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# Regularity Criteria of the Solutions to Axisymmetric Magnetohydrodynamic System

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**Abstract** In this paper, we consider Cauchy problem of the axially symmetric Magnetohydrodynamic (MHD) system. By using energy method, we establish some regularity criteria of the solutions for the axisymmetric solutions of the three dimensional incompressible MHD system.

**Keywords** Axisymmetric solutions, Regularity criterion, Incompressible magnetohydrodynamics.

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## 1. Introduction

In this paper, we consider Cauchy problem of the incompressible MHD system:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \triangle u + \nabla p = (v\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B - \triangle B = (B\cdot\nabla)u, \\ \mathrm{div} u = 0,\ \mathrm{div} B = 0, \\ (u,B)(\mathbf{x},0) = (u_0, B_0), \end{cases} \tag{1.1}$$

where $u = u(\mathbf{x},t) \in \mathbb{R}^3$ denotes the velocity of the fluid, $B = B(\mathbf{x},t) \in \mathbb{R}^3$ stands for the magnetic field, the scalar function $p = p(\mathbf{x},t)$ is pressure and $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$.

Our main concern here is to establish a family of unique solutions of system (1.1) with the form

$$u(\mathbf{x},t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z,$$

$$B(\mathbf{x},t) = B_r(r,z,t)e_r + B_\theta(r,z,t)e_\theta + B_z(r,z,t)e_z,$$

in the cylindrical coordinate system. Here,

$$\begin{aligned} &e_r = (\frac{x}{r},\frac{y}{r},0),\ e_\theta = (-\frac{y}{r},\frac{x}{r},0),\ e_z = (0,0,1), \\ &r = \sqrt{x^2+y^2},\ (x,y,z) = (r\cos\theta, r\sin\theta, z). \end{aligned}$$

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In terms of $(u_r, u_\theta, u_z, B_r, B_\theta, B_z)$, the MHD system (1.1) can be rewritten as

$$\begin{cases} \partial_t u_r + (u\cdot\nabla)u_r - \triangle u_r + \frac{u_r}{r^2} - \frac{u_\theta^2}{r} + \partial_r p - (B\cdot\nabla)B_r + \frac{B_\theta^2}{r} = 0, \\ \partial_t u_\theta + (u\cdot\nabla)u_\theta - \triangle u_\theta + \frac{u_\theta}{r^2} + \frac{u_r u_\theta}{r} - (B\cdot\nabla)B_\theta - \frac{B_r B_\theta}{r} = 0, \\ \partial_t u_z + (u\cdot\nabla)u_z - \triangle u_z + \partial_z p - (B\cdot\nabla)B_z = 0, \\ \partial_t B_r + (u\cdot\nabla)B_r - \triangle B_r + \frac{B_r}{r^2} - (B\cdot\nabla)u_r = 0, \\ \partial_t B_\theta + (u\cdot\nabla)B_\theta - \triangle B_\theta + \frac{B_\theta}{r^2} + \frac{u_\theta B_r}{r} - (B\cdot\nabla)u_\theta - \frac{B_\theta u_r}{r} = 0, \\ \partial_t B_z + (u\cdot\nabla)B_z - \triangle B_z - (B\cdot\nabla)u_z = 0, \\ \frac{u_r}{r} + \partial_r u_r + \partial_z u_z = 0, \\ \frac{B_r}{r} + \partial_r B_r + \partial_z B_z = 0, \\ (u_r, u_\theta, u_z, B_r, B_\theta, B_z)(r,z,0) = (u_0^r, u_0^\theta, u_0^z, B_0^r, B_0^\theta, B_0^z)(r,z). \end{cases} \tag{1.2}$$

First, let us briefly review some results of the MHD system (1.1). For the 2D case, Jiu and Zhao [9] get a global regular solution of MHD with dissipation terms $-(-\triangle)^\alpha u$ and $-(-\triangle)^\beta B$, when $0 \le \alpha < \frac{1}{2}, \beta \ge 1, 3\alpha + 2\beta > 3$. In particular, they also prove the solution exists globally when $\alpha = 0$ and $\beta > \frac{3}{2}$. For more details, one may refer to [1,6,15,19,20,24]. In the 3D case, Zhou [23] gets the MHD system with dissipation terms $-(-\triangle)^\alpha u$ and $-(-\triangle)^\beta B$ satisfying the following conditions:

$$u(x,t) \in L^p(0,T;L^q(\mathbb{R}^3)),\ \frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - 1,\ \frac{3}{2\alpha-1} < q \le \infty,$$

or

$$\Lambda^\alpha u(x,t) \in L^p(0,T;L^q),\ \frac{2\alpha}{p} + \frac{3}{q} \le 3\alpha - 1,\ \frac{3}{3\alpha-1} < q \le \frac{3}{\alpha-1}.$$

Then, the solution is globally regular. Other regularity criteria were shown in [4,7,8,21,22,25].

When $B = 0$, system (1.1) reduce to axisymmetric Navier-Stokes system, there have been extensive studies on the regularity criteria for axisymmetric Navier-Stokes, cf. [2, 5, 10–12, 16, 18]. Here, we only mention some results that related to our main results. Firstly, Chae and Lee [2] obtain some regularity criteria for axisymmetric weak solutions of the 3D Navier-Stokes equations with nonzero swirl. In [18], Wei proves the global regularity of solutions to the axially symmetric Navier-Stokes equations, if $\|ru_\theta(r,z,0)\|_{L^\infty}$ or $\|ru_\theta(r,z,t)\|_{L^\infty(r\le r_0)}$ is smaller than some dimensionless quantity of the initial data. This result improves the one in Lei and Zhang [12]. Motivated by [18], we will study a global regularity for the axially symmetric MHD system.

For equation (1.2), Lei [10] proves the global well-posedness of classical solutions for a family of special axisymmetric initial data whose swirl components of the velocity field and magnetic vorticity field are trivial. In [16], Wang and Wu study the properties of solutions to axially symmetric incompressible MHD system in three dimensions, and construct a family of global smooth solutions by applying the one-dimensional solutions. Recently, Li and Yuan [13] have obtained regularity criteria for the axisymmetric solutions of MHD system, if $\omega_\theta \in L^q(0,T;L^p(\mathbb{R}^3))$, and $n_\theta \in L^q(0,T;L^p(\mathbb{R}^3))$ satisfy

$$\int_0^T (\|\omega_\theta\|_p^q + \|n_\theta\|_p^q)dt < \infty,\ \textit{with}\ \ \frac{3}{p} + \frac{2}{q} \le 2,\ \frac{3}{2} < p \le \infty,\ 0 < q < \infty.$$

It is well-known that finite energy smooth solutions of the MHD system satisfy the following basic energy identity

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2)(s)ds = \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2. \quad (1.3)$$

Denote $J = ru_\theta, L = rB_\theta$, we obtain

$$\begin{aligned}
&\partial_t(J+L) + (u\cdot\nabla)(J+L) - \triangle(J+L) + \frac{2}{r}\partial_r(J+L) - (B\cdot\nabla)(J+L) \\
&+2(u_\theta B_r - u_r B_\theta) = 0, \qquad (1.4)\\
&\partial_t(J-L) + (u\cdot\nabla)(J-L) - \triangle(J-L) + \frac{2}{r}\partial_r(J-L) - (B\cdot\nabla)(J-L) \\
&+2(u_r B_\theta - u_\theta B_r) = 0. \qquad (1.5)
\end{aligned}$$

By direct calculation, the vorticity $\omega$ =curl $u$ and $\omega$ =curl $B$ can be written as

$$\begin{aligned}
\omega(x,t) &= \nabla\times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,\\
n(x,t) &= \nabla\times B = n_r e_r + n_\theta e_\theta + n_z e_z,
\end{aligned}$$

with

$$\begin{aligned}
\omega_r = -\partial_z u_\theta,\ \omega_\theta = \partial_z u_r - \partial_r u_z,\ \omega_z = \frac{1}{r}\partial_r(ru_\theta),\\
n_r = -\partial_z B_\theta,\ n_\theta = \partial_r B_z - \partial_z B_r,\ n_z = \frac{1}{r}\partial_r(rB_\theta).
\end{aligned}$$

Denote

$$\begin{aligned}
\Omega = \frac{\omega_\theta}{r} = \frac{\partial_z u_r - \partial_r u_z}{r},\ R = \frac{\omega_r}{r} = -\frac{\partial_z u_\theta}{r},\\
\Gamma = \frac{n_\theta}{r} = \frac{\partial_z B_r - \partial_r B_z}{r},\ K = \frac{n_r}{r} = -\frac{\partial_z B_\theta}{r},
\end{aligned}$$

then

$$\begin{cases}
\partial_t\Omega + (u\cdot\nabla)\Omega - (\triangle + \frac{2}{r}\partial_r)\Omega - (B\cdot\nabla)\Gamma + \frac{2u_\theta}{r}R - \frac{2B_\theta}{r}K = 0,\\
\partial_t R + (u\cdot\nabla)R - (\triangle + \frac{2}{r}\partial_r)R - (B\cdot\nabla)K - (\omega_r\partial_r + \omega_z\partial_z)\frac{u_r}{r}\\
+(n_r\partial_r + n_z\partial_z)\frac{B_r}{r} = 0,
\end{cases} \quad (1.6)$$

and

$$\begin{cases}
\partial_t\Gamma + (u\cdot\nabla)\Gamma - (\triangle + \frac{2}{r}\partial_r)\Gamma - (B\cdot\nabla)\Omega = 0,\\
\partial_t K + (u\cdot\nabla)K - (\triangle + \frac{2}{r}\partial_r)K - (B\cdot\nabla)R - (\omega_r\partial_r + \omega_z\partial_z)\frac{B_r}{r}\\
+(n_r\partial_r + n_z\partial_z)\frac{u_r}{r} = 0.
\end{cases} \quad (1.7)$$

The initial data is given by

$$J_0(r,z) = J(r,z,0),\ \Omega_0(r,z) = \Omega(r,z,0),\ R_0(r,z) = R(r,z,0),$$

$$L_0(r,z) = L(r,z,0),\ \Gamma_0(r,z) = \Gamma(r,z,0),\ K_0(r,z) = K(r,z,0).$$

Note that $\Omega, \Gamma$ and $R, K$ are introduced by Chen [3] and Majda-Bertozzi [14].

Our main results are as follows: If $(u, B)$ are axially symmetric solutions to the MHD system (1.1), satisfying $r^3\omega_\theta(r,z,0), r^3 n_\theta(r,z,0), J_0, L_0 \in L^2(\mathbb{R}^3)$, then $r^3\omega_\theta \in L^\infty(0,T;L^2(\mathbb{R}^3))$ and $r^3 n_\theta \in L^\infty(0,T;L^2(\mathbb{R}^3))$.

