakly to a common point of* $\bigcap\limits_{n=1}^{N} F(T_i)$ *and* $GEP(G)$*.*

(2) $\{x_n\}$*,* $\{u_n\}$ *converges strongly to a common fixed point of* $\bigcap\limits_{n=1}^{N} F(T_i)$ *and* $GEP(G)$ *if and only if* $\lim\limits_{n\to\infty} d(x_n, \Omega) = 0$*, for any* $n \geq 1$*.*

Proof We prove this theorem in six steps.

Step 1 $\lim\limits_{n\to\infty} \|x_n - q\|$ exists for each $q \in \Omega$.

For any $q \in \Omega$, from the definition of S_r in Lemma 2.4, we have $u_n = S_{r_n}(x_n - r_n A x_n)$ and $q = S_{r_n}(q - r_n A q)$, therefore

$$\begin{aligned}
\|u_n - q\|^2 &= \|S_{r_n}(x_n - r_n A x_n) - S_{r_n}(q - r_n A q)\|^2 \\
&= \|(x_n - q) - r_n(Ax_n - Aq)\|^2 \\
&\leq \|x_n - q\|^2 - 2r_n\langle x_n - q, Ax_n - Aq\rangle + r_n^2\|Ax_n - Aq\|^2 \\
&\leq \|x_n - q\|^2 - 2r_n\alpha\|Ax_n - Aq\|^2 + r_n^2\|Ax_n - Aq\|^2 \\
&= \|x_n - q\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aq\|^2 \\
&\leq \|x_n - q\|^2, \quad \text{for any } n \geq 1. \qquad (3.1)
\end{aligned}$$

By (3.1), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n x_n + (1-\alpha_n)[T_{i(n)}^{k(n)} u_n - \lambda_n \mu f(T_{i(n)}^{k(n)} u_n)] - q\| \\
&\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|T_{i(n)}^{k(n)} u_n - q\| + (1-\alpha_n)\lambda_n\mu\|f(T_{i(n)}^{k(n)} u_n)\| \\
&\leq \alpha_n\|x_n - q\| + (1-\alpha_n)h_{k(n)}\|u_n - q\| \\
&\quad + (1-\alpha_n)\lambda_n\mu\|f(T_{i(n)}^{k(n)} u_n) - f(q)\| + (1-\alpha_n)\lambda_n\mu\|f(q)\| \\
&\leq \alpha_n\|x_n - q\| + (1-\alpha_n)h_{k(n)}\|x_n - q\| \\
&\quad + (1-\alpha_n)\lambda_n\mu L h_{k(n)}\|x_n - q\| + (1-\alpha_n)\lambda_n\mu\|f(q)\|. \qquad (3.2)
\end{aligned}$$

Since $h_{k(n)} \subset [1, \infty)$, $h_{k(n)} \to 1$ $(n \to \infty)$, $\{h_{k(n)}\}$ is bounded, and there exists a

$D \geq 1$ such that

$$h_{k(n)} \leq D. \tag{3.3}$$

And we also can set $v_n = h_{k(n)} - 1$, for any $n \geq 1$, by condition (ii) we have

$$\sum_{n=1}^{\infty} v_n < \infty. \tag{3.4}$$

Therefore we have

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n)(1 + v_n)\|x_n - q\| \\
&\quad + (1 - \alpha_n)\lambda_n \mu LD \|x_n - q\| + (1 - \alpha_n)\lambda_n \mu \|f(q)\| \\
&\leq \alpha_n \|x_n - q\| + (1 - \alpha_n + v_n)\|x_n - q\| \\
&\quad + \lambda_n \mu LD \|x_n - q\| + \lambda_n \mu \|f(q)\| \\
&\leq (1 + v_n + \lambda_n \mu LD)\|x_n - q\| + \lambda_n \mu \|f(q)\|.
\end{aligned} \tag{3.5}$$

From condition (iii) and (3.4), it is easy to see that

$$\sum_{n=1}^{\infty}(v_n + \lambda_n \mu LD) < \infty; \quad \sum_{n=1}^{\infty} \lambda_n \mu < \infty.$$

It follows from Lemma 2.2 that $\lim\limits_{n\to\infty} \|x_n - q\|$ exists.

