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ASYMPTOTIC BEHAVIOR OF WAVE EQUATION OF KIRCHHOFF TYPE WITH STRONG DAMPING*†

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Abstract

The paper deals with the strongly damped nonlinear wave equation of Kirchhoff type. The existence of a global attractor is proven by using the decomposition, and moreover, the structure of the global attractor is established. Our results improve the previous results.

Keywords wave equations; global attractors; critical nonlinearity
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1 Introduction

The nonlinear evolution equations have been investigated by many authors. We consider the following problem

$$\begin{aligned} &u_{tt} - M(\|\nabla u\|^2)\triangle u - \triangle u_t + f(u_t) + g(u) = h(x), \quad x \in \Omega,\ t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\ u|_{\partial\Omega} = 0, \end{aligned} \tag{1}$$

where $M(s) = 1 + s^{\frac{m}{2}}$, $m \geq 2$, $\Omega \subset \mathbf{R^3}$ is a bounded domain with smooth boundary $\partial\Omega$. The assumptions on $f(u_t)$, $g(u)$ and $h(x)$ will be specified below.

When $N = 1$, such an equation without the dissipative term $\triangle u_t$ is introduced to describe the vibration of an elastic string. The original equation is

$$\rho h u_{tt} + \delta u_t = \Big\{p_0 + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 \mathrm{d}x\Big\}\frac{\partial^2 u}{\partial x^2} + f,$$

for $0 < x < L,\ t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coor-

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dinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length, p_0 is the initial axial tension, δ is the resistance modulus, and f is the external force. When $\delta = f = 0$, the equation is firstly introduced by Kirchhoff [6].

Equation (1) is also mathematically interesting and has been extensively investigated by many authors. By using asymptotic compactness the authors dealt with some absorbing properties of global attractor of the Kirchhoff type equation

$$u_{tt} - M(\|\nabla u\|^2)\triangle u - \triangle u_t + g(x,u) = h(x),$$

where g does not exhibit a critical growth [8].

The paper [13] studied the longtime behavior of the Kirchhoff type equation

$$u_{tt} - M(\|\nabla u\|^2)\triangle u - \triangle u_t + u + u_t + g(x,u) = h(x)$$

on R^n. It showed that the related continuous semigroup possesses a global attractor which is connected and has finite fractal and Hausdorff dimensions.

In [9] by using two half invariant sets, the author proved the existence and some absorbing properties of an attractor in a local sense for the initial boundary value problem of a quasilinear wave equation of Kirchhoff type

$$u_{tt} - (1 + \|\nabla u\|_2^2)\triangle u + u_t + g(x,u) = h(x).$$

Nonlinear evolution equations have been investigated by many authors, see [1-5,8-14], but, there are relatively few results on the global attractor for problem (1), where the functions f and g exhibit a critical growth. The problem considered in this manuscript is more difficult to be dealt with than those considered in [8,9] because the difficulty is caused not only by the critical growth of f and g but also by the nonlinearity of M. The aim of this paper is to improve the main results of [8,9], that is, by utilizing the decomposition idea [10] we prove the existence of a global attractor of (1). Still, the structure of the global attractor is established.

2 Preliminary

We first introduce the following notations:

$$L^p = L^p(\Omega), \quad H^k = H^k(\Omega), \quad H_0^1 = H_0^1(\Omega), \quad \|\cdot\|_p = \|\cdot\|_{L^p}, \quad \|\cdot\| = \|\cdot\|_{L^2},$$

with $p \geq 1$. The notations $(\cdot,\cdot)$ and $[\cdot,\cdot]$ will be used as the L^2-inner product and the duality pairing between dual spaces respectively. For brevity, we use the same letter C to denote different positive constants, and $C(\cdots)$ to denote positive constants depending on the quantities appearing in the parenthesis. In L^2 we introduce the operator $-\triangle$ with the domain $D(-\triangle) = H^2 \cap H_0^1$, where $-\triangle$ is the Laplace operator in Ω with the Dirichlet boundary condition. Below we denote by e_k the orthonormal basis in L^2 consisting of eigenfunctions of the operator $-\triangle$:

$$-\triangle e_k = \lambda_k e_k, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Some assumptions on f, g and h are necessary for formulating problem (1).

(D1) Let $f(s) = \phi(s) - \lambda s$, with $\phi \in C^1(\mathbf{R})$, $0 \le \phi'(s) \le Cs^4$, $\lambda < \lambda_1$ and $\phi(0) = 0$.

(D2) Let $g(s) = \gamma(s) - \theta s$, with $\gamma \in C^1(\mathbf{R})$, $\theta < \lambda_1$, $0 \le \int_0^s \gamma(y)\mathrm{d}y \le s\gamma(s)$, $|\gamma'(s)| \le Cs^4$ and $\gamma(0) = 0$.

(D3) Let $h \in L^2$.

