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Discrete Morse Flow for Yamabe Type Heat Flows

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Abstract. In this paper, we study the discrete Morse flow for either Yamabe type heat flow or nonlinear heat flow on a bounded regular domain in the whole space. We show that under suitable assumptions on the initial data g one has a weak approximate discrete Morse flow for the Yamabe type heat flow on any time interval. This phenomenon is very different from the smooth Yamabe flow, where the finite time blow up may exist.

AMS Subject Classifications: 53C44, 35K50, 93C55

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Key Words: Discrete Morse flow; Yamabe type flow; critical exponent; nonlinear heat flow.

1 Introduction

The aim of this note is to develop the discrete Morse flows both for Yamabe type heat flow and for the nonlinear heat flow for any finite time (see [1–3]). The phenomenon for weak solutions is very different from that of the smooth Yamabe flow, where the finite time blowup may exist (see [4] and [2]). We use the idea from [5], where the 2-dimensional Yamabe flow has been studied, to develop the discrete flow. We now recall definition of the weak solution to the Yamabe type heat flow. Let $\Omega \subset R^n$ be a regular bounded domain with smooth boundary. For any $T>0$, we let $Q=Q_T=\Omega\times[0,T]$. Assume that the initial data $g\in C^{2,1}(\overline{Q_T})$ is regular and $g_\nu=0$ on the boundary $\partial\Omega\times\{t\}$, where ν is the outward unit normal to $\partial\Omega$. Assume that $\psi\in C(\overline{\Omega})$ is a non-negative regular function. Hereafter, for a smooth function $u:\overline{\Omega}\to R$, we use the following notations,

$$\nabla u=(\partial u/\partial x^i), \qquad Lu=\Delta u-\psi(x)u.$$

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Assume $n\geq 3$. Recall that for $p=\frac{n+2}{n-2}$, the Yamabe type flow equation is

$$u^{p-1}\partial_t u = Lu, \quad \text{on Q}, \tag{1.1}$$

with the initial data $u=g$ and with the Neumann boundary condition $u_\nu=0$. Let $s_*=\frac{p+1}{2}=\frac{n}{n-2}$, which is the half of the Sobolev critical exponent [6]. We say that the non-negative $u\in C([0,T],H^1(\Omega))$ is a *weak solution* to the Yamabe type flow (1.1) if for any $\eta\in C_0^\infty(Q)$, we have

$$\int_Q \frac{1}{s_*}u^{s_*-1}\partial_t u^{s_*}\eta \mathrm{d}x\mathrm{d}t + \int_Q (\nabla u,\nabla\eta)\mathrm{d}x\mathrm{d}t = \int_Q \psi(x)u\eta \mathrm{d}x\mathrm{d}t$$

and $u(0,\cdot)-g\in H_0^1(\Omega)$ in weak sense that $\lim_{t\to 0}\|u(t,\cdot)-g(\cdot)\|_{L^2}\to 0$. Note that for any $p>1$ being a fixed exponent, one may propose the corresponding nonlinear heat flow and study the weak solution to it.

We have the following conclusion.

Theorem 1.1. *Assume that $\psi=0$ on Ω. For any $T>0$ and any initial-boundary data $g\in C^1(\Omega)$ with $g\geq 0$ and $g_\nu=0$ on the boundary, there exists a discrete Morse flow $\{\hat{u}_N(t)\}$ to the Yamabe type flow* (1.1) *with the initial data $\hat{u}_N(0)=g\geq 0$ and the lateral boundary condition $(\hat{u}_N)_\nu=0$. The limit of the discrete Morse flow is a weak solution to* (1.1) *on* $[0,T]\times\Omega$.

For the precise meaning of the discrete Morse flow, which is the triple $(u_N,\hat{u}_N,\partial_t\hat{u}_N^s)$, one may see the definition under Eq. (2.5) in Section 2. We may use the standard notation $\|u\|_p$ for the norm of the Lebesgue space $L^p(\Omega)$. Other notations are from the famous books [6] and [7].

The use of the discrete Morse flow method (also called Rothe's method) to the study of parabolic problems has a long history and this field is still very active. The discrete Morse flow method was introduced by E.Rothe in the paper [8]. This method for initial boundary value problems, consists of a time variable discretization by finite differences and leads to a sequence of boundary value problems for elliptic equations [9–11]. The method is also known as the horizontal line method for numerical purposes [12]. One may see the book [13] for a friendly introduction of Rothe's method. Only in recent decades, we can see some applications of discrete Morse flows to other geometric flows such as harmonic map heat flows. In [14], the authors applied the discrete Morse flow method to the problem of the heat flow for surfaces of prescribed mean curvature. In the recent work [15], the authors applied the discrete Morse flow method to the parabolic p-Laplacian systems. In the interesting work [16], the discrete Morse flow method had been used to construct infinitely many weak solutions to harmonic map heat flows to spheres. One may also see the references [17–19] for discrete Morse flows for harmonic map heat flows. As one can expect, it is possible to use this method to study weak solutions to Yang-Mills heat flow.