Step 2 $\lim\limits_{n\to\infty} \|x_n - u_n\| = 0$.

Since $\lim\limits_{n\to\infty} \|x_n - q\|$ exists, $\{\|x_n - q\|\}_{n=1}^{\infty}$ is bounded, so are $\|u_n - q\|$ and $\|f(T_{i(n)}^{k(n)} u_n)\|$. Then there exists a $K > 0$ such that

$$\max\{\|u_n - q\|, \|f(T_{i(n)}^{k(n)} u_n)\|\} \leq K.$$

Let $\sigma_n = T_{i(n)}^{k(n)} u_n - \lambda_n \mu f(T_{i(n)}^{k(n)} u_n)$, then we have

$$\begin{aligned}
\|\sigma_n - q\| &\leq \|[T_{i(n)}^{k(n)} u_n - q\| + \lambda_n \mu \|f(T_{i(n)}^{k(n)} u_n)\| \\
&\leq h_{k(n)} \|u_n - q\| + \lambda_n \mu \|f(T_{i(n)}^{k(n)} u_n)\| \\
&\leq \|u_n - q\| + v_n \|u_n - q\| + \lambda_n \mu \|f(T_{i(n)}^{k(n)} u_n)\| \\
&\leq [1 - (v_n + \lambda_n \mu)]\|u_n - q\| + (v_n + \lambda_n \mu)2K.
\end{aligned} \tag{3.6}$$

Form (3.1) and (3.6),

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n (x_n - q) + (1 - \alpha_n)(\sigma_n - q)\|^2 \leq \alpha_n \|x_n - q\|^2 + (1 - \alpha_n)\|\sigma_n - q\|^2 \\
&\leq \alpha_n \|x_n - q\|^2 + (1 - \alpha_n)[1 - (v_n + \lambda_n \mu)]\|u_n - q\|^2 + (1 - \alpha_n)(v_n + \lambda_n \mu)(2K)^2
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n-q\|^2+(1-\alpha_n)[1-(v_n+\lambda_n\mu)][\|x_n-q\|^2+r_n(r_n-2\alpha)\|Ax_n-Aq\|^2] \\
&\quad +(1-\alpha_n)(v_n+\lambda_n\mu)4K^2 \\
&\leq \|x_n-q\|^2+4(v_n+\lambda_n\mu)K^2+(1-\alpha_n)[1-(v_n+\lambda_n\mu)]r_n(r_n-2\alpha)\|Ax_n-Aq\|^2.
\end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned}
&(1-b)[1-(v_n+\lambda_n\mu)]c(2\alpha-d)\|Ax_n-Aq\|^2 \\
\leq\ &(1-\alpha_n)[1-(v_n+\lambda_n\mu)]r_n(r_n-2\alpha)\|Ax_n-Aq\|^2 \\
\leq\ &\|x_n-q\|^2-\|x_{n+1}-q\|^2+4(v_n+\lambda_n\mu)K^2.
\end{aligned} \tag{3.8}$$

Taking $n\to\infty$ in (3.8), we know that

$$\lim_{n\to\infty}\|Ax_n-Aq\|=0. \tag{3.9}$$

Since $u_n=S_{r_n}(x_n-r_nAx_n)$ and $q=S_{r_n}(q-r_nAq)$, we get that

$$\begin{aligned}
\|u_n-q\|^2&=\|S_{r_n}(x_n-r_nAx_n)-S_{r_n}(q-r_nAq)\| \\
&\leq \langle (x_n-r_nAx_n)-(q-r_nAq),u_n-q\rangle \\
&=\frac{1}{2}[\|(x_n-r_nAx_n)-(q-r_nAq)\|^2+\|u_n-q\|^2 \\
&\quad -\|(x_n-r_nAx_n)-(q-r_nAq)-(u_n-q)\|^2] \\
&=\frac{1}{2}[\|x_n-q\|^2+\|u_n-q\|^2-\|(x_n-u_n)-r_n(Ax_n-Aq)\|^2] \\
&=\frac{1}{2}[\|x_n-q\|^2+\|u_n-q\|^2-\|x_n-u_n\|^2 \\
&\quad +2r_n\langle x_n-u_n,Ax_n-Aq\rangle-r_n^2\|Ax_n-Aq\|^2].
\end{aligned}$$