Due to D1 and D2, the functions given by

$$\Phi_0(u) = 2\int_\Omega \int_0^{u(x)} \phi(y)\mathrm{d}y\mathrm{d}x, \quad \Phi_1(u) = [\phi(u), u],$$

$$\Gamma_0(u) = 2\int_\Omega \int_0^{u(x)} \gamma(y)\mathrm{d}y\mathrm{d}x \quad \text{and} \quad \Gamma_1(u) = [\gamma(u), u]$$

fulfill for every $u \in H_0^1$ the inequalities

$$0 \le \Phi_0(u) \le 2\Phi_1(u) \quad \text{and} \quad 0 \le \Gamma_0(u) \le 2\Gamma_1(u).$$

Moreover, since

$$|\phi(s)|^{\frac{6}{5}} = |\phi(s)|^{\frac{1}{5}}|\phi(s)| \le C|s||\phi(s)| = C\phi(s)s,$$

we deduce that for some $C > 0$ sufficiently large

$$\|\phi(u)\|_{L^{\frac{6}{5}}} \le C[\Phi_1(u)]^{\frac{5}{6}}, \quad u \in H_0^1.$$

We rewrite (1) in the equivalent form

$$u_{tt} - M(\|\nabla u\|^2)\triangle u - \triangle u_t + \phi(u_t) + \gamma(u) - \lambda u_t - \theta u = h. \tag{2}$$

Given a constant $\varepsilon > 0$, we set

$$\Pi_\varepsilon(u) = \|u_t\|^2 + (1+\varepsilon)\|\nabla u\|^2 - (\theta + \lambda\varepsilon)\|u\|^2 + \frac{2}{m+2}\|\nabla u\|^{m+2} + 2\varepsilon(u, u_t) \tag{3}$$

and

$$\Sigma_\varepsilon(u) = \|\nabla u_t\|^2 + M(\|\nabla u\|^2)\|\triangle u\|^2 - (\theta + \lambda\varepsilon)\|\nabla u\|^2 + \varepsilon\|\triangle u\|^2 + 2\varepsilon(\nabla u, \nabla u_t). \tag{4}$$

Lemma 2.1[4] *Given $k \ge 1$ and $C \ge 0$, let $\Lambda_\varepsilon : [0, \infty) \to [0, \infty)$ be a family of absolutely continuous functions satisfying the following inequalities for every $\varepsilon > 0$ small enough*

$$\frac{1}{k}\Lambda_0 \le \Lambda_\varepsilon \le k\Lambda_0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon(t) + \varepsilon\Lambda_\varepsilon(t) \le C\varepsilon^6\Lambda_\varepsilon^3(t) + C.$$

Then there are constants $\delta > 0$, $R \geq 0$ and an increasing function $J \geq 0$ such that

$$\Lambda_0(t) \leq J(\Lambda_0(0))\mathrm{e}^{-\delta t} + R.$$

Lemma 2.2[5] *Let $\Lambda : R^+ \to R^+$ be an absolutely continuous function satisfying, for some $\nu > 0$ and $k \geq 0$, the following inequality holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\nu\Lambda(t) \leq \mu(t)\Lambda(t) + k,$$

where $\mu : R^+ \to R^+$ fulfills

$$\int_t^T \mu(\tau)\mathrm{d}\tau \leq \nu(T - t) + m$$

for every $T > t \geq 0$ and some $m \geq 0$. Then

$$\Lambda(t) \leq \Lambda(0)\mathrm{e}^{m-\nu t} + \frac{k\mathrm{e}^m}{\nu}.$$

Lemma 2.3[10] *Let V, H, V' be three Hilbert spaces, each space included and dense in the following one as in $V \subset H \subset V'$, V' being the dual of V. If a function u belongs to $L^2(0, T; V)$ and its derivative u' belongs to $L^2(0, T; V')$, then u is almost everywhere equal to a function continuous from $[0, T]$ into H and we have the following equality which holds in the scalar distribution sense on $(0, t)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = 2(u, u').$$

Lemma 2.4[7] *Assume that X, B and Y are Banach spaces with $X \subset B \subset Y$, where the imbedding $X \subset B$ is compact.*

(1) *Let the set F be bounded in $L^p(0, T; X)$ where $1 \leq p < \infty$, and the set $Q = \{f' | f \in F\}$ be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$.*

(2) *Let the set F be bounded in $L^\infty(0, T; X)$ and the set $Q = \{f' | f \in F\}$ be bounded in $L^r(0, T; Y)$ where $r > 1$. Then F is relatively compact in $C(0, T; B)$.*

Definition 2.1 A function $u(t)$ is said to be a weak solution of (1) on $[0, T]$ if

$$u \in L^\infty(0, T; H_0^1) \cap L^2(0, T; H_0^1), \quad u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1),$$

and for almost every $t \in [0, T]$ and every $\varphi \in H_0^1$ the equality

$$[u_{tt}, \varphi] + M(\|u\|^2)(\nabla u, \nabla\varphi) + (\nabla u_t, \nabla\varphi) + [f(u_t), \varphi] + [g(u), \varphi] = (h, \varphi).$$