The plan of this paper is below. The main result, Theorem 1.1, will be proved in Section 3. In Section 2, we present the proof of the existence of discrete Morse flow for nonlinear heat flow and the conclusion is stated in Theorem 2.1.

2 Discrete Morse flow for nonlinear heat flow

In this section, we take any fixed $s\in(1,s_*)$ and study the nonlinear heat flow

$$u^{s-1}\partial_t u = Lu \quad \text{on Q}$$

with the boundary data $u_\nu=0$ on $\partial\Omega\times[0,T]$ and the initial data g.

We may assume that there is a positive number L such that $|g(x)|\le L$ in $\overline{\Omega}$. We let

$$H:=\{u\in H^1(\Omega);|u(x)|\le L+1,\forall\ x\in\Omega\}.$$

With the help of Rellich compactness [6], we know that H is a weakly closed subset of $H^1:=H^1(\Omega)$.

We may define

$$\lambda=\inf\left\{\int_\Omega(|\nabla u|^2+\psi(x)u^2)\mathrm{d}x;u\in H^1(\Omega),\ \int_\Omega|u|^2\mathrm{d}x=1\right\},$$

and we may assume that $\lambda>0$. Then we have for any $u\in H^1(\Omega)$,

$$\int_\Omega(|\nabla u|^2+\psi(x)u^2)\mathrm{d}x\ge\lambda\int_\Omega|u|^2\mathrm{d}x.$$

We define, for $u\in H$, the functional

$$J(u)=\frac{1}{2}\int_\Omega(|\nabla u|^2+\psi(x)u^2)\mathrm{d}x.$$

From the definition of λ we get,

$$J(u)\ge\frac{1}{2}\lambda\int_\Omega u^2\mathrm{d}x>0 \tag{2.1}$$

for any $u\neq0$, $u\in H$. Recall the Poincare inequality that for any $u\in H^1(\Omega)$,

$$\int_\Omega|\nabla u|^2\mathrm{d}x\ge\lambda_1\int_\Omega|u-\bar{u}|^2\mathrm{d}x,$$

for some uniform constant $\lambda_1>0$, where $\bar{u}=\frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x$. For $u\in H$, we have $|\bar{u}|\le L+1$. Then we have for any $u\in H^1(\Omega)$,

$$\|u\|_2\le\|u-\bar{u}\|_2+(L+1)\sqrt{|\Omega|}\le\|\nabla u\|_2+(L+1)\sqrt{|\Omega|}.$$

The above remark is useful in the estimation of L^2 norm of the weak solution.

We now consider discrete Morse flow for the nonlinear heat flow

$$u^{s-1}\partial_t u = Lu \quad \text{on Q} \tag{2.2}$$

with the boundary condition $u_\nu=0$ on $\partial\Omega\times[0,T]$ and the initial data g. We have the following result.

Theorem 2.1. *Assume that the function $\psi\in C(\bar{\Omega})$ is non-negative. Fix any $T>0$ and any given non-negative data $g\in H^1(\Omega)$. For the nonlinear heat flow* (2.2) *with the initial data $u_0=g$ and the lateral boundary condition $u_\nu=0$, there exists a discrete Morse flow $\{\hat{u}_N(t)\}$ and its limit is a weak solution to* (2.2) *on $[0,T]\times\Omega$.*

Proof. We now prove Theorem 2.1. Let $h=T/N$ for some positive integer $N>1$. For $n=1,\dots,N$, we let $t_n=nh$. So, the sequence $\{0<t_1<\dots<t_N\}$ is the division of the time interval $[0,T]$.

To introduce the discrete Morse flow for the flow (2.2), we define $u_0=g$. We may define the functional $F_1(\cdot)$ on H for the 1-step by

$$F_1(u)=\frac{1}{2s^2h}\int_\Omega ||u|^s-|u_0|^s|^2\mathrm{d}x+J(u).$$

One may use Theorem 1.8.2 in [20] to get a minimizer of $F_1(\cdot)$ on H, and we shall call this process the direct method as usual.