So we have

$$\begin{aligned}
\|u_n-q\|^2=\|x_n-q\|^2-\|x_n-u_n\|^2 \\
+2r_n\langle x_n-u_n,Ax_n-Aq\rangle-r_n^2\|Ax_n-Aq\|^2.
\end{aligned} \tag{3.10}$$

From (3.7) and (3.10),

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
\leq\ &\alpha_n\|x_n-q\|^2+(1-\alpha_n)[1-(v_n+\lambda_n\mu)]\|u_n-q\|^2+(1-\alpha_n)(v_n+\lambda_n\mu)4K^2 \\
\leq\ &\alpha_n\|x_n-q\|^2+(1-\alpha_n)[1-(v_n+\lambda_n\mu)][\|x_n-q\|^2-\|x_n-u_n\|^2 \\
&+2r_n\langle x_n-u_n,Ax_n-Aq\rangle-r_n^2\|Ax_n-Aq\|^2]+(1-\alpha_n)(v_n+\lambda_n\mu)4K^2 \\
\leq\ &\|x_n-q\|^2+(v_n+\lambda_n\mu)4K^2-(1-\alpha_n)[1-(v_n+\lambda_n\mu)]\|x_n-u_n\|^2 \\
&+2r_n(1-\alpha_n)[1-(v_n+\lambda_n\mu)]\|x_n-u_n\|\|Ax_n-Aq\|,
\end{aligned} \tag{3.11}$$

hence,

$$\begin{aligned}
&(1-b)[1-(v_n+\lambda_n\mu)]\|x_n-u_n\|^2 \\
\le\ &(1-\alpha_n)[1-(v_n+\lambda_n\mu)]\|x_n-u_n\|^2 \\
\le\ &\|x_n-q\|^2-\|x_{n+1}-q\|^2+4(v_n+\lambda_n\mu)K^2 \\
&+2r_n(1-\alpha_n)[1-(v_n+\lambda_n\mu)]\|x_n-u_n\|\|Ax_n-Aq\|.
\end{aligned} \tag{3.12}$$

Letting $n\to\infty$ in (3.12), form (3.9) we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.13}$$

<ins>Step 3</ins> $\lim\limits_{n\to\infty}\|x_n-T_lx_n\|=0$, $\lim\limits_{n\to\infty}\|u_n-T_lu_n\|=0$ $(l=1,2,\cdots,N)$.

Since $\{\|x_n-q\|\}_{n=1}^{\infty}$ is bounded, there exists an $M>0$ such that

$$\|x_n-q\|\le M, \quad \text{for any } n\ge 1. \tag{3.14}$$

We can assume that

$$\lim_{n\to\infty}\|x_n-q\|=d, \tag{3.15}$$

where $d\ge 0$ is some number. Then

$$\|x_{n+1}-q\|=\|\alpha_n(x_n-q)+(1-\alpha_n)(\sigma_n-q)\|. \tag{3.16}$$

By condition (iii) and (3.3)-(3.6), (3.14), (3.15),

$$\begin{aligned}
\limsup_{n\to\infty}\|\sigma_n-q\| &\le \limsup_{n\to\infty}\|[T_{i(n)}^{k(n)}u_n-q\|+\lambda_n\mu\|f(T_{i(n)}^{k(n)}u_n)\| \\
&\le \limsup_{n\to\infty}h_{k(n)}\|x_n-q\|+\lambda_n\mu(LDM+\|f(q)\|) \\
&\le \limsup_{n\to\infty}\|x_n-q\|+v_n\|x_n-q\|+\lambda_n\mu(LDM+\|f(q)\|) \\
&\le \limsup_{n\to\infty}\|x_n-q\|+v_nM+\lambda_n\mu(LDM+\|f(q)\|)\le d.
\end{aligned} \tag{3.17}$$