Definition 2.2 A function $u(t)$ is said to be a strong solution of (1) on $[0, T]$ if

$$u \in L^\infty(0, T; H_0^1 \cap H^2), \quad u_t \in L^\infty(0, T; H_0^1) \cap L^2(0, T; H_0^1 \cap H^2), \quad u_{tt} \in L^2(0, T; L^2),$$

and for almost every $t \in [0, T]$ and every $\varphi \in L^6$ the equality

$$(u_{tt}, \varphi) - M(\|u\|^2)(\Delta u, \varphi) - (\Delta u_t, \varphi) + [f(u_t), \varphi] + [g(u), \varphi] = (h, \varphi).$$

3 The Existence of A Strong Solution

We first make a prior estimate to the solution of (1).

Lemma 3.1 *Assume that* (D1), (D2) *and* (D3) *hold. If* $(u_0; u_1) \in (H^2 \cap H_0^1) \times H_0^1$, *we have*

$$\|u_t\|^2 + \|\nabla u\|^2 \leq C(\|\nabla u_0\|, \|u_1\|)\mathrm{e}^{-\delta t} + R.$$

Proof Taking the dual product of (2) with $2u_t + 2\varepsilon u$, we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon(u) + \varepsilon\Lambda_\varepsilon(u) + \frac{\lambda_1 - \lambda}{2\lambda_1}\|\nabla u_t\|^2 + \frac{\lambda_1 - \theta}{4\lambda_1}\varepsilon\|\nabla u\|^2 + \Phi_1(u_t) \leq C\varepsilon^6\Lambda_\varepsilon^3(u) + C, \quad (5)$$

where Definition 2.2 and the facts

$$\Lambda_\varepsilon(u) = \Pi_\varepsilon(u) + \Gamma_0(u),$$

$$2\varepsilon[\phi(u_t), u] \leq 2\varepsilon\|u\|_{L^6}\|\phi(u_t)\|_{L^{\frac{6}{5}}} \leq C\varepsilon[\Phi_1(u_t)]^{\frac{5}{6}}\|\nabla u\| \leq \Phi_1(u_t) + C\varepsilon^6\Lambda_\varepsilon^3(u)$$

and

$$2(h, u_t + \varepsilon u) \leq \frac{\lambda_1 - \lambda}{2\lambda_1}\|\nabla u_t\|^2 + \frac{\lambda_1 - \theta}{4\lambda_1}\varepsilon\|\nabla u\|^2 + C$$

are used. And (5) implies

$$\|u_t\|^2 + \|\nabla u\|^2 \leq C(\|\nabla u_0\|, \|u_1\|)\mathrm{e}^{-\delta t} + R,$$

according to Lemma 2.1.

Lemma 3.2 *Assume that* (D1), (D2) *and* (D3) *hold. If* $(u_0; u_1) \in (H^2 \cap H_0^1) \times H_0^1$, *we have*

$$\int_0^t \|\nabla u_t(\tau)\|^2 \mathrm{d}\tau \leq C.$$

Proof Taking the dual product of (2) with $2u_t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|u_t\|^2 + \int_0^{\|\nabla u\|^2} M(s)\mathrm{d}s + \Gamma_0(u) - 2(u, h) - \theta\|u\|^2\Big] + \frac{2\lambda_1 - 2\lambda}{\lambda_1}\|\nabla u_t\|^2 + 2\Phi_1(u_t) \leq 0, \quad (6)$$

where we use (D1), (D2) and Definition 2.2. And (6) implies

$$\int_0^t [\|\nabla u_t(\tau)\|^2 + \Phi_1(u_t(\tau))]\mathrm{d}\tau \leq C.$$

Lemma 3.3 *Assume that* (D1), (D2) *and* (D3) *hold. If* $(u_0; u_1) \in (H^2 \cap H_0^1) \times H_0^1$ *and* $h = 0$, *we have*

$$\int_0^t \|\nabla u(\tau)\|^2 \mathrm{d}\tau \leq C.$$

Proof Taking the dual product of (2) with $2u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla u\|^2 + 2(u, u_t) - \lambda\|u\|^2] + \frac{\lambda_1 - \theta}{\lambda_1}\|\nabla u\|^2 + 2\|\nabla u\|^{m+2} + 2[\gamma(u), u] \leq C\Phi_1(u_t) + 2\|u_t\|^2, \quad (7)$$

where we use Definition 2.2 and the fact

$$2[\phi(u_t),u] \le C[\Phi_1(u_t)]^{\frac{5}{6}}\|\nabla u\| \le C[\Phi_1(u_t)]^{\frac{5}{6}}\|\nabla u\|^{\frac{1}{3}} \le \frac{\lambda_1-\theta}{\lambda_1}\|\nabla u\|^2 + C\Phi_1(u_t).$$

It is apparent that (6) and (7) imply $\int_0^t \|\nabla u(\tau)\|^2 \mathrm{d}\tau \le C$, since $[\gamma(u),u] \ge 0$.