Under the assumption of Theorem 1, let $r_0 > 0$, $u_r B_\theta - u_\theta B_r \in L^1(0,T;L^\infty(\mathbb{R}^3))$, $u_0 \in H^2, B_0 \in H^2(\mathbb{R}^3)$ such that $J_0 \in L^\infty(\mathbb{R}^3),\ L_0 \in L^\infty(\mathbb{R}^3)$. Denote

$$P_1 = (2 + \|J_0\|_{L^\infty} + \|L_0\|_{L^\infty})(\|u_0\|_{L^2} + \|B_0\|_{L^2}),$$

and

$$P_0 = (\|\Omega_0\|_{L^2} + \|R_0\|_{L^2} + \|\Gamma_0\|_{L^2} + \|K_0\|_{L^2})P_1^3,$$

then there exists an absolute positive constant $C_0 > 0$ such that if

$$\|J\|_{L^\infty(r\le r_0)} + \|L\|_{L^\infty(r\le r_0)} \le (1 + \ln(C_0 \max\{P_0^{\frac{1}{4}}, r_0^{-1/3}P_1\} + 1))^{-\frac{3}{2}}, \qquad (a)$$

the axially symmetric MHD system are globally well-posed. If $u_r B_\theta - u_\theta B_r = 0$, the condition $u_r B_\theta - u_\theta B_r \in L^1(0,T;L^\infty(\mathbb{R}^3))$ is naturally satisfied. Then, theorem 1 is a global existence result. For example $u_r = B_r = 0$ or $B_\theta = B_r = 0$ or $B_\theta = u_\theta = 0$ etc.. By choosing $r_0 > 0$ such that $P_0^{\frac{1}{4}} \ge r_0^{-\frac{1}{3}}P_1$ and using $\|J\|_{L^\infty} + \|L\|_{L^\infty} \le \|J_0\|_{L^\infty} + \|L_0\|_{L^\infty}$, we obtain the following regularity condition

$$\|J_0\|_{L^\infty} + \|L_0\|_{L^\infty} \le (1 + \ln(C_0 P_0^{\frac{1}{4}} + 1))^{-\frac{3}{2}}. \qquad (b)$$

This condition depends only on initial values and is useful especially when $\|J_0\|_{L^\infty} + \|L_0\|_{L^\infty}$ is small. Denote

$$M(\varepsilon) = e^{\sqrt{1-2(1-\varepsilon^{-\frac{4}{3}})}-1} = e^{\sqrt{2\varepsilon^{-\frac{4}{3}}-1}-1}, \quad M_0(\varepsilon) = e^{\varepsilon^{-\frac{2}{3}}-1},$$

for $0 < \varepsilon \le 1$, then we can easily check that

$$1 + \ln M(\varepsilon) + \frac{1}{2}(\ln M(\varepsilon))^2 = \varepsilon^{-\frac{4}{3}}, \quad \varepsilon^{\frac{4}{3}} M(\varepsilon) \ge \frac{M_0}{C_0} > 0,$$

for some absolute positive constant $C_0$ and $0 < \varepsilon \le 1$, the functions $M$ and $M_0$ are given in Lemma 2.

This paper is organized as follows: In Section 2, we will give some notations and lemmas which will be used in the proof of our main theorems, and in Section 3, we will give the proofs of our main theorems.

## 2. Notations and lemmas

The Laplacian operator $\triangle$ and the gradient operator $\nabla$ in the cylindrical coordinate are

$$\triangle = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2, \quad \nabla = e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z,$$

we will denote $\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + u_r\partial_r + u_z\partial_z$.

Now, we begin with the elementary lemmas (see [18]).

$$\|\nabla \frac{u_r}{r}\|_{L^2}^2 \le \|\Omega\|_{L^2}^2,\ \|\nabla^2 \frac{u_r}{r}\|_{L^2}^2 \le \|\partial_z\Omega\|_{L^2}^2, \qquad (2.1)$$

$$\|\nabla \frac{B_r}{r}\|_{L^2}^2 \leq \|\Gamma\|_{L^2}^2,\ \|\nabla^2 \frac{B_r}{r}\|_{L^2}^2 \leq \|\partial_z \Gamma\|_{L^2}^2. \tag{2.2}$$

Denote

$$h(r,z,t) = \int_0^r |u_\theta(r',z,t)|dr',\ \ for\ r>0,\ a(t) = \|\frac{h}{r}(t)\|_{L^\infty},$$

$$s(r,z,t) = \int_0^r |B_\theta(r',z,t)|dr',\ \ for\ r>0,\ b(t) = \|\frac{s}{r}(t)\|_{L^\infty},$$

then we have the following inequality:

$$a(t)^2 \leq \|R(t)\|_{L^2}\|\frac{u_\theta}{r}(t)\|_{L^2}, \tag{2.3}$$

$$b(t)^2 \leq \|K(t)\|_{L^2}\|\frac{B_\theta}{r}(t)\|_{L^2}. \tag{2.4}$$

(see [18]) For the $a(t)$ and the $b(t)$ are given in the Lemma 2, $\|J\|_{L^\infty(r\leq r_1)} \leq \varepsilon \leq 1$, $\|L\|_{L^\infty(r\leq r_1)} \leq \varepsilon \leq 1$, and $0 < r_1 \leq \frac{\varepsilon M(\varepsilon)}{a(t)+b(t)}$. Then,

$$\int_{\mathbb{R}^3} \frac{|u_\theta(t)|}{r}|f|^2dx \leq \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{-\frac{1}{3}} + \|J\|_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx, \tag{2.5}$$

$$\int_{\mathbb{R}^3} |u_\theta(t)|^2|f|^2dx \leq \varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|J\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx, \tag{2.6}$$

$$\int_{\mathbb{R}^3} \frac{|B_\theta(t)|}{r}|f|^2dx \leq \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{-\frac{1}{3}} + \|L\|_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx, \tag{2.7}$$

$$\int_{\mathbb{R}^3} |B_\theta(t)|^2|f|^2dx \leq \varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|L\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx, \tag{2.8}$$

for all axially symmetric scalar and vector functions $f \in \mathrm{H}^1(\mathbb{R}^3)$. The proofs of the above lemmas follow the same idea as [18] and is omitted here. According to the Lemma 2, we immediately have the following. Assume that $t > 0$, $\|J\|_{L^\infty(r\leq r_1)} \leq \varepsilon \leq 1, \|L\|_{L^\infty(r\leq r_1)} \leq \varepsilon \leq 1$, and $0 < r_1 \leq \frac{\varepsilon M(\varepsilon)}{a(t)+b(t)}$, then

$$\int_{\mathbb{R}^3} |u_\theta(t)|^2|f|^2dx \leq \frac{1}{16}\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|J\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx, \tag{2.9}$$

$$\int_{\mathbb{R}^3} |B_\theta(t)|^2|f|^2dx \leq \frac{1}{16}\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|L\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx. \tag{2.10}$$

**Proof.** By using (2.7) and (2.8), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |4u_\theta(t)|^2|\frac{1}{4}f|^2dx \\
\leq\ &\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\frac{1}{4}\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|J\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |\frac{1}{4}f|^2dx \\
\leq\ &\frac{1}{16}\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|J\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2dx,
\end{aligned}$$

similarly, we also have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |4B_\theta(t)|^2 |\frac{1}{4} f|^2 dx \\
\leq\ & \varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\frac{1}{4}\partial_r f|^2 dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|L\|^2_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |\frac{1}{4} f|^2 dx \\
\leq\ & \frac{1}{16}\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2 dx + C\frac{(\varepsilon^{\frac{2}{3}} + \|L\|^2_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2 dx.
\end{aligned}
$$

The corollary is proved. □

## 3. Proof of the main theorems

Before proving our main theorems, we give the following proposition that is crucial in this paper. Suppose that $(u, B)$ are axially symmetric solutions to the MHD system (1.2), $J, L$ satisfy (1.4)-(1.5), with initial data $J_0 \in L^2(\mathbb{R}^3)$, $L_0 \in L^2(\mathbb{R}^3)$, then $J \in L^2(0, T; L^2(\mathbb{R}^3))$, $L \in L^2(0, T; L^2(\mathbb{R}^3))$, $J + L \in L^2(0, T; \dot{H}^1(\mathbb{R}^3))$ and $J - L \in L^2(0, T; \dot{H}^1(\mathbb{R}^3))$ for any $T > 0$. **Proof.** Multiplying the both sides of (1.4) by $(J + L)$ and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} (J+L)^2 dx + \int_{\mathbb{R}^3} |\nabla (J+L)|^2 dx \\
&= -\int_{\mathbb{R}^3} \frac{2}{r}\partial_r (J+L)(J+L) dx - 2\int_{\mathbb{R}^3} (u_\theta B_r - B_\theta u_r)(J+L) dx \\
&= K_1 + K_2.
\end{aligned} \tag{3.1}
$$

Direct calculation we have

$$
\begin{aligned}
K_1 &= -\int_{\mathbb{R}^3} \frac{2}{r}\partial_r (J+L)(J+L) dx, \\
&= -2\pi \int_{-\infty}^{\infty}\int_0^{\infty} \partial_r (J+L)^2 dr dz \\
&= 0.
\end{aligned}
$$

Here, we have used $J(0, z, t) = L(0, z, t) = 0$. As for $K_2$, by applying the Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality and the Basic Energy estimate, we have

$$
\begin{aligned}
K_2 &\leq \int_{\mathbb{R}^3} (u_\theta B_r + B_\theta u_r)(J+L) dx \\
&= \int_{\mathbb{R}^3} u_\theta B_r (J+L) dx + \int_{\mathbb{R}^3} u_r B_\theta (J+L) dx \\
&\leq \|(J+L)\|_{L^6} \|u_\theta\|_{L^3} \|B_r\|_{L^2} + \|(J+L)\|_{L^6} \|B_\theta\|_{L^3} \|u_r\|_{L^2} \\
&\leq \|\nabla (J+L)\|_{L^2} \|u_\theta\|^{\frac{1}{2}}_{L^2} \|\nabla u_\theta\|^{\frac{1}{2}}_{L^2} \|B_r\|_{L^2} \\
&\quad + \|\nabla (J+L)\|_{L^2} \|B_\theta\|^{\frac{1}{2}}_{L^2} \|\nabla B_\theta\|^{\frac{1}{2}}_{L^2} \|u_r\|_{L^2} \\
&\leq C\|\nabla (J+L)\|_{L^2} \|\nabla u_\theta\|^{\frac{1}{2}}_{L^2} + C\|\nabla (J+L)\|_{L^2} \|\nabla B_\theta\|^{\frac{1}{2}}_{L^2}
\end{aligned}
$$

$$\leq \frac{1}{2}\|\nabla(J+L)\|_{L^2}^2 + C(\|\nabla u_\theta\|_{L^2} + \|\nabla B_\theta\|_{L^2}).$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(J+L)^2dx + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla(J+L)|^2dx \leq C(\|\nabla u_\theta\|_{L^2} + \|\nabla B_\theta\|_{L^2}).$$

By using Gronwall's inequality, we can obtain

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|J+L\|_{L^2}^2 + \int_0^T \|\nabla(J+L)\|_{L^2}^2dt\\
&\leq C(\|J_0+L_0\|_{L^2}^2) + C\int_0^T(\|\nabla u_\theta\|_{L^2} + \|\nabla B_\theta\|_{L^2})dt\\
&\leq C(\|J_0+L_0\|_{L^2}^2) + C_T(\|u_0\|_{L^2}, \|B_0\|_{L^2}).
\end{aligned} \tag{3.2}$$

Similarly, multiplying the both sides of (1.5) by $(J-L)$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(J-L)^2dx + \int_{\mathbb{R}^3}|\nabla(J-L)|^2dx\\
&= -\int_{\mathbb{R}^3}\frac{2}{r}\partial_r(J-L)(J-L)dx - 2\int_{\mathbb{R}^3}(B_\theta u_r - u_\theta B_r)(J-L)dx.
\end{aligned} \tag{3.3}$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(J-L)^2dx + \int_{\mathbb{R}^3}|\nabla(J-L)|^2dx \leq C(\|\nabla u_\theta\|_{L^2} + \|\nabla B_\theta\|_{L^2}).$$

Hence,

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|J-L\|_{L^2}^2dx + C\int_0^T \|\nabla(J-L)\|_{L^2}^2dt\\
&\leq C(\|J_0-L_0\|_{L^2}^2) + C_T(\|u_0\|_{L^2}, \|B_0\|_{L^2}).
\end{aligned} \tag{3.4}$$

Combining the (3.2) and (3.4), we obtain

$$\begin{aligned}
\sup_{0\leq t\leq T}(\|J\|_{L^2} + \|L\|_{L^2}) &\leq C(\|J_0\|_{L^2} + \|L_0\|_{L^2}) + C_T(\|u_0\|_{L^2}, \|B_0\|_{L^2})\\
&\leq C(\|J_0\|_{L^2}, \|L_0\|_{L^2}, \|u_0\|_{L^2}, \|B_0\|_{L^2}, T).
\end{aligned} \tag{3.5}$$

Now, we finish our proof. □

Using Proposition 3, we prove Theorem 1 now.