Thus from (3.15)-(3.17) and Lemma 2.5 we know

$$\lim_{n\to\infty}\|\sigma_n-x_n\|=0. \tag{3.18}$$

By (3.18), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(\alpha_n-1)x_n+(1-\alpha_n)\sigma_n\| \\
&\le (1-\alpha_n)\|\sigma_n-x_n\|\to 0, \quad n\to\infty.
\end{aligned} \tag{3.19}$$

It follows from (3.13) and (3.18) that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-T_{i(n)}^{k(n)}x_n\| &\le \lim_{n\to\infty}\|x_n-\sigma_n\|+\|\sigma_n-T_{i(n)}^{k(n)}x_n\| \\
&\le \lim_{n\to\infty}\|x_n-\sigma_n\|+\|T_{i(n)}^{k(n)}u_n-T_{i(n)}^{k(n)}x_n\|+\lambda_n\mu\|f(T_{i(n)}^{k(n)}u_n)\| \\
&\le \lim_{n\to\infty}\|x_n-\sigma_n\|+h_{k_n}\|u_n-x_n\|+\lambda_n\mu K=0,
\end{aligned} \tag{3.20}$$

and from (3.19) that

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \quad \text{for any } j = 1, 2, \cdots, N. \tag{3.21}$$

Since for any positive integer $n > N$, it can be written as $n = (k(n) - 1)N + i(n)$, $i(n) \in [1, 2, \cdots, N]$. Let $\tau_n = \|x_n - T_{i(n)}^{k(n)} x_n\|$, then $\tau_n \to 0$ and

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - T_{i(n)}^{k(n)} x_n\| + \|T_{i(n)}^{k(n)} x_n - T_n x_n\| \\ &= \tau_n + \|T_{i(n)}^{k(n)} x_n - T_{i(n)} x_n\| \le \tau_n + D\|T_{i(n)}^{k(n)-1} x_n - x_n\| \\ &\le \tau_n + D(\|T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N}\| \\ &\quad + \|T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|).\end{aligned} \tag{3.22}$$

Since for each $n > N$, $n = (n - N)(\text{mod} N)$, again since $n = (k(n) - 1)N + i(n)$, we obtain $n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N)$, that is,

$$k(n - N) = k(n) - 1, \quad i(n - N) = i(n).$$

Therefore we have

$$\begin{aligned}\|T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N}\| &= \|T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N}\| \\ &\le D\|x_n - x_{n-N}\|\end{aligned} \tag{3.23}$$

and

$$\|T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N}\| = \|T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\| = \tau_{n-N}. \tag{3.24}$$

Substituting (3.23) and (3.24) to (3.22), we have

$$\|x_n - T_n x_n\| \le \tau_n + D^2\|x_n - x_{n-N}\| + D\tau_{n-N} + D\|x_{n-N} - x_n\|.$$

By (3.20) and (3.21) we know that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.25}$$

Consequently, for any $j = 1, 2, \cdots, N$, from (3.12) and (3.16) we have

$$\begin{aligned}\|x_n - T_{n+j} x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\le (1 + h_1)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \to 0, \quad n \to \infty.\end{aligned} \tag{3.26}$$

This implies that the sequence

$$\bigcup_{j=1}^{N} \{\|x_n - T_{n+j} x_n\|\}_{n=1}^{\infty} \to 0, \quad n \to \infty.$$

Since for each $l = 1, 2, \cdots, N$, $\{\|x_n - T_l x_n\|\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N} \{\|x_n - T_{n+j} x_n\|\}_{n=1}^{\infty}$, therefore we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \text{for any } l = 1, 2, \cdots, N. \tag{3.27}$$

By (3.13) and (3.27) we have

$$\begin{aligned}\lim_{n\to\infty}\|u_n-T_lu_n\| &\le \|u_n-x_n\|+\|x_n-T_lx_n\|+\|T_lx_n-T_lu_n\| \\ &\le \|x_n-T_lx_n\|+(1+D)\|u_n-x_n\|=0, \quad \text{for any } l=1,2,\cdots,N.\end{aligned} \tag{3.28}$$

Step 4 $\{\omega_w(x_n)\}\subset\Omega$.

