Ann. of Appl. Math.
32:1(2016), 30-35

EXISTENCE OF PERIODIC SOLUTIONS OF A CLASS OF SECOND-ORDER NON-AUTONOMOUS SYSTEMS*†

Yuan Qi‡, Kun Liu
(*College of Math. and Statistics, Longdong University, Gansu 745000, PR China*)

Abstract

In this paper, we are concerned with the existence of periodic solutions of second-order non-autonomous systems. By applying the Schauder's fixed point theorem and Miranda's theorem, a new existence result of periodic solutions is established.

Keywords second-order non-autonomous systems; periodic solutions; Schauder's fixed point theorem; Miranda's theorem

2000 Mathematics Subject Classification 34B15; 34C25; 37J45

1 Introduction

In the past few years, there has been considerable interest in the existence of periodic solutions of the following second-order periodic boundary value problems

$$\begin{aligned} &u''(t) + a(t)u(t) = f(t, u(t)) + c(t), \\ &u(0) = u(T), \quad u'(0) = u'(T), \end{aligned}$$

where $a, c \in L^1(0,T)$ and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is continuous. For more details please see [1-6] and the references therein. In particular, many authors mentioned above paid their attention to the non-resonant case, that is, the unique solution of the following linear problem

$$u''(t) + a(t)u(t) = 0, \quad u(0) = u(T), \ u'(0) = u'(T) \tag{1.1}$$

is the trivial one. To the end, the function a is supposed to satisfy the basic assumption:

*Supported by the Fundamental Research Funds for the Gansu Universities (2015A-150), the PhD scientific research start-up capital funded projects of Longdong University (XYBY05).

†Manuscript received November 16, 2015

‡Corresponding author. E-mail: mathqy@163.com

(H0) The Green's function $G(t,s)$ of the linear problem (1.1) is nonnegative for every $(t,s) \in [0,T] \times [0,T]$. See [3] for the details.

It is well known that if (1.1) is non-resonant and h is a L^1-function then the Fredholm's alternative theorem implies that the nonhomogeneous problem

$$u''(t) + a(t)u(t) = h(t), \quad u(0) = u(T),\ u'(0) = u'(T)$$

always has a unique solution, which can be written as

$$u(t) = \int_0^T G(t,s)h(s)\mathrm{d}s.$$

And consequently, the linear problem (1.1) is non-resonant. On the other hand, several authors have focused their attention to the existence of periodic solutions of the second-order nonlinear systems. Here we refer the readers to Chu, Torres and Zhang [7], Franco and Webb [8], Cao and Jiang [9] and Wang [10]. Especially in [9], Cao and Jiang obtained several existence results of periodic solutions of the following second order coupled systems

$$\begin{cases} u''(t) + a_1(t)u(t) = f_1(t, v(t)) + e_1(t), \\ v''(t) + a_2(t)v(t) = f_2(t, u(t)) + e_2(t), \end{cases} \tag{1.2}$$

where $f, g : (\mathbf{R}/T\mathbf{Z}) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous and $a_i, e_i \in C(\mathbf{R}/T\mathbf{Z}, \mathbf{R}),\ i = 1, 2$.

Clearly, the above mentioned papers all dealt with the non-resonant problems, that is $a_i(t) \not\equiv 0,\ i = 1, 2$. Now, the natural question is whether or not there is a periodic solution of (1.2) if $a_i(t) \equiv 0,\ i = 1, 2$?

In this paper, we shall establish a new existence result of periodic solutions of the resonant coupled systems

$$\begin{cases} u''(t) = f(t, u(t), v(t)) + e_1(t), \\ v''(t) = g(t, u(t), v(t)) + e_2(t). \end{cases} \tag{1.3}$$

To the best of our knowledge, the existence results of periodic solutions of the above systems are relatively little, and our result shall fill this gap.

The main result of this paper is as follows.

Theorem 1.1 *Suppose that*

(H1) $f, g \in C((\mathbf{R}/T\mathbf{Z}) \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ *are bounded. There are two positive constants* l_1 *and* l_2 *such that for each* $(t, x, y) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$,

$$\begin{aligned} f(t,x,y)x < 0, \qquad |x| \ge l_1, \\ g(t,x,y)y < 0, \qquad |y| \ge l_2; \end{aligned}$$

(H2) $e_i \in C(\mathbf{R}/T\mathbf{Z}, \mathbf{R})$ *with mean value*

$$\bar{e}_i = \frac{1}{T}\int_0^T e_i(s)\mathrm{d}s = 0, \quad i = 1, 2.$$

Then (1.3) *has a periodic solution.*

The Miranda's theorem below will be crucial in our arguments.

Theorem A[11] *Let*

$$G = \{x \in \mathbf{R}^n : |x_i| < L,\ i = 1, 2, \cdots, n\}.$$

Suppose that the mapping $F = (F_1, F_2, \cdots, F_n) : \overline{G} \to \mathbf{R}^n$ *is continuous on the closure* $\overline{G}$ *of* G*, such that* $F(x) \neq \theta = (0, 0, \cdots, 0)$ *for* x *on the boundary* ∂G *of* G*. In addition,*

(i) $F_i(x_1, x_2, \cdots, x_{i-1}, -L, x_{i+1}, \cdots, x_n) \geq 0$ for $1 \leq i \leq n$

and

(ii) $F_i(x_1, x_2, \cdots, x_{i-1}, +L, x_{i+1}, \cdots, x_n) \leq 0$ for $1 \leq i \leq n$.