**Proof of Theorem1.1** Applying the operator $(\partial_z, -\partial_r)$ to $u_r$ and $u_z$ of (1.2), we have

$$\begin{aligned}
&\partial_t\omega_\theta + (u_r\partial_r + u_z\partial_z)\omega_\theta - (\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r)\omega_\theta + \frac{1}{r^2}\omega_\theta - \frac{u_r\omega_\theta}{r} - \frac{\partial_z u_\theta^2}{r}\\
&= (B_r\partial_r + B_z\partial_z)n_\theta + \frac{B_r n_\theta}{r} - \frac{\partial_z B_\theta^2}{r}.
\end{aligned} \tag{3.6}$$

Suppose that $r^3\omega_\theta \in L^2(\mathbb{R}^3)$, multiplying the both sides of (3.6) by $r^6\omega_\theta$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx + \int_{\mathbb{R}^3}(u_r\partial_r + u_z\partial_z)(r^3\omega_\theta)(r^3\omega_\theta)dx$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} u_r[\partial_r(r^3)]\omega_\theta(r^3\omega_\theta)dx - [\int_{\mathbb{R}^3}(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)(r^3\omega_\theta)(r^3\omega_\theta)dx\\
&-2\int_{\mathbb{R}^3}[\partial_r(r^3)](\partial_r\omega_\theta)(r^3\omega_\theta)dx - \int_{\mathbb{R}^3}(r^3\omega_\theta^2)(\partial_r^2+\frac{1}{r}\partial_r)r^3dx]\\
&+\int_{\mathbb{R}^3}\frac{1}{r^2}\omega_\theta(r^6\omega_\theta)dx - \int_{\mathbb{R}^3}\frac{u_r\omega_\theta}{r}(r^6\omega_\theta)dx - \int_{\mathbb{R}^3}\frac{[\partial_z(u_\theta^2)]}{r}(r^6\omega_\theta)dx\\
&=\int_{\mathbb{R}^3}(B_r\partial_r+B_z\partial_z)(r^3n_\theta)(r^3\omega_\theta)dx - \int_{\mathbb{R}^3}B_r[\partial_r(r^3)]n_\theta(r^3\omega_\theta)dx\\
&\quad+\int_{\mathbb{R}^3}\frac{B_rn_\theta}{r}(r^6\omega_\theta)dx - \int_{\mathbb{R}^3}\frac{[\partial_z(B_\theta^2)]}{r}(r^6\omega_\theta)dx.
\end{aligned}
$$

After integration by parts and more elementary computations, where $\int_{\mathbb{R}^3}(u_r\partial_r + u_z\partial_z)(r^3\omega_\theta)(r^3\omega_\theta)dx = 0$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx + \int_{\mathbb{R}^3}|\nabla(r^3\omega_\theta)|^2dx\\
&=\int_{\mathbb{R}^3}(B_r\partial_r+B_z\partial_z)(r^3n_\theta)(r^3\omega_\theta)dx + 4\int_{\mathbb{R}^3}r^5\omega_\theta^2u_rdx\\
&\quad+8\int_{\mathbb{R}^3}r^4\omega_\theta^2dx - 2\int_{\mathbb{R}^3}B_rr^5\omega_\theta n_\theta dx + \int_{\mathbb{R}^3}r^5\omega_\theta[\partial_z(u_\theta^2)]dx\\
&\quad-\int_{\mathbb{R}^3}r^5\omega_\theta[\partial_z(B_\theta^2)]dx.
\end{aligned}
\tag{3.7}
$$

Similarly, applying the operator $(\partial_z, -\partial_r)$ to $B_r$ and $B_z$ of (1.2), we have

$$
\begin{aligned}
&\partial_tn_\theta + (u_r\partial_r+u_z\partial_z)n_\theta - (\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)n_\theta + \frac{1}{r^2}n_\theta - \frac{u_rn_\theta}{r}\\
&=(B_r\partial_r+B_z\partial_z)\omega_\theta + \frac{B_r\omega_\theta}{r}.
\end{aligned}
\tag{3.8}
$$

Suppose that $r^3n_\theta \in L^2(\mathbb{R}^3)$, multiplying the both sides of (3.6) by $r^6n_\theta$ and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(r^3n_\theta)^2dx + \int_{\mathbb{R}^3}(u_r\partial_r+u_z\partial_z)(r^3n_\theta)(r^3n_\theta)dx\\
&-\int_{\mathbb{R}^3}u_r[\partial_r(r^3)]n_\theta(r^3n_\theta)dx - [\int_{\mathbb{R}^3}(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)(r^3n_\theta)(r^3n_\theta)dx\\
&-2\int_{\mathbb{R}^3}[\partial_r(r^3)](\partial_rn_\theta)(r^3n_\theta)dx - \int_{\mathbb{R}^3}(r^3n_\theta^2)(\partial_r^2+\frac{1}{r}\partial_r)r^3dx]\\
&+\int_{\mathbb{R}^3}\frac{1}{r^2}n_\theta(r^6n_\theta)dx - \int_{\mathbb{R}^3}\frac{u_rn_\theta}{r}(r^6n_\theta)dx\\
&=\int_{\mathbb{R}^3}(B_r\partial_r+B_z\partial_z)(r^3\omega_\theta)(r^3n_\theta)dx - \int_{\mathbb{R}^3}B_r[\partial_r(r^3)]\omega_\theta(r^3n_\theta)dx\\
&\quad+\int_{\mathbb{R}^3}\frac{B_rn_\theta}{r}(r^6n_\theta)dx.
\end{aligned}
$$

After integration by parts and more elementary computations, where $\int_{\mathbb{R}^3}(u_r\partial_r + u_z\partial_z)(r^3n_\theta)(r^3n_\theta)dx = 0$, we have

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(r^3n_\theta)^2dx + \int_{\mathbb{R}^3}|\nabla(r^3n_\theta)|^2dx
$$

$$= \int_{\mathbb{R}^3} (B_r\partial_r + B_z\partial_z)(r^3\omega_\theta)(r^3 n_\theta)dx + 4\int_{\mathbb{R}^3} r^5 n_\theta^2 u_r dx$$
$$+ 8\int_{\mathbb{R}^3} r^4 n_\theta^2 dx - 2\int_{\mathbb{R}^3} B_r r^5 \omega_\theta n_\theta dx. \tag{3.9}$$

Combining the equations (3.7) and (3.9), and by $(B\cdot\nabla n, \omega) = -(B\cdot\nabla\omega, n)$ with the condition $\mathrm{div} u = 0$, we have

$$\frac{1}{2}\frac{d}{dt}[\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx + \int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx] + \int_{\mathbb{R}^3}|\nabla(r^3\omega_\theta)|^2dx + \int_{\mathbb{R}^3}|\nabla(r^3 n_\theta)|^2dx$$
$$= 4\int_{\mathbb{R}^3} r^5\omega_\theta^2 u_r dx + 4\int_{\mathbb{R}^3} r^5 n_\theta^2 u_r dx + 8\int_{\mathbb{R}^3} r^4\omega_\theta^2 dx + 8\int_{\mathbb{R}^3} r^4 n_\theta^2 dx$$
$$- 4\int_{\mathbb{R}^3} B_r r^5\omega_\theta n_\theta dx + \int_{\mathbb{R}^3} r^5\omega_\theta[\partial_z(u_\theta^2)]dx - \int_{\mathbb{R}^3} r^5\omega_\theta[\partial_z(B_\theta^2)]dx$$
$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{3.10}$$

We will estimate $I_1 - I_7$, by use of the Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality and Basic Energy estimate, and choosing $\epsilon$ sufficiently small, we obtain

$$\begin{aligned}
I_1 &\le C\|u_r\|_{L^2}(\int_{\mathbb{R}^3}(r^5\omega_\theta^2)^2dx)^{\frac{1}{2}} \\
&\le C(\int_{\mathbb{R}^3}(\omega_\theta)^2dx)^{\frac{1}{6}}(\int_{\mathbb{R}^3}(r^3\omega_\theta)^5dx)^{\frac{1}{3}} \\
&\le C\int_{\mathbb{R}^3}(\omega_\theta)^2dx + C(\int_{\mathbb{R}^3}(r^3\omega_\theta)^5dx)^{\frac{2}{5}} \\
&\le C\|\omega_\theta\|_{L^2}^2 + C\|r^3\omega_\theta\|_{L^2}^{\frac{1}{5}}\|\nabla(r^3\omega_\theta)\|_{L^2}^{\frac{9}{5}} \\
&\le C\|\omega_\theta\|_{L^2}^2 + \epsilon\|\nabla(r^3\omega_\theta)\|_{L^2}^2 + C_\epsilon\|r^3\omega_\theta\|_{L^2}^2.
\end{aligned}$$

Similarly, we estimate $I_2$,

$$I_2 \le C\|n_\theta\|_{L^2}^2 + \epsilon\|\nabla(r^3 n_\theta)\|_{L^2}^2 + C_\epsilon\|r^3 n_\theta\|_{L^2}^2.$$

For $I_3$ and $I_4$, we estimate

$$\begin{aligned}
I_3 &\le C\int_{\mathbb{R}^3}(r^3\omega_\theta)^{\frac{4}{3}}(\omega_\theta)^{\frac{2}{3}}dx \\
&\le C(\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx)^{\frac{2}{3}}(\int_{\mathbb{R}^3}(\omega_\theta)^2dx)^{\frac{1}{3}} \\
&\le C\|\omega_\theta\|_{L^2}^2 + C_\epsilon\|r^3\omega_\theta\|_{L^2}^2.
\end{aligned}$$

Similarly, we estimate $I_4$,

$$\begin{aligned}
I_4 &\le C(\int_{\mathbb{R}^3}(r^3 n_\theta)^2dx)^{\frac{2}{3}}(\int_{\mathbb{R}^3}(n_\theta)^2dx)^{\frac{1}{3}} \\
&\le C\|n_\theta\|_{L^2}^2 + C_\epsilon\|r^3 n_\theta\|_{L^2}^2.
\end{aligned}$$