We now do the induction construction. Given any minimizer $u_{n-1}\subset H$ of the functional F_{n-1} and we may assume $|u_{n-1}(x)|\le L$ on Ω. Define, the functional for the n-step,

$$F_n(u)=\frac{1}{2s^2h}\int_\Omega ||u|^s-|u_{n-1}|^s|^2\mathrm{d}x+J(u).$$

Before taking minimizing sequence, we need the truncation process. Note that for $0\le u\in H^1$, we may let $u_L(x)=u(x)$ for $u(x)\le L$ and $u_L(x)=L$ for $u(x)>L$. Then it is clear that $F_n(u_L)\le F_n(u)$. Clearly, the functional F_n is non-negative, i.e., $F_n(\cdot)\ge 0$ on H. Using the Sobolev compactness embedding theorem, we know the functional

$$\int_\Omega ||u|^s-|u_{n-1}|^s|^2\mathrm{d}x,$$

is continuous in H (since $2s=p+1<\frac{2n}{n-2}$). By the direct method, we may take a minimizing sequence $(u_k)\subset H$ with $0\le u_k\le L$, whose weak H^1 limit is a non-negative minimizer $u_n\in H$: $0\le u_n(x)\le L$ on Ω, of the functional $F_n(u)$. Then u_n is an interior point of H and u_n solves the following equation

$$u_n^{s-1}\frac{1}{sh}(|u_n|^s-|u_{n-1}|^s)=Lu_n \quad \text{on } \Omega \tag{2.3}$$

with the boundary condition $(u_n)_\nu=0$, in the weak sense. That is, for any $\eta\in C_0^\infty(\Omega)$, we have

$$\frac{1}{sh}\int_\Omega u_n^{s-1}(u_n^s-u_{n-1}^s)\eta\mathrm{d}x+\int_\Omega \nabla u_n\cdot\nabla\eta\mathrm{d}x=-\int_\Omega \psi(x)u_n\eta\mathrm{d}x.$$

By the minimality of u_n, we have

$$\frac{1}{2s^2h}\int_\Omega ||u_n|^s-|u_{n-1}|^s|^2\mathrm{d}x+J(u_n)\le J(u_{n-1}).$$

Adding them together, we have

$$\frac{1}{2}\int_0^T\int_\Omega\left|\frac{u_n^s-u_{n-1}^s}{sh}\right|^2\mathrm{d}x+J(u_n)\leq J(u_0). \tag{2.4}$$

By (2.1), we know that for some $\lambda_0>0$,

$$\frac{1}{2s^2}\sum_0^n\int_\Omega\left|\frac{u_k^s-u_{k-1}^s}{h}\right|^2\mathrm{d}x+\frac{1}{2}\lambda_0\int_\Omega|\nabla u_n|^2\mathrm{d}x\leq J(u_0),$$

which implies that there is a uniform constant $C(u_0)>0$ such that

$$\int_\Omega|\nabla u_n|^2\mathrm{d}x\leq C(u_0).$$

We now define $u_N(t)\in H$ and $\hat{u}_N(t)\in H$ for $t\in[0,T]$ in such a way that, for $n=1,\dots,N$, $t_n=nh$,

$$u_N(t)=\frac{t_n-t}{h}u_{n-1}+\frac{t-t_{n-1}}{h}u_n;\quad \hat{u}_N(t)=u_n,\quad \text{for } t\in(t_{n-1},t_n].$$

We further define, for $n=1,\dots,N$,

$$\partial_t\hat{u}_N^s(t)=\frac{1}{h}(u_n^s-u_{n-1}^s),\quad \text{for } t\in[t_{n-1},t_n].$$

Then Eq. (2.3) can be written as

$$\hat{u}_N^{s-1}\partial_t u_N^s=L\hat{u}_N\quad \text{on } Q, \tag{2.5}$$

which holds in weak sense and we call the triple $(u_N,\hat{u}_N,\partial_t\hat{u}_N^s)$ the discrete Morse flow for the evolution Eq. (2.2).

Note that $\partial_t u_N=\frac{1}{h}(u_n-u_{n-1})$ for $t\in(t_{n-1},t_n]$,

$$|u_N-\hat{u}_N|^s\leq|u_N^s-\hat{u}_N^s|\leq h|\partial_t\hat{u}_N^s|,$$

and for t small,

$$|u_N(t,x)-g(x)|\leq\frac{t}{h}|u_1(x)-g(x)|\leq|\hat{u}_N(t_1,x)-g(x)|.$$

We remark that it holds the energy bound

$$\int_0^T\mathrm{d}t\int_\Omega|\partial_t\hat{u}_N^s(t)|^2\mathrm{d}x=\frac{1}{h}\sum_1^N\int_\Omega|u_n^s-u_{n-1}^s|^2\mathrm{d}x.$$

The relation (2.4) implies that

$$\frac{1}{2}\int_0^T\mathrm{d}t\int_\Omega|u_N(t)^s-\hat{u}_N^s(t)|^2\mathrm{d}x\leq s^2h^2J(u_0)\to 0$$

as $h\to 0$. By (2.4) we have for some uniform constant $C(T)>0$,

$$\int_0^T |\partial_t u_N^s(t)|_{L^2}^2 + J(u_N) \le C(T),$$

and in particular, $\int_\Omega |\nabla u_N(t)|^2 \mathrm{d}x \le C(u_0,T)$ for some uniform constant $C(u_0,T)$.