Lemma 3.4 *Assume that* (D1), (D2) *and* (D3) *hold. If* $(u_0;u_1) \in (H^2 \cap H_0^1) \times H_0^1$, *we have*

$$\|\nabla u_t(t)\| \le C.$$

Proof Using Definition 2.2 and taking the dual product of (2) with $2u_{tt}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u) - 2(h,u_{tt}) + 2\theta\|u_t\|^2 + 2\|u_{tt}\|^2 - 2[\gamma'(u)u_t,u_t]\\ &= 2M(\|\nabla u\|^2)\|\nabla u_t\|^2 + 4M'(\|\nabla u\|^2)(\nabla u,\nabla u_t)^2,\end{aligned}$$

where

$$\begin{aligned}\Psi(u) = \|\nabla u_t\|^2 + 2[\gamma(u),u_t] - \lambda\|u_t\|^2 + 2\int_\Omega\int_0^{u_t}\phi(s)\mathrm{d}s\mathrm{d}x\\ -2\theta(u,u_t) + 2M(\|\nabla u\|^2)(\nabla u,\nabla u_t) + D,\end{aligned}$$

with $D>0$ being sufficiently large. The estimates

$$2[\gamma'(u)u_t,u_t] + 2(h,u_{tt}) \le C\|\nabla u_t\|^2 + \|u_{tt}\|^2 + C$$

and

$$2M(\|\nabla u\|^2)\|\nabla u_t\|^2 + 4M'(\|\nabla u\|^2)(\nabla u,\nabla u_t)^2 \le C\|\nabla u_t\|^2$$

give

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u) \le C\|\nabla u_t\|^2 + C.$$

Thus, for every fixed $T>0$, integrating the above inequality over $[t,T]$ for some positive $t \ge T-1$ and using Lemma 3.2, we arrive at

$$\|\nabla u_t(T)\|^2 \le 2\Psi(u(T)) \le 2\Psi(u(t)) + C \le C(1+\|\nabla u_t(t)\|^6),$$

by noting $\|\nabla u_t(t)\|^2 \le 2\Psi(u(t)) \le C(1+\|\nabla u_t(t)\|^6)$. If $T \le 1$ we choose $t=0$. Otherwise we observe that, in view of $\int_0^t \|\nabla u_t(\tau)\|^2\mathrm{d}\tau \le C$, there exists a $K = K(\|\nabla u_0\|,\|u_1\|) \ge 0$ such that, for some $t_T \in [T-1,T]$, $\|\nabla u_t(t_t)\| \le K$. Choosing $t = t_T$ the proof is finished.

Lemma 3.5 *Assume that* (D1), (D2) *and* (D3) *hold. If* $(u_0;u_1) \in (H^2 \cap H_0^1) \times H_0^1$, *we have*

$$\|\nabla u_t\|^2 + \|\triangle u\|^2 \le C, \qquad \int_0^T [\|\triangle u(t)\|^2 + \|\triangle u_t(t)\|^2]\mathrm{d}t \le C(T). \tag{8}$$

Proof Using Definition 2.2 and taking the dual product of (2) with $-2\triangle u_t - \varepsilon\triangle u$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Theta(u)+\varepsilon\Theta(u)+\frac{\lambda_1-\theta}{2\lambda_1}\varepsilon\|\triangle u\|^2+\frac{\lambda_1-\lambda}{\lambda_1}\|\triangle u_t\|^2+2[\gamma'(u)u_t,\triangle u]\\ \le\ &2[\phi(u_t),\triangle u_t+\varepsilon\triangle u]-2(h,\triangle u_t+\varepsilon\triangle u)+2M'(\|\nabla u\|^2)\|\triangle u\|^2(\nabla u,\nabla u_t),\end{aligned}$$

where

$$\Theta(u)=\Sigma_\varepsilon(u)-2[\gamma(u),\triangle u].$$

Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta(u)+\varepsilon\Theta(u)+\frac{\lambda_1-\theta}{2\lambda_1}\varepsilon\|\triangle u\|^2+\frac{\lambda_1-\lambda}{\lambda_1}\|\triangle u_t\|^2\le C+C\|\triangle u\|^2\|\nabla u_t\|, \tag{9}$$

where the estimates

$$2M'(\|\nabla u\|^2)\|\triangle u\|^2(\nabla u,\nabla u_t)\le C\|\triangle u\|^2\|\nabla u_t\|,$$

$$-2(h,\triangle u_t+\varepsilon\triangle u)\le\frac{\lambda_1-\theta}{8\lambda_1}\varepsilon\|\triangle u\|^2+\frac{\lambda_1-\lambda}{4\lambda_1}\|\triangle u_t\|^2+C,$$