As for $I_5$, we obtain

$$I_5 \le C\|B_r\|_{L^2}\int_{\mathbb{R}^3}(r^3\omega_\theta)^{\frac{5}{6}}(\omega_\theta)^{\frac{1}{6}}(r^3 n_\theta)^{\frac{5}{6}}(n_\theta)^{\frac{1}{6}}dx$$

$$
\begin{aligned}
&\le C\int_{\mathbb{R}^3}((r^3\omega_\theta)^5dx)^{\frac{1}{6}}(\int_{\mathbb{R}^3}((r^3n_\theta)^5dx)^{\frac{1}{6}}(\int_{\mathbb{R}^3}(\omega_\theta)^2dx)^{\frac{1}{12}}(\int_{\mathbb{R}^3}(n_\theta)^2dx)^{\frac{1}{12}}\\
&\le C\|r^3\omega_\theta\|_{L^5}^{\frac{5}{6}}\|r^3n_\theta\|_{L^5}^{\frac{5}{6}}\|\omega_\theta\|_{L^2}^{\frac{1}{6}}\|n_\theta\|_{L^2}^{\frac{1}{6}}\\
&\le C\|r^3\omega_\theta\|_{L^2}^{\frac{1}{12}}\|\nabla(r^3\omega_\theta)\|_{L^2}^{\frac{3}{4}}\|r^3n_\theta\|_{L^2}^{\frac{1}{12}}\|\nabla(r^3n_\theta)\|_{L^2}^{\frac{3}{4}}\|\omega_\theta\|_{L^2}^{\frac{1}{6}}\|n_\theta\|_{L^2}^{\frac{1}{6}}\\
&\le C\|\omega_\theta\|_{L^2}^2+\epsilon\|\nabla(r^3\omega_\theta)\|_{L^2}^2+C_\epsilon\|r^3\omega_\theta\|_{L^2}^2\\
&\quad +C\|n_\theta\|_{L^2}^2+\epsilon\|\nabla(r^3n_\theta)\|_{L^2}^2+C_\epsilon\|r^3n_\theta\|_{L^2}^2.
\end{aligned}
$$

For $I_6$ and $I_7$, we estimate

$$
\begin{aligned}
I_6+I_7 &\le \int_{R^3}(r^3\omega_\theta)[\partial_z(J^2-L^2)]dx\\
&\le \int_{R^3}(r^3\omega_\theta)\partial_z(J+L)(J-L)dx\\
&= \int_{R^3}(r^3\omega_\theta)(J-L)\partial_z(J+L)dx+\int_{R^3}(r^3\omega_\theta)(J+L)\partial_z(J-L)dx\\
&\le \|r^3\omega_\theta\|_{L^2}\|(J-L)\|_{L^\infty}\|\partial_z(J+L)\|_{L^2}\\
&\quad +\|r^3\omega_\theta\|_{L^2}\|(J-L)\|_{L^\infty}\|\partial_z(J+L)\|_{L^2}\\
&\le C_\varepsilon\|r^3\omega_\theta\|_{L^2}^2+C\|\nabla(J+L)\|_{L^2}^2+C\|\nabla(J-L)\|_{L^2}.
\end{aligned}
$$

Inserting the above estimate into (3.10), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx+\int_{\mathbb{R}^3}(r^3\omega_\theta)^2dx)\\
&+\int_{\mathbb{R}^3}|\nabla(r^3\omega_\theta)|^2dx+\int_{\mathbb{R}^3}|\nabla(r^3n_\theta)|^2dx\\
&\le C(\|\omega_\theta\|_{L^2}^2+\|n_\theta\|_{L^2}^2)+C_\epsilon(\|r^3\omega_\theta\|_{L^2}^2+\|r^3n_\theta\|_{L^2}^2)\\
&\quad +C(\|\nabla(J-L)\|_{L^2}^2+\|\nabla(J+L)\|_{L^2}^2).
\end{aligned}\tag{3.11}
$$

Using the Gronwall's inequality and Basic Energy estimate, we have

$$
\begin{aligned}
&\sup_{0\le t\le T}[\|r^3\omega_\theta\|_{L^2}^2+\|r^3n_\theta\|_{L^2}^2]\\
&+C[\int_0^T\int_{\mathbb{R}^3}|\nabla(r^3\omega_\theta)|^2dxdt+\int_0^T\int_{\mathbb{R}^3}|\nabla(r^3n_\theta)|^2dxdt]\\
&\le C(T)[\|r^3\omega_\theta(0)\|_{L^2}^2+\|r^3n_\theta(0)\|_{L^2}^2]+C[\int_0^T\|\omega_\theta\|_{L^2}^2dt+\int_0^T\|n_\theta\|_{L^2}^2dt]\\
&\quad +C[\int_0^T\|\nabla(J-L)\|_{L^2}^2+\|\nabla(J+L)\|_{L^2}^2dt]\\
&\le C(T,\|r^3\omega_\theta(0)\|_{L^2},\|r^3n_\theta(0)\|_{L^2},\|u_0\|_{L^2},\|B_0\|_{L^2},\|J_0\|_{L^2},\|L_0\|_{L^2}).
\end{aligned}
$$

Thus, Theorem 1 have been proved.

Now, we give the following proposition before proving the Theorem 1. Suppose that $(u,B)$ are axially symmetric solutions to the MHD system (1.2). Let $\int_0^T\|u_rB_\theta-u_\theta B_r\|_{L^\infty}dt\le C$, $J$, $L$ satisfy (1.4)-(1.5), with initial data $J_0,L_0\in L^\infty(\mathbb{R}^3)$, then $J\in L^\infty(0,T;L^\infty(\mathbb{R}^3))$, $L\in L^\infty(0,T;L^\infty(\mathbb{R}^3))$. **Proof.** Multiplying the both sides of (1.4) by $|J+L|^{p-2}(J+L)$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|J+L|^p dx+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla|J+L|^{\frac{p}{2}}|^2dx\\
&=-\int_{\mathbb{R}^3}\frac{2}{r}\partial_r(J+L)|J+L|^{p-2}(J+L)dx\\
&\quad-2\int_{\mathbb{R}^3}(u_\theta B_r-B_\theta u_r)|J+L|^{p-2}(J+L)dx\\
&=K_3+K_4. \qquad (3.12)
\end{aligned}$$

For $K_3$, we have

$$\begin{aligned}
K_3&=-\int_{\mathbb{R}^3}\frac{2}{r}\partial_r(J+L)|J+L|^{p-2}(J+L)dx\\
&=-\frac{4\pi}{p}\int_{-\infty}^{\infty}\int_0^{\infty}\partial_r|J+L|^p drdz=0.
\end{aligned}$$

For $K_4$, we estimate

$$\begin{aligned}
K_4&\le\int_{\mathbb{R}^3}|J+L|^{p-1}|u_\theta B_r-B_\theta u_r|dx\\
&\le\|J+L\|_{L^p}^{p-1}\|u_\theta B_r-B_\theta u_r\|_{L^p}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|J+L\|_{L^p}^p+\frac{4(p-1)}{p^2}\|\nabla|J+L|^{\frac{p}{2}}\|_{L^2}^2\\
&\le\|J+L\|_{L^p}^{p-1}\|u_\theta B_r-B_\theta u_r\|_{L^p}.
\end{aligned}$$

Based on direct calculation, we obtain

$$\frac{d}{dt}\|J+L\|_{L^p}\le\|u_\theta B_r-B_\theta u_r\|_{L^p}. \qquad (3.13)$$

Let $p\to\infty$ and use the assumption $\int_0^T\|u_\theta B_r-B_\theta u_r\|_{L^\infty}dt\le C$, we obtain

$$\begin{aligned}
\sup_{0\le t\le T}\|J+L\|_{L^\infty}&\le\|J_0+L_0\|_{L^\infty}+\int_0^T\|u_\theta B_r-B_\theta u_r\|_{L^\infty}dt\\
&\le\|J_0+L_0\|_{L^\infty}+C. \qquad (3.14)
\end{aligned}$$

Similarly, multiplying the both sides of (3.25) by $|J-L|^{p-2}(J-L)$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|J-L|^p dx+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla|J-L|^{\frac{p}{2}}|^2dx\\
&=-\int_{\mathbb{R}^3}\frac{2}{r}\partial_r(J-L)|J-L|^{p-2}(J-L)dx\\
&\quad-2\int_{\mathbb{R}^3}(u_rB_\theta-u_\theta B_r)|J-L|^{p-2}(J-L)dx.
\end{aligned}$$

Then, we have

$$\frac{1}{p}\frac{d}{dt}\|J-L\|_{L^p}^p+\frac{4(p-1)}{p^2}\|\nabla|J-L|^{\frac{p}{2}}\|_{L^2}^2$$
$$\leq \|J-L\|_{L^p}^{p-1}\|B_\theta u_r-uB_\theta B_r\|_{L^p}.$$

Hence, we obtain

$$\sup_{0\leq t\leq T}\|J-L\|_{L^\infty}\leq \|J_0-L_0\|_{L^\infty}+C. \tag{3.15}$$

Combining the estimates (3.14) and (3.15), we obtain

$$\sup_{0\leq t\leq T}(\|J\|_{L^\infty}+\|L\|_{L^\infty})\leq C(\|J_0\|_{L^\infty}+\|L_0\|_{L^\infty}). \tag{3.16}$$

Thus, we complete the proof of Proposition 3. $\square$

For $p\in[2,\infty)$, by using Hölder's inequality, Young's inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
K_4 &\leq \int_{\mathbb{R}^3}|J+L|^{p-1}|u_\theta B_r-B_\theta u_r|dx\\
&=\int_{\mathbb{R}^3}|J+L|^{\frac{p}{2}}|J+L|^{\frac{p}{2}-1}|u_\theta B_r-B_\theta u_r|dx\\
&\leq \||J+L|^{\frac{p}{2}}\|_{L^6}\||J+L|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}}\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}\\
&\leq \|\nabla|J+L|^{\frac{p}{2}}\|_{L^2}\|J+L\|_{L^p}^{\frac{p-2}{2}}\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}\\
&\leq \frac{2(p-1)}{p^2}\|\nabla|J+L|^{\frac{p}{2}}\|_{L^2}^2+\frac{Cp^2}{p-1}\|J+L\|_{L^p}^{\frac{p-2}{2}}\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}^2.
\end{aligned}$$

Thus, we have

$$\frac{1}{p}\frac{d}{dt}\|J+L\|_{L^p}^p+\frac{2(p-1)}{p^2}\|\nabla|J+L|^{\frac{p}{2}}\|_{L^2}^2$$
$$\leq \frac{Cp^2}{p-1}\|J+L\|_{L^p}^{\frac{p-2}{2}}\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}^2,$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|J+L\|_{L^p}^2\leq \frac{Cp^2}{p-1}\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}^2. \tag{3.17}$$

Hence, we have

$$\sup_{0\leq t\leq T}\|J+L\|_{L^p}^2\leq \|J_0+L_0\|_{L^p}^2+C(p)\int_0^T\|u_\theta B_r-B_\theta u_r\|_{L^{\frac{3p}{p+3}}}^2dt. \tag{3.18}$$

Similarly, we obtain

$$\frac{1}{p}\frac{d}{dt}\|J-L\|_{L^p}^p+\frac{2(p-1)}{p^2}\|\nabla|J-L|^{\frac{p}{2}}\|_{L^2}^2$$
$$\leq \frac{Cp^2}{p-1}\|J+L\|_{L^p}^{\frac{p-2}{2}}\|B_\theta u_r-u_\theta B_r\|_{L^{\frac{3p}{p+3}}}^2,$$

which implies

$$\sup_{0\le t\le T}\|J-L\|_{L^p}^2 \le \|J_0-L_0\|_{L^p}^2 + C(p)\int_0^T \|B_\theta u_r - u_\theta B_r\|^2_{L^{\frac{3p}{p+3}}}\,dt.. \quad (3.19)$$

where $\int_0^T \|u_\theta B_r - B_\theta u_r\|^2_{L^{\frac{3p}{p+3}}}\,dt \le C$ satisfied, we could obtain $J\in L^\infty(0,T;L^p(\mathbb{R}^3))$, $L\in L^\infty(0,T;L^p(\mathbb{R}^3))$.