Then, $F(x) = \theta$ *has a solution in* G.

2 Proof of Theorem 1.1

Define an operator $L : D(L) \to E$ by

$$Lx := x'',$$

where E is a Banach space composed of continuous T-periodic functions with the norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|$$

and

$$D(L) = \{x \in C^2[0, T] : x(0) = x(T),\ x'(0) = x'(T)\}.$$

Then it is not difficult to check that L is not invertible since $\mathrm{Ker}(L) = \{c\}$, $c \in \mathbf{R}$.

Let $V = \mathrm{Ker}(L)$. Then $L^2(0, T) = V \oplus V^\perp$, where

$$V^\perp = \left\{ y \in L^2(0, T) : \int_0^T y(s)\mathrm{d}s = 0 \right\}.$$

Now, $u, v \in L^2(0, T)$ can be rewritten as

$$\begin{aligned} u &= s + w, \quad s \in V,\ w \in V^\perp, \\ v &= \rho + \psi, \quad \rho \in V,\ \psi \in V^\perp. \end{aligned}$$

And from (H2) it follows that (1.3) is equivalent to the following equations

$$w''(t) = f(t, s + w(t), \rho + \psi(t)) + e_1(t), \quad t \in (0, T), \tag{2.1}$$

$$\Phi_1(s, \rho, w, \psi) := \int_0^T f(\tau, s + w(\tau), \rho + \psi(\tau))\mathrm{d}\tau = 0; \tag{2.2}$$

$$\psi''(t) = g(t, s + w(t), \rho + \psi(t)) + e_2(t), \quad t \in (0, T), \tag{2.3}$$

$$\Phi_2(s, \rho, w, \psi) := \int_0^T g(\tau, s + w(\tau), \rho + \psi(\tau))\mathrm{d}\tau = 0. \tag{2.4}$$

By (2.1) and (2.3), we get

$$w(t) = (L\big|_{V^\perp})^{-1}\big(f(t, s + w(t), \rho + \psi(t)) + e_1(t)\big) =: T_{s,\rho,\psi}(w(t)), \tag{2.5}$$

$$\psi(t) = (L\big|_{V^\perp})^{-1}\big(g(t, s + w(t), \rho + \psi(t)) + e_2(t)\big) =: T_{s,\rho,w}(\psi(t)). \tag{2.6}$$

Moreover, (H1) implies there exist $M_1 > 0,\ M_2 > 0$ such that

$$f(t, x, y) \le M_1, \quad g(t, x, y) \le M_2, \quad (t, x, y) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}.$$

Thus, by Schauder's fixed point theorem, equations (2.5) and (2.6) have solutions $\widetilde{w}(t)$ and $\widetilde{\psi}(t)$, respectively. Furthermore, each possible solution w of (2.5) is bounded, so there is a constant $L_1 > 0$ such that $\|w\| \le L_1$. Similarly, each possible solution ψ of (2.6) satisfies $\|\psi\| \le L_2$ for some constant $L_2 > 0$.

Substituting $\widetilde{w}(t)$ and $\widetilde{\psi}(t)$ into (2.2) and (2.4), respectively. Then it is not difficult to see the proof will be completed, if we could find some real numbers s_0 and ρ_0 such that

$$\Phi_1(s_0, \rho_0, \widetilde{w}(t), \widetilde{\psi}(t)) = 0, \tag{2.7}$$

and

$$\Phi_2(s_0, \rho_0, \widetilde{w}(t), \widetilde{\psi}(t)) = 0. \tag{2.8}$$

Clearly, it follows from (H1) that there exists a constant $s_1 > 0$ sufficiently large such that

$$s_1 + \widetilde{w}(t) \ge l_1,$$

therefore

$$f(t, s_1 + \widetilde{w}(t), \rho + \widetilde{\psi}(t)) < 0, \quad t, \rho \in \mathbf{R}. \tag{2.9}$$

On the other hand, there exists a constant $s_2 < 0$ with $|s_2|$ large enough such that

$$s_2 + \widetilde{w}(t) \le -l_1,$$

therefore

$$f(t, s_2 + \widetilde{w}(t), \rho + \widetilde{\psi}(t)) > 0, \quad t, \rho \in \mathbf{R}. \tag{2.10}$$

Similarly, we may choose $\rho_1 > 0$ and $\rho_2 < 0$ with ρ_1 and $|\rho_2|$ sufficiently large such that

$$\rho_1 + \widetilde{\psi}(t) \geq l_2, \qquad \rho_2 + \widetilde{\psi}(t) \leq -l_2,$$

and accordingly,

$$g(t, s + \widetilde{w}(t), \rho_1 + \widetilde{\psi}(t)) < 0, \qquad t, s \in \mathbf{R}, \tag{2.11}$$

$$g(t, s + \widetilde{w}(t), \rho_2 + \widetilde{\psi}(t)) > 0, \qquad t, s \in \mathbf{R}. \tag{2.12}$$