$$2[\gamma'(u)u_t,\triangle u]\le C(\|\triangle u\|^{\frac12}\|\nabla u\|^{\frac12})^2\|\nabla u\|^2\|\triangle u\|\|\nabla u_t\|\le C\|\triangle u\|^2\|\nabla u_t\|$$

and

$$2\varepsilon[\phi(u_t),\triangle u]\le C\varepsilon\|\nabla u_t\|^4\|\triangle u\|\|\triangle u_t\|\le\frac{\lambda_1-\theta}{8\lambda_1}\varepsilon\|\triangle u\|^2+\frac{\lambda_1-\lambda}{4\lambda_1}\|\triangle u_t\|^2$$

are used. Thus (9) implies (8), according to Lemma 2.2 and the fact

$$C\int_t^T\|\nabla u_t(\tau)\|\mathrm{d}\tau\le\sqrt{T-t}\Big[\int_t^T\|\nabla u_t(\tau)\|^2\mathrm{d}\tau\Big]^{\frac12}\le\frac{\varepsilon}{2}(T-t)+C.$$

Theorem 3.1 *Assume that* (D1), (D2) *and* (D3) *hold. For every* $T>0$ *and every* $(u_0;u_1)\in(H^2\cap H_0^1)\times H_0^1$, *there is a strong solution* u *of problem* (1) *on* $[0,T]$.

Proof To prove the existence of strong solutions, we use the standard Galerkin method. We seek for approximate solutions of the form

$$u^n(t)=\sum_{k=1}^n T_{jn}(t)e_j,\quad n=1,2,\cdots,$$

where $-\triangle e_j=\lambda_j e_j$ and $T_{jn}(t)=(u^n,e_j)$ with

$$(u_{tt}^n,e_j)+(M(\|\nabla u^n\|^2)\nabla u^n,\nabla e_j)+(\nabla u_t^n,\nabla e_j)+[\phi(u_t^n)-\lambda u_t^n+\gamma(u^n)-\theta u^n-h,e_j]=0, \tag{10}$$

where

$$(u^n(0);u_t^n(0))\to(u_0;u_1)\ \text{ in }\ (H^2\cap H_0^1)\times H_0^1\ \text{ as }\ n\to\infty.$$

Obviously, in Lemmas 3.1-3.5, the estimates with respect to u still hold for u^n. Hence, we can extract a subsequence, still denoted by u^n, such that

$$u^n \to u \ \text{ in } \ L^\infty(0,\infty; H^2\cap H_0^1) \ \text{ weak}^\star, \tag{11}$$
$$u_t^n \to u_t \ \text{ in } \ L^\infty(0,\infty; H_0^1)\cap L^2(0,\infty; H_0^1) \ \text{ weak}^\star, \tag{12}$$
$$u_{tt}^n \to u_{tt} \ \text{ in } \ L^2(0,T; L^2) \ \text{ weak} \tag{13}$$

and

$$M(\|\nabla u^n\|^2)\triangle u^n \to \chi \ \text{ in } \ L^\infty(0,\infty; L^2) \ \text{ weak}^\star. \tag{14}$$

We must show $M(\|\nabla u\|^2)\triangle u = \chi$. In fact the facts

$$\gamma(u^n) \to \gamma(u) \ \text{ in } \ L^{\frac{6}{5}}(\Omega_T) \ \text{ weak}$$

and

$$\phi(u_t^n) \to \phi(u_t) \ \text{ in } \ L^{\frac{6}{5}}(\Omega_T) \ \text{ weak}$$

follow from (11)-(13), where $\Omega_T = \Omega\times(0,T)$.

For any $\eta\in C_0(0,\infty; L^2)$ and $T>0$, using Lemma 2.4 we have as $n\to\infty$

$$\begin{aligned}
&\int_0^T (\chi - M(\|\nabla u(\tau)\|^2)\triangle u(\tau), \eta)\mathrm{d}\tau \\
=&\int_0^T (M(\|\nabla u^n(\tau)\|^2) - M(\|\nabla u(\tau)\|^2))(\triangle u^n(\tau), \eta)\mathrm{d}\tau \\
&+\int_0^T M(\|\nabla u(\tau)\|^2)(\triangle u^n(\tau) - \triangle u(\tau), \eta)\mathrm{d}\tau \\
&+\int_0^T (\chi - M(\|\nabla u^n(\tau)\|^2)\triangle u^n(\tau), \eta)\mathrm{d}\tau \to 0,
\end{aligned}$$

since

$$\begin{aligned}
&\Big|\int_0^T [M(\|\nabla u^n(\tau)\|^2) - M(\|\nabla u(\tau)\|^2)](\triangle u^n(\tau), \eta)\mathrm{d}\tau\Big| \\
\le& C\int_0^T |(\nabla u^n(\tau) - \nabla u(\tau), \nabla u^n(\tau) + \nabla u(\tau))|\mathrm{d}\tau \\
\le& C\int_0^T \|\nabla u^n(\tau) - \nabla u(\tau)\|\mathrm{d}\tau \to 0 \quad \text{as} \ \ n\to\infty.
\end{aligned}$$

4 The Semigroup

In this section we give the existence and uniqueness of a weak solution of (1).