In particular, if $p=3$, the assumption $\int_0^T \|u_\theta B_r - B_\theta u_r\|^2_{L^{\frac{3}{2}}}\,dt \le C$ is satisfied naturally. Hence, we have

$$\sup_{0\le t\le T}\|J\|_{L^3} + \|L\|_{L^3} \le C(\|J_0\|_{L^3} + \|L_0\|_{L^3}).$$

In fact, by using Hölder's inequality and Gagliardo-Nirenberg inequality, we get

$$\|u_\theta B_r - B_\theta u_r\|^2_{L^{\frac{3}{2}}} \le \|\nabla u\|^2_{L^2}\|B\|^2_{L^2}.$$

By applying Proposition 3, we give the proof of Theorem 1 now.

**Proof of Theorem 1** By applying standard energy estimate to $\Omega,\Gamma$ equations, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Omega\|^2_{L^2} + \int_{\mathbb{R}^3}\Omega(u\cdot\nabla)\Omega dx - \int_{\mathbb{R}^3}\Omega(\triangle + \frac{2}{r}\partial_r)\Omega dx + \int_{\mathbb{R}^3}\Omega(B\cdot\nabla)\Gamma dx\\ &+\int_{\mathbb{R}^3}\Omega(\frac{2u_\theta}{r})Rdx + \int_{\mathbb{R}^3}\Omega(\frac{2B_\theta}{r})Kdx = 0, \end{aligned}\quad (3.20)$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Gamma\|^2_{L^2} + \int_{\mathbb{R}^3}\Gamma(u\cdot\nabla)\Gamma dx - \int_{\mathbb{R}^3}\Gamma(\triangle + \frac{2}{r}\partial_r)\Gamma dx\\ &+\int_{\mathbb{R}^3}\Gamma(B\cdot\nabla)\Omega dx = 0.\end{aligned}\quad (3.21)$$

Using $\mathrm{div}u=0$, we have

$$\int_{\mathbb{R}^3}\Omega(u\cdot\nabla)\Omega dx = \int_{\mathbb{R}^3}\Omega u_i\partial_i\Omega dx = -\frac{1}{2}\int_{\mathbb{R}^3}(\nabla\cdot u)\Omega^2 dx = 0.$$

Similarly, we have

$$\int_{\mathbb{R}^3}\Gamma(u\cdot\nabla)\Gamma dx = -\frac{1}{2}\int(\nabla\cdot u)\Gamma^2 dx = 0.$$

By direct calculations, we get

$$\begin{aligned}-\int_{\mathbb{R}^3}\Omega(\triangle+\frac{2}{r}\partial_r)\Omega dx &= -[\int_{\mathbb{R}^3}\Omega\triangle\Omega dx + \int_{\mathbb{R}^3}\Omega\frac{2}{r}\partial_r\Omega dx]\\ &= -[\int_{\mathbb{R}^3}\Omega\partial_i\partial_i\Omega dx + \int_{\mathbb{R}^3}\frac{1}{r}(\partial_r\Omega^2)rdrdz]\\ &= \|\nabla\Omega\|^2_{L^2} + \int_{\mathbb{R}^3}|\Omega(0,z,t)|^2dz.\end{aligned}$$

Similarly, we have

$$-\int_{\mathbb{R}^3}\Gamma(\triangle+\frac{2}{r}\partial_r)\Gamma dx=\|\nabla\Gamma\|_{L^2}^2+\int_{\mathbb{R}^3}|\Gamma(0,z,t)|^2dz.$$

We can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+(\|\nabla\Omega\|_{L^2}^2+\|\nabla\Gamma\|_{L^2}^2)+\int_{\mathbb{R}^3}|\Omega(0,z,t)|^2dz\\&=-2\int_{\mathbb{R}^3}(\frac{u_\theta}{r}\Omega R+\frac{B_\theta}{r}\Omega K)dx.\end{aligned}\tag{3.22}$$

Now, we estimate the right hand side of (3.22) under the assumption of condition (a). Let

$$\varepsilon=(1+\ln(C_0\max\{P^{\frac{1}{4}},r_0^{-1/3}P_1\}+1))^{-\frac{3}{2}},\tag{3.23}$$

then $\|J\|_{L^\infty(r\le r_1)}\le\varepsilon\le1$, $\|L\|_{L^\infty(r\le r_1)}\le\varepsilon\le1$, $0<r\le r_1$, and we can apply Lemma 2 with

$$r_1=r(t)=\min\{\frac{\varepsilon M(\varepsilon)}{a(t)+b(t)},r_0\}.$$

By Lemma 2, we have

$$\begin{aligned}&-2\int_{\mathbb{R}^3}\frac{u_\theta}{r}\Omega Rdx\\&\le2\int_{\mathbb{R}^3}\sqrt{|\frac{u_\theta}{r}|}|\Omega|\sqrt{|\frac{u_\theta}{r}|}|R|dx\\&\le\frac{1}{4}\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}^3}\frac{|u_\theta|}{r}|\Omega|^2dx+4\varepsilon^{-\frac{1}{3}}\int_{\mathbb{R}^3}\frac{|u_\theta|}{r}|R|^2dx\\&\le\frac{1}{4}\varepsilon^{\frac{1}{3}}[\varepsilon^{-\frac{1}{3}}\int_{\mathbb{R}^3}|\partial_r\Omega|^2dx+\frac{C(\varepsilon^{-\frac{1}{3}}+\|J\|_{L^\infty})}{r_1^2}]\int_{r\ge\frac{r_1}{2}}|\Omega|^2dx]\\&\quad+4\varepsilon^{-\frac{1}{3}}[\varepsilon^{-\frac{1}{3}}\int_{\mathbb{R}^3}|\partial_rR|^2dx+\frac{C(\varepsilon^{-\frac{1}{3}}+\|J\|_{L^\infty})}{r_1^2}\int_{r\ge\frac{r_1}{2}}|R|^2dx]\\&\le\frac{1}{4}\|\partial_r\Omega\|_{L^2}^2+\frac{C(1+\varepsilon^{\frac{1}{3}}\|J\|_{L^\infty})}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\Omega|^2dx\\&\quad+4\varepsilon^{-\frac{2}{3}}\|\partial_rR\|_{L^2}^2+\frac{C\varepsilon^{-\frac{2}{3}}(1+\varepsilon^{\frac{1}{3}}\|J\|_{L^\infty})}{r_1^2}\int_{r\ge\frac{r_1}{2}}|R|^2dx,\end{aligned}\tag{3.24}$$

Similarly, we have

$$\begin{aligned}&-2\int_{\mathbb{R}^3}\frac{B_\theta}{r}\Omega Kdx\\&\le\frac{1}{4}\|\partial_r\Omega\|_{L^2}^2+\frac{C(1+\varepsilon^{\frac{1}{3}}\|L\|_{L^\infty})}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\Omega|^2dx+4\varepsilon^{-\frac{2}{3}}\|\partial_rK\|_{L^2}^2\\&\quad+\frac{C\varepsilon^{-\frac{2}{3}}(1+\varepsilon^{\frac{1}{3}}\|L\|_{L^\infty})}{r_1^2}\int_{r\ge\frac{r_1}{2}}|K|^2dx.\end{aligned}\tag{3.25}$$

Inserting (3.24) and (3.25) into (3.22), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+(\|\nabla\Omega\|_{L^2}^2+\|\nabla\Gamma\|_{L^2}^2)\\
&+\int_{\mathbb{R}^3}|\Omega(0,z,t)|^2dz+\int_{\mathbb{R}^3}|\Gamma(0,z,t)|^2dz\\
&\le\frac{1}{2}\|\partial_r\Omega\|_{L^2}^2+4\varepsilon^{-\frac{2}{3}}(\|\partial_rR\|_{L^2}^2+\|\partial_rK\|_{L^2}^2)\\
&\quad+\frac{C[1+\varepsilon^{\frac{1}{3}}(\|J\|_{L^\infty}+\|L\|_{L^\infty})]}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\Omega|^2dx\\
&\quad+\frac{C\varepsilon^{-\frac{2}{3}}[1+\varepsilon^{\frac{1}{3}}(\|J\|_{L^\infty}+\|L\|_{L^\infty})]}{r_1^2}\int_{r\ge\frac{r_1}{2}}|R|^2+|K|^2dx.
\end{aligned}\tag{3.26}
$$

As for $R, K$ equations, we have the following estimates

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|R\|_{L^2}^2+\int_{\mathbb{R}^3}R(u\cdot\nabla)Rdx-\int_{\mathbb{R}^3}R(\triangle+\frac{2}{r}\partial_r)Rdx+\int_{\mathbb{R}^3}R(B\cdot\nabla)Kdx\\
&-\int_{\mathbb{R}^3}R(\omega_r\partial_r+w_z\partial_z)\frac{u_r}{r}dx+\int_{\mathbb{R}^3}R(n_r\partial_rdx+n_z\partial_z)\frac{B_r}{r}dx=0,
\end{aligned}\tag{3.27}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|K\|_{L^2}^2+\int_{\mathbb{R}^3}K(u\cdot\nabla)Kdx-\int_{\mathbb{R}^3}K(\triangle+\frac{2}{r}\partial_r)Kdx+\int_{\mathbb{R}^3}K(B\cdot\nabla)Rdx\\
&-\int_{\mathbb{R}^3}K(\omega_r\partial_r+w_z\partial_z)\frac{u_r}{r}dx+\int_{\mathbb{R}^3}K(n_r\partial_r+n_z\partial_z)\frac{B_r}{r}dx=0.
\end{aligned}\tag{3.28}
$$

Similarly, using $\mathrm{div}u=0$, we get

$$
\int_{\mathbb{R}^3}R(u\cdot\nabla)Rdx=-\frac{1}{2}\int_{\mathbb{R}^3}(\nabla\cdot u)R^2dx=0,
$$

$$
\int_{\mathbb{R}^3}K(u\cdot\nabla)Kdx=-\frac{1}{2}\int_{\mathbb{R}^3}(\nabla\cdot u)K^2dx=0.
$$

and by direct calculations, we obtain

$$
-\int_{\mathbb{R}^3}R(\triangle+\frac{2}{r}\partial_r)Rdx=\|\nabla R\|_{L^2}^2+\int_{\mathbb{R}^3}|R(0,z,t)|^2dz,
$$

$$
-\int_{\mathbb{R}^3}K(\triangle+\frac{2}{r}\partial_r)Kdx=\|\nabla R\|_{L^2}^2+\int_{\mathbb{R}^3}|K(0,z,t)|^2dz.
$$