Since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $u\in\omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ convergent weakly to u. Hence, from (3.13) we know that $u_{n_i}\rightharpoonup u$. As $\|u_n-T_lu_n\|\to 0$, we obtain that $T_lu_{n_i}\rightharpoonup u$. Next we show $u\in\Omega$. Since

$$G(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0, \quad \text{for any } y\in C.$$

From (A2), we have

$$\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge G(y,u_n).$$

Replacing n by n_i, we have

$$\langle Ax_{n_i},y-u_{n_i}\rangle+\Big\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle\ge G(y,u_{n_i}). \tag{3.29}$$

For $t\in(0,1]$ and $y\in C$, let $y_t=ty+(1-t)u$. Since $y\in C$ and $u\in C$, we have $y_t\in C$. So from (3.29) we know that

$$\begin{aligned}\langle y_t-u_{n_i},Ay_t\rangle &\ge \langle y_t-u_{n_i},Ay_t\rangle-\langle y_t-u_{n_i},Ax_{n_i}\rangle-\Big\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle+G(y_t,u_{n_i}) \\ &= \langle y_t-u_{n_i},Ay_t-Au_{n_i}\rangle+\langle y_t-u_{n_i},Au_{n_i}-Ax_{n_i}\rangle \\ &\quad -\Big\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle+G(y_t,u_{n_i}).\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|Au_{n_i}-Ax_{n_i}\|\to 0$. Further, from the monotonicity of A, we have $\langle y_t-u_{n_i},Ay_t-Au_{n_i}\rangle\ge 0$. Therefore, from (A4) we obtain that

$$\langle y_t-v,Ay_t\rangle\ge G(y_t,v), \tag{3.30}$$

as $i\to\infty$. From (A1), (A4) and (3.30), we also have

$$\begin{aligned}0=G(y_t,y_t) &\le tG(y_t,y)+(1-t)G(y_t,v)\le tG(y_t,y)+(1-t)\langle y_t-v,Ay_t\rangle \\ &= tG(y_t,y)+(1-t)t\langle y-v,Ay_t\rangle.\end{aligned}$$

And hence

$$0\le G(y_t,y)+(1-t)\langle y-v,Ay_t\rangle.$$

Letting $t\to 0$, for each $y\in C$, we have

$$0 \leq G(v, y) + \langle y - v, Av \rangle, \tag{3.31}$$

which implies $v \in GEP(G)$.

On the other hand, since T is a asymptotically nonexpansive mapping, by Lemma 2.6 we know that the mapping $I - T$ is demiclosed at zero. Note that $u_n - T_l u_n \to 0$ and $u_{n_i} \rightharpoonup v$. Thus, $v \in \bigcap_{n=1}^{N} F(T_n)$. Consequently, we deduce that $v \in \Omega$. Since v is an arbitrary element, we conclude that $\omega_w(x_n) \subset \Omega$.

Step 5 $\{x_n\}$, $\{u_n\}$ converge weakly to an element of Ω.

It is sufficient to show that $\omega_w(x_n)$ is a single-point set. Let $u, v \in \Omega$, then $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively.

Assume that $u \neq v$. Since H is a Hilbert space, it satisfies $Opial's$ condition. Thus we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{j\to\infty} \|x_{n_j} - u\| < \lim_{j\to\infty} \|x_{n_j} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{k\to\infty} \|x_{n_k} - v\| \\ &< \lim_{k\to\infty} \|x_k - u\| = \lim_{n\to\infty} \|x_n - u\|,\end{aligned}$$

which is a contradiction. This shows that $\omega_w(x_n)$ is a single-point set. Note that $\lim_{n\to\infty} \|x_n - u_n\| = 0$, hence, both $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of Ω.