Theorem 4.1 *Assume that* (D1), (D2) *and* (D3) *hold. For every* $T>0$ *and every* $(u_0;u_1)\in H_0^1\times L^2$, *there is a unique weak solution* u *of* (1) *on* $[0,T]$. *Moreover, there exists a constant* $C(T,\varepsilon)$ *such that the difference* $w=u-v$ *satisfies the estimate*

$$\|w_t\|^2 + \|\nabla w\|^2 \le C(T,\varepsilon)[\|w_t(0)\|^2 + \|\nabla w(0)\|^2],$$

where u and v are two solutions of (1) *in the space* $C(\mathbf{R}^+; H_0^1\times L^2)$ *with initial datas* $(u_0; u_1)$ *and* $(v_0; v_1)$ *respectively.*

Proof Let u and v be two solutions of (1) with to initial datas $(u_0; u_1)$ and $(v_0; v_1)$ respectively. Then $w = u - v$ satisfies

$$\begin{aligned} &w_{tt} - \Delta w_t - M(\|\nabla u\|^2)\Delta w + \phi(u_t) - \phi(v_t) + \gamma(u) - \gamma(v) - \lambda w_t - \theta w \\ &= [M(\|\nabla u\|^2) - M(\|\nabla v\|^2)]\Delta v. \end{aligned} \tag{15}$$

Using Definition 2.1 and taking the dual product of (15) with $2w_t + 2\varepsilon w$, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|w_t\|^2 + \varepsilon\|\nabla w\|^2 + 2\varepsilon(w, w_t)] + 2\|\nabla w_t\|^2 + 2M(\|\nabla u\|^2)(\nabla w, \nabla w_t + \varepsilon\nabla w) \\ &= 2(\varepsilon+\lambda)\|w_t\|^2 + 2[\phi(v_t) - \phi(u_t) + \gamma(v) - \gamma(u), w_t + \varepsilon w] + 2\theta\varepsilon\|w\|^2 \\ &\quad +2[M(\|\nabla v\|^2) - M(\|\nabla u\|^2)](\nabla v, \nabla w_t + \varepsilon\nabla w) + 2(\theta + \lambda\varepsilon)(w, w_t). \end{aligned} \tag{16}$$

Using (16), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|w_t\|^2 + \varepsilon\|\nabla w\|^2 + 2\varepsilon(w, w_t)] \le C[\|w_t\|^2 + \varepsilon\|\nabla w\|^2 + 2\varepsilon(w, w_t)], \tag{17}$$

where the facts

$$2[M(\|\nabla v\|^2) - M(\|\nabla u\|^2)](\nabla v, \nabla w_t + \varepsilon\nabla w) \le \frac{1}{2}\|\nabla w_t\|^2 + C\|\nabla w\|^2,$$

$$2\varepsilon[\phi(u_t) - \phi(v_t), w] \le \frac{1}{2}\|\nabla w_t\|^2 + C\|\nabla w\|^2,$$

$$2M(\|\nabla u\|^2)(\nabla w, \nabla w_t) \le \frac{1}{4}\|\nabla w_t\|^2 + C\|\nabla w\|^2$$

and

$$2[\gamma(u) - \gamma(v), w_t + \varepsilon w] \le \frac{1}{4}\|\nabla w_t\|^2 + C\|\nabla w\|^2$$

are used. And (17) implies

$$\|w_t\|^2 + \|\nabla w\|^2 \le C(T, \varepsilon)[\|w_t(0)\|^2 + \|\nabla w(0)\|^2]. \tag{18}$$

Therefore, we arrive at the uniqueness of a strong solution of (1).

By using limit process and (18) we can prove the existence of a weak solution with initial data $(u_0; u_1) \in H_0^1\times L^2$. Indeed, we can choose a sequence $(u_0^n; u_1^n)$ from $(H^2\cap H_0^1)\times H_0^1$ such that $(u_0^n; u_1^n) \to (u_0; u_1)$ in $H_0^1\times L^2$. Due to (18), the solutions $u^n(t)$ with initial data $(u_0^n; u_1^n)$ converge to a function $u(t)$ in the sense that

$$\max_{t\in[0,T]}[\|u_t^n(t) - u_t(t)\|^2 + \|\nabla u^n(t) - \nabla u(t)\|^2] \to 0 \quad \text{as} \quad n\to\infty.$$

From the boundedness provided by Lemmas 3.1 and 3.2 we also have weak* convergence of $(u^n; u_t^n)$ to $(u; u_t)$ in the space

$$[L^\infty(0, T; H_0^1)\cap L^2(0, T; H_0^1)]\times[L^\infty(0, T; L^2)\cap L^2(0, T; H_0^1)].$$

This implies that $u(t)$ is a weak solution. By (18) we get the uniqueness of a weak solution of (1). Therefore, we have

$$(u(t); u_t(t)) \in C([0,T]; H_0^1 \times L^2),$$

according to Lemma 2.3. The arbitrariness of T proves the existence of the solution when $t \in (0,\infty)$.