Combining the equations (3.27) and (3.28) and applying $(B\cdot\nabla\Gamma,\Omega)=-(B\cdot\nabla\Omega,\Gamma)$ ,$(B\cdot\nabla K,R)=-(B\cdot\nabla R,K)$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|R\|_{L^2}^2+\|K\|_{L^2}^2)+(\|\nabla R\|_{L^2}^2+\|\nabla K\|_{L^2}^2)\\
&+\int_{\mathbb{R}^3}|R(0,z,t)|^2dz+\int_{\mathbb{R}^3}|K(0,z,t)|^2dz\\
&=\int_{\mathbb{R}^3}R(\omega_r\partial_r+\omega_z\partial_z)\frac{u_r}{r}dx-\int_{\mathbb{R}^3}R(n_r\partial_rdx+n_z\partial_z)\frac{B_r}{r}dx
\end{aligned}
$$

$$+\int_{\mathbb{R}^3} K(\omega_r\partial_r+\omega_z\partial_z)\frac{u_r}{r}dx-\int_{\mathbb{R}^3} K(n_r\partial_r dx+n_z\partial_z)\frac{B_r}{r}dx. \tag{3.29}$$

Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^3} R(\omega_r\partial_r+\omega_z\partial_z)\frac{u_r}{r}dx\\
=&\int_{\mathbb{R}^3}(\partial_z R)u_\theta\partial_r(\frac{u_r}{r})dx-\int_{\mathbb{R}^3}(\partial_r R)\frac{1}{r}(ru_\theta)\partial_r(\frac{u_r}{r})dx\\
\le&\ C\|\nabla R\|_{L^2}\|u_\theta\nabla(\frac{u_r}{r})\|_{L^2}\\
\le&\ \frac{1}{4}\|\nabla R\|_{L^2}^2+\|u_\theta\nabla(\frac{u_r}{r})\|_{L^2}.
\end{aligned}\tag{3.30}$$

Similarly, we have

$$-\int_{\mathbb{R}^3} R(n_r\partial_r+n_z\partial_z)\frac{B_r}{r}dx\le\frac{1}{4}\|\nabla R\|_{L^2}^2+\|B_\theta\nabla(\frac{B_r}{r})\|_{L^2}^2, \tag{3.31}$$

$$\int_{\mathbb{R}^3} K(w_r\partial_r+w_z\partial_z)\frac{B_r}{r}dx\le\frac{1}{4}\|\nabla K\|_{L^2}^2+\|u_\theta\nabla(\frac{B_r}{r})\|_{L^2}^2, \tag{3.32}$$

$$-\int_{\mathbb{R}^3} K(n_r\partial_r+n_z\partial_z)\frac{u_r}{r}dx\le\frac{1}{4}\|\nabla K\|_{L^2}^2+\|B_\theta\nabla(\frac{u_r}{r})\|_{L^2}^2. \tag{3.33}$$

Hence, using (2.9) and (2.10), when we choose $f=\partial_r(\frac{u_r}{r}),\partial_z(\frac{u_r}{r})$, we obtain

$$\begin{aligned}
\|u_\theta\nabla(\frac{u_r}{r})\|_{L^2}^2&\le\int_{\mathbb{R}^3}|4u_\theta|^2|\frac{1}{4}\nabla(\frac{u_r}{r})|^2dx\\
&\le\frac{1}{16}\varepsilon^{\frac{2}{3}}\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{u_r}{r})|^2dx+\frac{C\varepsilon^{\frac{2}{3}}(1+\varepsilon^{-\frac{2}{3}}\|J\|_{L^\infty}^2)}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\nabla(\frac{u_r}{r})|^2dx,\\
\|B_\theta\nabla(\frac{u_r}{r})\|_{L^2}^2&\le\int_{\mathbb{R}^3}|4B_\theta|^2|\frac{1}{4}\nabla(\frac{u_r}{r})|^2dx\\
&\le\frac{1}{16}\varepsilon^{\frac{2}{3}}\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{u_r}{r})|^2dx+\frac{C\varepsilon^{\frac{2}{3}}(1+\varepsilon^{-\frac{2}{3}}\|L\|_{L^\infty}^2)}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\nabla(\frac{u_r}{r})|^2dx.
\end{aligned}$$

Similarly, when we choose $f=\partial_r(\frac{B_r}{r}),\partial_z(\frac{B_r}{r})$, we obtain

$$\|u_\theta\nabla(\frac{B_r}{r})\|_{L^2}^2\le\frac{1}{16}\varepsilon^{\frac{2}{3}}\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{B_r}{r})|^2dx+\frac{C\varepsilon^{\frac{2}{3}}(1+\varepsilon^{-\frac{2}{3}}\|J\|_{L^\infty}^2)}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\nabla(\frac{B_r}{r})|^2dx,$$

$$\|B_\theta\nabla(\frac{B_r}{r})\|_{L^2}^2\le\frac{1}{16}\varepsilon^{\frac{2}{3}}\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{B_r}{r})|^2dx+\frac{C\varepsilon^{\frac{2}{3}}(1+\varepsilon^{-\frac{2}{3}}\|L\|_{L^\infty}^2)}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\nabla(\frac{B_r}{r})|^2dx.$$

By using Lemma 2, we have

$$\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{u_r}{r})|^2dx\le\int_{\mathbb{R}^3}|\nabla^2(\frac{u_r}{r})|^2dx\le\|\nabla^2(\frac{u_r}{r})|^2\|_{L^2}^2\le\|\partial_z\Omega\|_{L^2}^2,$$

$$\int_{\mathbb{R}^3}|\partial_r\nabla(\frac{B_r}{r})|^2dx\le\int_{\mathbb{R}^3}|\nabla^2(\frac{B_r}{r})|^2dx\le\|\nabla^2(\frac{B_r}{r})|^2\|_{L^2}^2\le\|\partial_z\Gamma\|_{L^2}^2.$$

Combining the above estimates, we have

$$\frac{1}{2}\frac{d}{dt}(\|R\|_{L^2}^2+\|K\|_{L^2}^2)+\frac{1}{2}(\|\nabla R\|_{L^2}^2+\|\nabla K\|_{L^2}^2)$$

$$+\int_{\mathbb{R}^3}|R(0,z,t)|^2dz+\int_{\mathbb{R}^3}|K(0,z,t)|^2dz$$
$$\le\frac{C\varepsilon^{\frac{2}{3}}[1+\varepsilon^{-\frac{2}{3}}(\|J\|_{L^\infty}^2+\|L\|_{L^\infty}^2)]}{r_2^1}\int_{r\ge\frac{r_2}{2}}|\nabla(\frac{B_r}{r})|^2+|\nabla(\frac{u_r}{r})|^2dx$$
$$+\frac{1}{8}\varepsilon^{\frac{2}{3}}(\|\partial_z\Omega\|_{L^2}^2+\|\partial_z\Gamma\|_{L^2}^2). \tag{3.34}$$

Multiplying the both sides of (3.26) by $\frac{1}{8}\varepsilon^{\frac{2}{3}}$ and combining the (3.34) with it, then we can obtain

$$\frac{d}{dt}[\frac{1}{16}\varepsilon^{\frac{2}{3}}(\|\Omega\|_{L^2}^2+\|\Gamma|_{L^2}^2)+\frac{1}{2}(\|R\|_{L^2}^2+\|K\|_{L^2}^2)]+\frac{1}{8}\varepsilon^{\frac{2}{3}}(\int_{\mathbb{R}^3}|\Omega(0,z,t)|^2dz$$
$$+\int_{\mathbb{R}^3}|\Gamma(0,z,t)|^2dz)+\int_{\mathbb{R}^3}|R(0,z,t)|^2dz+\int_{\mathbb{R}^3}|K(0,z,t)|^2dz$$
$$\le\frac{C\varepsilon^{\frac{2}{3}}[1+\varepsilon^{\frac{1}{3}}(\|J\|_{L^\infty}+\|L\|_{L^\infty})]}{r_1^2}\int_{r\ge\frac{r_1}{2}}|\Omega|^2dx$$
$$+\frac{C[1+\varepsilon^{\frac{1}{3}}(\|J\|_{L^\infty}+\|L\|_{L^\infty})]}{r_1^2}\int_{r\ge\frac{r_1}{2}}|R|^2+|K|^2dx$$
$$+\frac{C\varepsilon^{\frac{2}{3}}[1+\varepsilon^{-\frac{2}{3}}(\|J\|_{L^\infty}^2+\|L\|_{L^\infty}^2)]}{r_2^2}\int_{r\ge\frac{r_1}{2}}|\nabla(\frac{B_r}{r})|^2+|\nabla(\frac{u_r}{r})|^2dx. \tag{3.35}$$

Denote

$$P_2=1+\varepsilon^{\frac{1}{3}}(\|J\|_{L^\infty}+\|L\|_{L^\infty})+\varepsilon^{-\frac{2}{3}}(\|J\|_{L^\infty}^2+\|L\|_{L^\infty}^2),$$

we have

$$\frac{d}{dt}[\frac{1}{16}\varepsilon^{\frac{2}{3}}(\|\Omega\|_{L^2}^2+\|\Gamma|_{L^2}^2)+\frac{1}{2}(\|R\|_{L^2}^2+\|K\|_{L^2}^2)]$$
$$\le\frac{CP_2}{r(t)^2}\int_{r\ge\frac{r(t)}{2}}\varepsilon^{\frac{2}{3}}(|\nabla(\frac{u_r}{r})|^2+|\Omega|^2+|\nabla(\frac{B_r}{r})|^2)+|R|^2+|K|^2dx. \tag{3.36}$$

By Lemma 2 and the fact that

$$\nabla(\frac{u_r}{r})=e_r\partial_r(\frac{u_r}{r})+\frac{e_\theta}{r}\partial_\theta(\frac{u_r}{r})+e_z\partial_z(\frac{u_r}{r})$$
$$=-(\frac{u_r}{r^2})e_r+\frac{e_r}{r}\partial_ru_r+\frac{e_z}{r}\partial_zu_r=\frac{\nabla u_r}{r}-(\frac{u_r}{r^2})e_r,$$
$$|\nabla u|^2=|\nabla u_r|^2+|\nabla u_\theta|^2+|\nabla u_z|^2+|\frac{u_r}{r}|^2+|\frac{u_\theta}{r}|^2.$$

Similarly, we have

$$\nabla(\frac{B_r}{r})=\frac{\nabla B_r}{r}-(\frac{B_r}{r^2})e_r,$$
$$|\nabla B|^2=|\nabla B_r|^2+|\nabla B_\theta|^2+|\nabla B_z|^2+|\frac{B_r}{r}|^2+|\frac{B_\theta}{r}|^2.$$

By Lemma 2, we have

$$\int_{r\ge\frac{r(t)}{2}}|\nabla(\frac{u_r}{r})|^2+|\Omega|^2dx\le\|\nabla(\frac{u_r}{r})\|_{L^2}^2+\|\Omega\|_{L^2}^2\le2\|\Omega\|_{L^2}^2,$$

$$\int_{r\geq\frac{r(t)}{2}}|\nabla(\frac{u_r}{r})|^2+|\Omega|^2+|R|^2dx\leq\frac{C_1}{r(t)^2}\int_{\mathbb{R}^3}|\nabla u|^2dx.$$