Let

$$L := \max\big\{ \max\{s_1, |s_2|\},\ \max\{\rho_1, |\rho_2|\}\big\}$$

and

$$G = \{(s, \rho) \in \mathbf{R}^2 : |s| < L,\ |\rho| < L\}.$$

Define

$$F_1(s, \rho) := \Phi_1(s, \rho, \widetilde{w}(t), \widetilde{\psi}(t)), \quad F_2(s, \rho) := \Phi_2(s, \rho, \widetilde{w}(t), \widetilde{\psi}(t)).$$

Then (H1) implies $F = (F_1, F_2) : \overline{G} \to \mathbf{R}^2$ is continuous on the closure $\overline{G}$ of G. In addition, (2.9)-(2.12) and the definition of G yield

$$F((s, \rho)) \neq \theta = (0, 0) \quad \text{for } (s, \rho) \in \partial G.$$

Finally, let us show that assumptions (i) and (ii) of Theorem A are also satisfied. To prove Theorem A (i), it is equivalent to prove

$$F_1(-L, \rho) = \Phi_1(-L, \rho, \widetilde{w}(t), \widetilde{\psi}(t)) = \int_0^T f(\tau, -L + \widetilde{w}(\tau), \rho + \widetilde{\psi}(\tau))\mathrm{d}\tau \geq 0, \tag{2.13}$$

$$F_2(s, -L) = \Phi_2(s, -L, \widetilde{w}(t), \widetilde{\psi}(t)) = \int_0^T f(\tau, s + \widetilde{w}(\tau), -L + \widetilde{\psi}(\tau))\mathrm{d}\tau \geq 0. \tag{2.14}$$

Since

$$-L + \widetilde{w}(\tau) \leq s_2 + \widetilde{w}(\tau) \leq -l_1,$$

the first inequality in (H1) implies (2.13) holds. On the other hand, the second inequality in (H1) and the fact

$$-L + \widetilde{\psi}(\tau) \leq \rho_2 + \widetilde{\psi}(\tau) \leq -l_2$$

yield (2.14) is also satisfied.

To prove Theorem A (ii), it is equivalent to prove

$$F_1(L, \rho) = \Phi_1(L, \rho, \widetilde{w}(t), \widetilde{\psi}(t)) = \int_0^T f(\tau, L + \widetilde{w}(\tau), \rho + \widetilde{\psi}(\tau))\mathrm{d}\tau \leq 0, \tag{2.15}$$

$$F_2(s, L) = \Phi_2(s, L, \widetilde{w}(t), \widetilde{\psi}(t)) = \int_0^T f(\tau, s + \widetilde{w}(\tau), L + \widetilde{\psi}(\tau))\mathrm{d}\tau \leq 0. \tag{2.16}$$

By the similar arguments as in the proof of (2.13) and (2.14), we can show that (2.15) and (2.16) are also satisfied.

Consequently, Theorem A implies there exists an $(s_0, \rho_0) \in G$ such that

$$F((s_0, \rho_0)) = \theta.$$

Therefore, (2.7) and (2.8) hold, and the proof is completed.

References

[1] I. Rachunková, M. Tvrdý, I. Vrkoč, Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems, *J. Diff. Equs.*, **176**(2001),445-469.

[2] P.J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Diff. Equs.*, **190**(2003),643-662.

[3] P.J. Torres, Weak singularities may help periodic solutions to exist, *J. Diff. Equs.*, **232**(2007),277-284.

[4] J. Chu, M. Li, Positive periodic solutions of Hill's equations with singular nonlinear perturbations, *Nonlinear Anal.*, **69**(2008),276-286.

[5] X. Li, Z. Zhang, Periodic solutions for second-order differential equations with a singular nonlinearity, *Nonlinear Anal.*, **69**(2008),3866-3876.

[6] H. Li, Positive periodic solutions of second-order differential equations with singularities, *Bull. Austral. Math. Soc.*, **78**(2008),163-169.

[7] J. Chu, P.J. Torres, M. Zhang, Periodic solutions of second order non-autonomous singular dynamical systems, *J. Diff. Equs.*, **239**(2007),196-212.

[8] D. Franco, J.R.L. Webb, Collisionless orbits of singular and nonsingular dynamical systems, *Discrete Contin. Dyn. Syst.*, **15**(2006),747-757.

[9] Z. Cao, D. Jiang, Periodic solutions of second order singular coupled systems, *Nonlinear Anal.*, **71**(2009),3661-3667.

[10] H. Wang, Positive periodic solutions of singular systems with a parameter, *J. Diff. Equs.*, **249**(2010),2986-3002.

[11] C. Miranda, Un'osservatione su un teorema di Brouwer, *Boll. Un. Mat. Ital.*, **3**:2(1940), 5-7.

(*edited by Mengxin He*)