A semigroup of operators is a family of operators $S(t)$, $t \geq 0$, that map the space E into itself and enjoy the properties: $S(t+s) = S(t)S(s)$, $s,t \geq 0$ and $S(0) = I$. The main consequence of Theorem 4.1 is that the mappings

$$S : (u_0; u_1) \to (u(t); u_t(t)), \quad t \geq 0,$$

define a strongly continuous semigroup (or dynamical system) on $E = H_0^1 \times L^2$, where u is the weak solution of (1).

5 The Existence of A Global Attractor

In this section we shall give the definition of the global attractor and prove that the dynamical system possesses a global attractor.

Definition 5.1 The global attractor of a semigroup $S : X \to X$ is a compact set $A \subset X$ satisfying

(i) A is fully invariant for $S(t)$, that is, $S(t)A = A$ for every $t \geq 0$;

(ii) A is an attracting set for $S(t)$, that is,

$$\lim_{t\to+\infty} \delta_X(S(t)B, A) = 0,$$

for every bounded set $B \subset X$, where δ_X denotes the usual Hausdorff semidistance in Banach space X.

Remark 5.1 Definition 5.1 implies that the global attractor is a compact set, but in [3] the global attractor is a bounded closed set.

In order to obtain the existence of a global attractor we need a lemma.

Lemma 5.1[9] *Let X be a Banach space and $S(t)$ be a continuous semigroup on X. Then $S(t)$ possesses a global attractor which is connected if the following conditions are satisfied:*

(i) *There exists a bounded absorbing set B_0 in X, that is*

$$\lim_{t\to+\infty} \delta_X(S(t)B, B_0) = 0,$$

for every bounded set $B \subset X$.

(ii) *$S(t) = S_1(t) + S_2(t)$, where $S_i : X \to X$, $i = 1, 2$, $S_1(t)$ is precompact for $t > T_0$, for some T_0, and $S_2(t)$ is a continuous mapping from X into itself with the property that, for any bounded set $B \subset X$,*

$$r_B(t)=\sup_{\varphi\in B}\|S_2(t)\varphi\|\to 0 \quad as \quad t\to+\infty.$$

Theorem 5.1 *Assume that* (D1), (D2) *and* (D3) *hold. Let* $h=0$. *Then the dynamical system* $S(t)$ *on* E *admits a global attractor* A *contained and bounded in* $(H^2\cap H_0^1)\times H_0^1$.

Proof From Lemma 3.1 we conclude that

$$B_0=\{(u;v)\in E|\ \|(u;v)\|_E^2=\|\nabla u\|^2+\|v\|^2\le R+1\}$$

is an absorbing set of $S(t)$.

We define operators $S_1: E\to E$, $S_1(t)(0;0)=(w(t);w_t(t))$ with

$$w_{tt}-M(\|\nabla w\|^2)\triangle w-\triangle w_t+\phi(w_t)+\gamma(w)-\lambda w_t-\theta w=0, \tag{19}$$
$$w(x,0)=0,\quad w_t(x,0)=0,\quad x\in\Omega,\quad w|_{\partial\Omega}=0. \tag{20}$$

We learn from Lemmas 3.2, 3.3 and 3.5 that

$$\|\triangle w(t)\|+\|\nabla w_t(t)\|\le C,\quad \int_0^T[\|\triangle w_t(\tau)\|^2+\|\triangle w(\tau)\|^2]\mathrm{d}\tau\le C(T) \tag{21}$$

and

$$\int_0^t[\|\nabla w_t(\tau)\|^2+\|\nabla w(\tau)\|^2]\mathrm{d}\tau\le C.$$

And we define operators $S_2: E\to E$, $S_2(t)(u_0;u_1)=(v(t);v_t(t))$ with

$$\begin{aligned}&v_{tt}-M(\|\nabla u\|^2)\triangle v-\Delta v_t+\phi(u_t)-\phi(w_t)+\gamma(u)-\gamma(w)\\&=\lambda v_t+\theta v+[M(\|\nabla u\|^2)-M(\|\nabla w\|^2)]\triangle w, \qquad (22)\\&v(x,0)=u_0(x),\quad v_t(x,0)=u_1(x),\quad x\in\Omega,\quad v|_{\partial\Omega}=0. \qquad (23)\end{aligned}$$