Similarly, we have

$$\int_{r\geq\frac{r(t)}{2}}|\nabla(\frac{B_r}{r})|^2dx\leq\|\Gamma\|_{L^2}^2,$$
$$\int_{r\geq\frac{r(t)}{2}}|\nabla(\frac{B_r}{r})|^2+|K|^2dx\leq\frac{C_1}{r(t)^2}\int_{\mathbb{R}^3}|\nabla B|^2dx.$$

Hence, we can obtain

$$\begin{aligned}&\frac{CP_2}{r(t)^2}\int_{r\geq\frac{r(t)}{2}}\varepsilon^{\frac{2}{3}}(|\nabla(\frac{u_r}{r})|^2+|\Omega|^2+|\nabla(\frac{u_r}{r})|^2)+|R|^2+|K|^2dx\\&\leq\frac{CP_2}{r(t)^2}[\varepsilon^{\frac{2}{3}}(2\|\Omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+\|R\|_{L^2}^2+\|K\|_{L^2}^2],\end{aligned}$$

$$\begin{aligned}&\frac{CP_2}{r(t)^2}\int_{r\geq\frac{r(t)}{2}}\varepsilon^{\frac{2}{3}}(|\nabla(\frac{u_r}{r})|^2+|\Omega|^2+|\nabla(\frac{u_r}{r})|^2)+|R|^2+|K|^2dx\\&\leq\frac{CP_2}{r(t)^2}\frac{C_1}{r(t)^2}\int_{r\geq\frac{r(t)}{2}}|\nabla u|^2+|\nabla B|^2dx,\end{aligned}$$

$C_1$ is a absolute positive constant.

Denote

$$A(t)=\frac{1}{16}\varepsilon^{\frac{2}{3}}(\|\Omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+\frac{1}{2}(\|R\|_{L^2}^2+\|K\|_{L^2}^2),$$

then $A(t)\in C[0,T)\cap C^1(0,T)$. Hence, we have

$$\begin{aligned}\frac{d}{dt}A(t)\leq\ &\frac{CP_2}{r(t)^2}\min\{\varepsilon^{\frac{2}{3}}(\|\Omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+\frac{1}{2}(\|R\|_{L^2}^2+\|K\|_{L^2}^2),\\&\frac{C_1}{r(t)^2}\int_{r\geq\frac{r(t)}{2}}|\nabla u|^2+|\nabla B|^2dx\}\\\leq\ &\frac{CP_2}{r(t)^2}\min\{A(t),\frac{C_1}{r(t)^2}(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2)\}.\end{aligned}\tag{3.37}$$

Fix $t\in(0,T)$. If $0<r_1=r(t)=\frac{\varepsilon M(\varepsilon)}{a(t)+b(t)}\leq r_0$, by Lemma 2, we have

$$a(t)^2\leq\|R(t)\|_{L^2}\|\frac{u_\theta}{r}(t)\|_{L^2}\leq\|R(t)\|_{L^2}\|\nabla u(t)\|_{L^2}\leq A(t)^{\frac{1}{2}}\|\nabla u(t)\|_{L^2},$$
$$b(t)^2\leq\|K(t)\|_{L^2}\|\frac{B_\theta}{r}(t)\|_{L^2}\leq\|K(t)\|_{L^2}\|\nabla B(t)\|_{L^2}\leq A(t)^{\frac{1}{2}}\|\nabla B(t)\|_{L^2},$$

$$\frac{1}{r_2(t)^2}=\frac{(a(t)+b(t))^2}{(\varepsilon M(\varepsilon))^2}\leq\frac{2(a(t)^2+b(t)^2)}{(\varepsilon M(\varepsilon))^2}\leq\frac{2A(t)^{\frac{1}{2}}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})}{(\varepsilon_1M(\varepsilon_1))^2}.$$

Hence, the equations (3.34) implies

$$\frac{d}{dt}A(t)\leq\frac{CP_2A(t)^{\frac{1}{2}}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})}{(\varepsilon M(\varepsilon))^2}$$

$$
\begin{aligned}
&\min\{A(t), \frac{2A(t)^{\frac{1}{2}}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})}{(\varepsilon M(\varepsilon))^2}(\|\nabla u(t)\|_{L^2}^2+\|\nabla B\|_{L^2}^2)\}\\
\leq\ & \frac{CP_2A(t)(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})}{(\varepsilon M(\varepsilon))^2}\min\{A(t)^{\frac{1}{2}}, \frac{(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^3}{(\varepsilon M(\varepsilon))^2}\}\\
\leq\ & \frac{CP_2A(t)^{\frac{4}{3}}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}.
\end{aligned}
$$

On the other hand, if $r(t)=r_0$ and we have

$$
\begin{aligned}
\frac{d}{dt}A(t) &\leq \frac{CP_2}{r(t)^2}\min\{A(t), \frac{C_1}{r(t)^2}(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2)\}\\
&\leq \frac{CP_2}{r_0^4}(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2). \qquad (3.38)
\end{aligned}
$$

Combining the above two hands, we have

$$
\begin{aligned}
\frac{d}{dt}A(t) &\leq \max\{\frac{CP_2A(t)^{\frac{4}{3}}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{CP_2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2}{r_0^4}\}\\
&\leq CP_2(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2)\max\{\frac{A(t)^{\frac{4}{3}}}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4}\}. \qquad (3.39)
\end{aligned}
$$

Denote

$$
F(y)=\int_y^{+\infty}[\max\{\frac{y^{\frac{4}{3}}}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4}\}]^{-1}dy,
$$

then we have

$$
F(A(t))=\int_{A(t)}^{+\infty}[\max\{\frac{y^{\frac{4}{3}}}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4}\}]^{-1}dy,
$$

there we have

$$
\frac{d}{dt}F(y)=\frac{dF}{dy}\frac{dy}{dt},\ \ \frac{dF}{dy}=-[\max\{\frac{y^{\frac{4}{3}}}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4}\}]^{-1}.
$$

Hence, we obtain

$$
\frac{d}{dt}F(A(t))\geq -CP_2(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2),
$$

we also get

$$
-F(A(t))\leq \int_0^t CP_2(\|\nabla u(s)\|_{L^2}^2+\|\nabla B(s)\|_{L^2}^2)ds,
$$

and we use the energy identity to obtain

$$
\begin{aligned}
F(A(0))-F(A(t)) &\leq \int_0^t CP_2(\|\nabla u(s)\|_{L^2}^2+\|\nabla B(s)\|_{L^2}^2)ds\\
&\leq C_2P_2(\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2),
\end{aligned}
$$

$C_2$ is a absolute positive constant.

Therefore, we have to obtain the result $F(A(0)) > C_2P_2(\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2)$. Now, we know

$$\inf_{0<t<T} F(A(t)) > 0, \ \sup_{0<t<T} A(t) < +\infty,$$

and

$$\sup_{0<t<T} (\|\Omega(t)\|_{L^2}^2 + \|\Gamma(t)\|_{L^2}^2) < +\infty.$$

Hence, we can prove the global regularity.

We notice that

$$\begin{aligned}F(A(0)) &\geq F(\max\{A(0), \frac{(\varepsilon M(\varepsilon))^2}{r_0^3}\}) \\ &= \int_{A(0)}^{+\infty} [\max\{\frac{y^{\frac{4}{3}}}{(\varepsilon M(\varepsilon))^{\frac{8}{3}}}, \frac{1}{r_0^4}\}]^{-1} dy \\ &= 3\max\{A(0), \frac{(\varepsilon M(\varepsilon))^2}{r_0^3}\}^{-\frac{1}{3}} (\varepsilon M(\varepsilon))^{\frac{8}{3}},\end{aligned}$$

and

$$\begin{aligned}A(0)^{\frac{1}{2}} &= [\frac{1}{16}\varepsilon^{\frac{2}{3}}(\|\Omega_0\|_{L^2}^2 + \|\Gamma_0\|_{L^2}^2) + \frac{1}{2}(\|R_0\|_{L^2}^2 + \|K_0\|_{L^2}^2)]^{\frac{1}{2}} \\ &\leq \|\Omega_0\|_{L^2} + \|\Gamma_0\|_{L^2} + \|R_0\|_{L^2} + \|K_0\|_{L^2}.\end{aligned}$$

Form the definition of $P_0$, $P_1$, $P_2$, $\varepsilon$, and Proposition 3, we have

$$P_2 < \varepsilon^{-\frac{2}{3}}[1 + (\|J\|_{L^\infty} + \|L\|_{L^\infty})]^2, \ P_2(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2) \leq \varepsilon^{-\frac{2}{3}} P_1^2,$$
$$P_1^3(A(0))^{\frac{1}{2}} \leq P_0, \ M_0(\varepsilon) > C_0 \max\{P_0^{\frac{1}{4}}, r_0^{-\frac{1}{3}} P_1\}.$$

For prove $F(A(0)) > C_2P_2(\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2)$, we claim that $C_0 > C_3 \max\{1, (\frac{C_1}{3})^{\frac{1}{2}}\}$, there $C_0$ and $C_3$ are absolute positive constants. Thus, Theorem 1 is true for this $C_0$.

$$\begin{aligned}&(\frac{P_2(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2)}{F(A(0))})^{\frac{3}{2}} \\ &\leq (\frac{\varepsilon^{-\frac{2}{3}} P_1^2}{3\max\{A(0), \frac{(\varepsilon M(\varepsilon))^2}{r_0^3}\}^{-1/3}(\varepsilon M(\varepsilon))^{8/3}})^{\frac{3}{2}} \\ &\leq \frac{\max\{P_1^3 A(0)^{\frac{1}{2}}, \frac{P_1^3\varepsilon M(\varepsilon)}{r_0^{3/2}}\}}{3^{3/2}\varepsilon(\varepsilon M(\varepsilon))^4} \\ &\leq 3^{-3/2}\max\{\frac{P_0}{[C_0\max\{P_0^{\frac{1}{4}}, r_0^{-\frac{1}{3}}P_1\}]^4}, \frac{P_1^3}{[C_0\max\{P_0^{\frac{1}{4}}, r_0^{-\frac{1}{3}}P_1\}]^3 r_0^{3/2}}\} \\ &< 3^{-3/2}\max\{\frac{C_3^4}{C_0^4}, \frac{C_3^3}{C_0^3}\} \\ &= 3^{-3/2}(\frac{C_3}{C_0})^3 \leq C_1^{-3/2}.\end{aligned}$$

Therefore, the claim is true this completes the proof of the Theorem 1, and the condition (a) in Theorem 1 is satisfied.

# 4. Appendix

In this appendix, we shall present the detailed proof of Lemma 2 in the previous sections.

**Proof.** First, we prove that if $f = 0 \ for \ r \geq r_1$, integrate in $z$ and then

$$\int_{\mathbb{R}^3} \frac{|u_\theta(t)|}{r}|f|^2 rdrdz \leq \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2 rdrdz, \tag{4.1}$$

$$\int_{\mathbb{R}^3} \frac{|B_\theta(t)|}{r}|f|^2 rdrdz \leq \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} |\partial_r f|^2 rdrdz. \tag{4.2}$$

In this case,

$$f(r', z) = -\int_{r'}^{r_1} r\frac{\partial_r f(r,z)}{r}dr,$$

there $0 < r' < r_1$.