Step 6 $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of Ω if and only if $\liminf_{n\to\infty} d(x_n, \Omega) = 0$. From (3.5),(3.14) we have

$$\begin{aligned}\|x_{n+1} - q\| &\leq (1 + v_n + \lambda_n \mu LD)\|x_n - q\| + \lambda_n \mu \|f(q)\| \\ &\leq \|x_n - q\| + v_n M + \lambda_n \mu (LDM + \|f(q)\|) = \|x_n - q\| + \delta_n,\end{aligned} \tag{3.32}$$

where $\delta_n = v_n M + \lambda_n \mu (LDM + \|f(q)\|)$. Hence, $d(x_{n+1}, \Omega) \leq d(x_n, \Omega) + \delta_n$. Since $\sum_{n=1}^{\infty} \delta_n < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, \Omega)$ exists.

If $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of $\{T_1, T_2, \cdots, T_N\}$, then $\lim_{n\to\infty} \|x_n - p\| = 0$. Since

$$0 \leq d(x_n, \Omega) \leq \|x_n - p\|,$$

we know that $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

Conversely, suppose $\liminf_{n\to\infty} d(x_n, \Omega) = 0$, then $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Moreover, we have $\sum_{n=1}^{\infty} \delta_n < \infty$, thus for arbitrary $\epsilon > 0$, there exists a positive integer N such that $d(x_n, \Omega) < \epsilon/4$ and $\sum\limits_{j=n}^{\infty} \delta_j < \epsilon/4$ for all $n \geq N$. It follows from (3.32) that for all $n, m \geq N$ and for all $p \in \Omega$, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_N - p\| + \sum_{j=N+1}^{n} \delta_j + \|x_N - p\| + \sum_{j=N+1}^{m} \delta_j \\ &\leq 2\|x_N - p\| + 2\sum_{j=N}^{\infty} \delta_j.\end{aligned}$$

Taking infimum over all $p \in \Omega$, we obtain

$$\|x_n - x_m\| \leq 2d(x_N, \Omega) + 2\sum_{j=N}^{\infty} \delta_j < \epsilon, \quad \text{for any } n, m \geq N.$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\lim\limits_{n\to\infty} x_n = u$, then from (3.27) and Lemma 2.6 we have $u \in \Omega$. In view of (3.13), we conclude that both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of Ω. This completes the proof.

Theorem 3.2 *Suppose that H is a real Hilbert space, and C is a nonempty closed convex subset of H. Let $\{T_1, T_2, \cdots, T_N\} : C \to C$ be N asymptotically nonexpansive mappings with $\Omega = \bigcap\limits_{n=1}^{N} F(T_i) \cap GEP(G) \neq \emptyset$ and there exists a T_l, $1 \leq l \leq N$ which is semicompact. Let $f : C \to C$ be L-Lipschitzian mapping. If hybrid iteration $\{x_n\}$ defined by (1.5), where μ is a positive fixed constant, $\{\alpha_n\}$, $\{\lambda_n\}$, $\{r_n\}$ and $\{h_n\}$ defined by (2.1) satisfy the following conditions:*

(i) $a \leq \alpha_n \leq b$ *for some* $a, b \in (0, 1)$;

(ii) $\sum\limits_{n=1}^{\infty} (h_n - 1) < \infty$;

(iii) $\{\lambda_n\} \subset [0.1)$, $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$;

(iv) $0 < c \leq r_n \leq d < 2\alpha$;

(v) $1 < h_n + \lambda_n \mu < 2$.

Then, $\{x_n\}$, $\{u_n\}$ converge strongly to an element of Ω.

Proof From the proof of Theorem 3.1, $\{x_n\}$ is bounded, and $\lim\limits_{n\to\infty} \|x_n - T_l x_n\| = 0$, for any $l = 1, 2, \cdots, N$. Especially, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{3.33}$$

By the assumption of Theorem 3.2, we may assume that T_1 is semicomptact, without loss of generality. It follows from (3.33) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $p \in C$ and

$$\|p - T_l p\| = \lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = \lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \text{for any } l = 1, 2, \cdots, N.$$

This implies that $p \in \Omega$. In addition, since $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $\lim_{n\to\infty} \|x_n - p\| = 0$, that is, $\{x_n\}$ converges strongly to $p \in \Omega$. By (3.13) we know that $\{u_n\}$ also converges strongly to $p \in \Omega$. The proof is completed.

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