Taking the dual product of (22) with $2v_t+2\varepsilon v$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Xi(v)+\varepsilon\Xi(v)+2\|\nabla v_t\|^2-(2\lambda+3\varepsilon)\|v_t\|^2+(\varepsilon-\varepsilon^2)\|\nabla v\|^2+(\lambda\varepsilon^2-\theta\varepsilon)\|v\|^2\\&=2\varepsilon^2(v,v_t)+2[\phi(w_t)-\phi(u_t)+\gamma(w)-\gamma(u),v_t+\varepsilon v]-2\|\nabla u\|^m(\nabla v,\nabla v_t+\varepsilon\nabla v)\\&\quad+2[M(\|\nabla w\|^2)-M(\|\nabla u\|^2)](\nabla w,\nabla v_t+\varepsilon\nabla v),\end{aligned}$$

where

$$\Xi(v)=\|v_t\|^2+(1+\varepsilon)\|\nabla v\|^2+2\varepsilon(v,v_t)-(\theta+\lambda\varepsilon)\|v\|^2.$$

Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Xi(v)+\varepsilon\Xi(v)\le C(\|\nabla u\|^2+\|\nabla w\|^2)\Xi(v), \tag{24}$$

where the estimates

$$
\begin{aligned}
&2\varepsilon[\phi(w_t)-\phi(u_t),v] \le C\varepsilon\|\nabla v_t\|\|\nabla v\| \le \frac{\lambda_1-\lambda}{4\lambda_1}\|\nabla v_t\|^2+\frac{\lambda_1-\theta}{4\lambda_1}\varepsilon\|\nabla v\|^2,\\
&2[\gamma(u)-\gamma(w),v_t] \le C\int_\Omega (u^4+w^4)|v||v_t|\mathrm{d}x\\
&\qquad\qquad \le \frac{\lambda_1-\lambda}{4\lambda_1}\|\nabla v_t\|^2+C(\|\nabla u\|^2+\|\nabla w\|^2)\|\nabla v\|^2,\\
&2\varepsilon[\gamma(u)-\gamma(w),v] \le C\varepsilon\int_\Omega (u^4+w^4)|v|^2\mathrm{d}x \le C(\|\nabla u\|^2+\|\nabla w\|^2)\|\nabla v\|^2,\\
&2\|\nabla u\|^m(\nabla v,\nabla v_t+\varepsilon\nabla v) \le \frac{\lambda_1-\lambda}{4\lambda_1}\|\nabla v_t\|^2+C\|\nabla v\|^2\|\nabla u\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&2[M(\|\nabla w\|^2)-M(\|\nabla u\|^2)](\nabla w,\nabla v_t+\varepsilon\nabla v)\\
\le{}& \frac{\lambda_1-\lambda}{4\lambda_1}\|\nabla v_t\|^2+C(\|\nabla u\|^2+\|\nabla w\|^2)\|\nabla v\|^2
\end{aligned}
$$

are used. And from Lemma 2.2 (24) implies

$$
\|(v(t);v_t(t))\|_E \le C\|(u_0;u_1)\|_E \mathrm{e}^{-w_0 t},
$$

since $w_0>0$.

Since the embedding $(H^2\times H_0^1)\times H_0^1 \hookrightarrow H_0^1\times L^2$ is compact, (21) implies that for any bounded set $B\subset E$, $\bigcup\limits_{t\ge 0} S_1(t)B$ is relatively compact in E, that is, $S_1(t)$ is precompact. Therefore, according to Lemma 5.1, $S(t)$ possesses a global attractor A contained and bounded in $(H^2\cap H_0^1)\times H_0^1$. Moreover A is connected.

6 Structure of the Global Attractor

We finally discuss the structure of the global attractor.

Definition 6.1 The continuous function $L(y)$ defined on Y is called the Lyapunov function of the dynamical system $(S(t),X)$ on Y if the following conditions hold:

(a) For any $y\in Y$ the function $L(S(t)y)$ is a nonincreasing function with respect to $t\ge 0$;

(b) if for some $t_0>0$ and $y\in X$ the equation $L(y)=L(S(t_0)y)$ holds, then $y=S(t)y$ for all $t\ge 0$, that is, y is a stationary point of the semigroup $S(t)$.

Let $Y\subset X$ be an invariant set of the dynamical system $(S(t),X)$. Then the unstable set $M_+(Y)$ of Y is the set of points u_* which belong to a complete orbit $\Upsilon=\{u(t)\mid t\in\mathbf{R}\}$ such that

$$
\lim_{t\to-\infty}\operatorname{dist}(u(t),Y)=0.
$$

Lemma 6.1[3] *Let the dynamical system $(S(t),X)$ possess a global attractor A. Assume also that the Lyapunov function $L(y)$ exists on A. Then $A=M_+(N)$, where*

N is the set of stationary points of the dynamical system.

Theorem 6.1 *Assume that* (D1), (D2) *and* (D3) *hold. Let $h = 0$. Assume that A is the global attractor for the dynamical system $(S(t), E)$. Then $A = M_+(N)$.*

Proof It follows from (6) and Theorem 5.1 that

$$L((u; v)) = \|v\|^2 + \int_0^{\|\nabla u\|^2} M(s)\mathrm{d}s + \Gamma_0(u) - \theta\|u\|^2$$

is a Lyapunov function on A. Therefore, we arrive at $A = M_+(N)$, where Lemma 6.1 is used.

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