By using Hölder's inequality we have

$$|f(r', z)|^2 \leq \int_{r'}^{r_1} \frac{1}{r}dr \int_{r'}^{r_1} r|\partial_r f(r,z)|^2 dr.$$

It is easy to estimate that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u_\theta(r', z, t)||f(r', z)|^2 dr' \\
&\leq \int_{\mathbb{R}^3} |u_\theta(r', z, t)|dr' \int_{r'}^{r_1} \frac{1}{r}dr \int_{r'}^{r_1} r|\partial_r f(r,z)|^2 dr \\
&\leq \int_{\mathbb{R}^3} r|\partial_r f(r,z)|^2 dr \int_0^{r_1} |u_\theta(r', z, t)|dr' \int_{r'}^{r_1} \frac{dr'}{r}.
\end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |B_\theta(r', z, t)||f(r', z)|^2 dr' \\
&\leq \int |B_\theta(r', z, t)|dr' \int_{r'}^{r_1} \frac{1}{r}dr \int_{r'}^{r_1} r|\partial_r f(r,z)|^2 dr \\
&\leq \int_{\mathbb{R}^3} r|\partial_r f(r,z)|^2 dr \int_0^{r_1} |B_\theta(r', z, t)|dr' \int_{r'}^{r_1} \frac{dr'}{r},
\end{aligned} \tag{4.4}$$

there $h(r, z, t) = \int_0^r |u_\theta(r', z, t)|dr'$, $r > 0$, and $s(r, z, t) = \int_0^r |B_\theta(r', z, t)|dr'$, $r > 0$, we have

$$\int_0^{r_1} |u_\theta(r', z, t)| \int_{r'}^{r_1} \frac{dr}{r}dr' = \int_0^{r_1} h(r, z, t)\frac{dr}{r}, \ 0 < r' < r_1, \tag{4.5}$$

$$\int_0^{r_1} |B_\theta(r', z, t)| \int_{r'}^{r_1} \frac{dr}{r}dr' = \int_0^{r_1} s(r, z, t)\frac{dr}{r}, \ 0 < r' < r_1, \tag{4.6}$$

there $a(t) = \|\frac{h(t)}{r}\|_{L^\infty}$, $b(t) = \|\frac{s(t)}{r}\|_{L^\infty}$, $|J| = |ru_\theta|$, $|L| = |rB_\theta|$, then we have

$$h(r, z, t) \leq ra(t) \leq r[a(t) + b(t)], |u_\theta| = \frac{|J|}{r} \leq \frac{\varepsilon}{r},$$

$$s(r,z,t) \le rb(t) \le r[a(t)+b(t)], |B_\theta| = \frac{|J|}{r} \le \frac{\varepsilon}{r},$$

for $0 < r < r_1$. Hence, if $\frac{\varepsilon M(\varepsilon)}{a(t)+b(t)} \le r \le r_1$, we obtain

$$\begin{aligned}
h(r,z,t) &= \int_0^r |u_\theta(r',z,t)|dr' \\
&= \int_0^{\frac{\varepsilon}{a(t)+b(t)}} |u_\theta(r',z,t)|dr' + \int_{\frac{\varepsilon}{a(t)+b(t)}}^r |u_\theta(r',z,t)|dr' \\
&= h(\frac{\varepsilon}{a(t)+b(t)}, z, t) + \int_{\frac{\varepsilon}{a(t)+b(t)}}^r |u_\theta(r',z,t)|dr' \\
&\le \frac{\varepsilon}{a(t)+b(t)}[a(t)+b(t)] + \int_{\frac{\varepsilon}{a(t)+b(t)}}^r \frac{\varepsilon}{r'}dr' \\
&= \varepsilon(1+\ln\frac{r[a(t)+b(t)]}{\varepsilon}).
\end{aligned}$$

Thus, we can get the above estimate of $h$ as

$$\begin{aligned}
&\int_0^{r_1} h(r,z,t)\frac{1}{r}dr \\
= &\int_0^{\frac{\varepsilon}{a(t)+b(t)}} h(r,z,t)\frac{1}{r}dr + \int_{\frac{\varepsilon}{a(t)+b(t)}}^{r_1} h(r,z,t)\frac{1}{r}dr \\
\le &\int_0^{\frac{\varepsilon}{a(t)+b(t)}} r[a(t)+b(t)]\frac{1}{r}dr + \int_{\frac{\varepsilon}{a(t)+b(t)}}^{r_1} \varepsilon(1+\ln\frac{r[a(t)+b(t)]}{\varepsilon})\frac{1}{r}dr \\
\le &\ \varepsilon + \int_1^{M(\varepsilon)} \varepsilon(1+\ln r)\frac{dr}{r} \\
= &\ \varepsilon(1+\ln M(\varepsilon) + \frac{1}{2}(\ln M(\varepsilon))^2 = \varepsilon^{-\frac{1}{3}}.
\end{aligned} \tag{4.7}$$

When $0 < r_1 \le \frac{\varepsilon M(\varepsilon)}{a(t)+b(t)}$ , we have

$$\int_{\mathbb{R}^3} |u_\theta(r',z,t)||f(r',z)|^2 dr' \le \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} r|\partial_r f|^2 dr.$$

Similarly, we have the estimate of $s$ as

$$\begin{aligned}
s(r,z,t) &= \int_0^r |B_\theta(r',z,t)|dr' \\
&\le s(\frac{\varepsilon}{a(t)+b(t)}, z, t) + \int_{\frac{\varepsilon}{a(t)+b(t)}}^r |B_\theta(r',z,t)|dr' \\
&\le \frac{\varepsilon}{a(t)+b(t)}[a(t)+b(t)] + \int_{\frac{\varepsilon}{a(t)+b(t)}}^r \frac{\varepsilon}{r'}dr' \\
&= \varepsilon(1+\ln\frac{r[a(t)+b(t)]}{\varepsilon}).
\end{aligned}$$

Hence, we have

$$\int_0^{r_1} h(r,z,t)\frac{1}{r}dr$$

$$\leq \int_0^{\frac{\varepsilon}{a(t)+b(t)}} r[a(t)+b(t)]\frac{1}{r}dr + \int_{\frac{\varepsilon}{a(t)+b(t)}}^{r_1} \varepsilon(1+\ln\frac{r[a(t)+b(t)]}{\varepsilon})\frac{1}{r}dr$$
$$\leq \varepsilon[1+\ln M(\varepsilon)+\frac{1}{2}(\ln M(\varepsilon)]^2 = \varepsilon^{-\frac{1}{3}}. \tag{4.8}$$

When $0 < r_1 \leq \frac{\varepsilon M(\varepsilon)}{a(t)+b(t)}$, we have

$$\int_{\mathbb{R}^3} |B_\theta(r',z,t)||f(r',z)|^2 dr' \leq \varepsilon^{-\frac{1}{3}} \int_{\mathbb{R}^3} r|\partial_r f|^2 dr.$$

Now, we discuss general $f$. Set a smooth cut-off function of $r$ such that$0 < r < r_1$, $(\imath)\phi \leq 0, (\imath\imath)\phi \equiv 1$, if $0 \leq r \leq \frac{1}{3}, (\imath\imath\imath)\phi \equiv 0$, if $r \geq 1$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\frac{u_\theta(t)}{r}||f|^2 rdrdz \\
=& \int_{\mathbb{R}^3} |u_\theta(t)||\phi(\frac{r}{r_1})f|^2 drdz - \int_{\mathbb{R}^3} |u_\theta(t)|(1-\phi(\frac{r}{r_1})^2)|f|^2 drdz \\
\leq& \int_{0\leq r\leq \frac{r_1}{2}} |\frac{u_\theta(t)}{r}||\phi(\frac{r}{r_1})f|^2 rdrdz + \int_{r\geq \frac{r_1}{2}} \frac{|J|}{r}(1-\phi(\frac{r}{r_1})^2)|f|^2 drdz \\
\leq& \varepsilon^{-\frac{1}{3}}(\int_{0\leq r\leq \frac{r_1}{2}} |\partial_r f|^2 rdrdz + \frac{C}{r_1^2}\int_{r\geq \frac{r_1}{3}} |f|^2 rdrdz) + \int_{r\geq \frac{r_1}{2}} \frac{|J|}{r}|f|^2 drdz \\
\leq& \varepsilon^{-\frac{1}{3}} \int_{0\leq r\leq \frac{r_1}{2}} |\partial_r f|^2 rdrdz + C\frac{(\varepsilon^{-\frac{1}{3}} + \|J\|_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2 rdrdz.
\end{aligned}$$

For all $0 < r < r_1$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\partial_r(\phi(\frac{r}{r_1})f)|^2 rdrdz \\
=& \int_{\mathbb{R}^3} (|\partial_r\phi(\frac{r}{r_1})|^2|f|^2 + |\phi(\frac{r}{r_1})|^2|\partial_r f|^2) rdrdz - \int_{\mathbb{R}^3} |f|^2 \partial_r[\phi(\frac{r}{r_1})\partial_r\phi(\frac{r}{r_1})r] drdz \\
\leq& \int_{0<r<\frac{r_1}{2}} |\partial_r f|^2 rdrdz + \frac{C}{r_1^2}\int_{r\geq \frac{r_1}{2}} |f|^2 rdrdz.
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u_\theta(t)|^2|f|^2 rdrdz \\
=& \int_{\mathbb{R}^3} |u_\theta(t)|^2|\phi(\frac{r}{r_1})f|^2 rdrdz + \int_{\mathbb{R}^3} |u_\theta(t)|^2(1-\phi(\frac{r}{r_1})^2)|f|^2 rdrdz \\
\leq& \|J\|_{L^\infty(r\leq r_1)} \int_{0\leq r\leq \frac{r_1}{2}} |u_\theta(t)||\phi(\frac{r}{r_1})f|^2 drdz + \int_{r\geq \frac{r_1}{2}} \frac{|J|^2}{r^2}|f|^2 rdrdz \\
\leq& \varepsilon^{\frac{2}{3}} \int_{0\leq r\leq \frac{r_1}{2}} |\partial_r f|^2 rdrdz + C\frac{(\varepsilon^{\frac{2}{3}} + \|J\|_{L^\infty}^2)}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2 rdrdz.
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^3} |\frac{B_\theta(t)}{r}||f|^2 rdrdz$$

$$\leq \varepsilon^{-\frac{1}{3}} \int_{0\leq r\leq \frac{r_1}{2}} |\partial_r f|^2 rdrdz + C\frac{(\varepsilon^{-\frac{1}{3}} + \|L\|_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2 rdrdz.$$

$$\begin{aligned} &\int_{\mathbb{R}^3} |B_\theta(t)|^2 |f|^2 rdrdz \\ &\leq \varepsilon^{\frac{2}{3}} \int_{0\leq r\leq \frac{r_1}{2}} |\partial_r f|^2 rdrdz + C\frac{(\varepsilon^{\frac{2}{3}} + \|L\|^2_{L^\infty})}{r_1^2} \int_{r\geq \frac{r_1}{2}} |f|^2 rdrdz. \end{aligned}$$

□

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