By the uniform bounds above and the standard results from Sobolev spaces [20] we may assume that as $h\to 0$, for any $r>1$, $u_N\to u_\infty$ in $L^r(Q)$, $\nabla u_N \to \nabla u_\infty$ weakly in $L^2(Q)$, and $\partial_t \hat{u}_N^s \to \partial_t u_\infty^s$ weakly in $L^2(Q)$. Then we may get the limit $u_\infty(t)$ which solves (2.2) weakly with the uniform estimate

$$\int_0^T |\partial_t u_\infty^s(t)|_{L^2}^2 + J(u_\infty) \le C(T).$$

Using the elementary inequality that $|a-b|^s \le |a^s - b^s|$ for any $a>0$ and $b>0$, we know that

$$|u_\infty^s(t) - g^s|_{L^2} \le \int_0^t |\partial_t u_\infty^s(t)|_{L^2} \le \sqrt{t}C(T) \to 0$$

as $t\to 0$, i.e., the initial data of the flow $(u_\infty(t))$ is g in the weak L^2 sense. Then the proof of Theorem 2.1 is complete. □

We remark that with a little more effort, we may extend the result above to any $p>1$ or any $\psi(x)\in L^q(\bar{\Omega})$ $(q\ge 2)$ by the method used in Chapter 5 in Ladyzhenskaya's book [13].

3 Discrete Morse flow for Yamabe type heat flow

As we remark above, with a minor modification, the method above may be applied to prove Theorem 1.1, but we prefer to give another proof. Recall that we have assumed that $p=\frac{n+2}{n-2}$ and $\phi=0$ on Ω. We now prove Theorem 1.1. We use the subspace H as above. As before, we let $h=T/N$ for any positive integer $N>1$. For $n=1,...,N$, we let $t_n=nh$ such that $0<t_1<...<t_N$ is the division of the time interval $[0,T]$.

To introduce the discrete Morse flow for flow (1.1), we define, for $u\in H$, the Dirichlet functional

$$D(u)=\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x.$$

We define $u_0=g$ and set $s=s_*$ for notation simpler. Note that $2s=p+1=\frac{2n}{n-2}$. We define the functional E_1 on H by

$$E_1(u)=\frac{1}{2s^2h}\int_\Omega ||u|^s - |u_0|^s|^2 \mathrm{d}x + D(u).$$

We now do the induction definition. Assume that the functional $E_{n-1}(u)$ is defined and there is a minimizer of the functional $E_{n-1}(\cdot)$ on H such that $0\le u_{n-1}(x)\le L$ on Ω.

Define, for the minimizer $u_{n-1}\subset H$ of the functional $E_{n-1}(\cdot)$, a new functional for the n-step,

$$E_n(u)=\frac{1}{2s^2h}\int_\Omega ||u|^s-|u_{n-1}|^s|^2\mathrm{d}x+D(u).$$

By the direct method, we know that there is a non-negative minimizer $u_n\in H$ of the functional $E_n(u)$. We now use a trick introduced in [2]. For any $\xi\in C_0^1(\Omega):\xi\geq 0$ and $t>0$ small, $(1-t\xi)u_n\in H$ and we have

$$E_n(u_n)\leq E_n((1-t\xi)u_n).$$

Then

$$0\leq\lim_{t\to 0_+}t^{-1}(E_n((1-t\xi)u_n)-E_n(u_n)),$$

which gives us

$$0\leq -\int u_n^s\frac{1}{sh}(|u_n|^s-|u_{n-1}|^s)\xi+\int\nabla u_n\cdot\nabla(u_n\xi).$$

Note that

$$a(a-b)\geq\frac{1}{2}(a^2-b^2),\ \ \forall a,b\in\mathbb{R}.$$

Then we have for $w_n:=u_n^2$,

$$\int\frac{1}{sh}(|w_n|^s-|w_{n-1}|^s)\xi+\int\nabla w_n\cdot\nabla\xi\leq 0.$$

We may take the test function

$$\xi=\max(w_n-L^2,0)$$

to conclude that

$$\int\frac{1}{sh}\left(|w_n|^s-L^{2s}\right)\xi+\int|\nabla\xi|^2\leq 0,$$

which implies that $u_n\leq L$ on Ω. Hence, u_n is a interior point in H.

Then we know that u_n solves the following equation

$$u_n^{s-1}\frac{1}{sh}(|u_n|^s-|u_{n-1}|^s)=\Delta u_n\quad\text{on }\Omega\subset R^n \tag{3.1}$$

with the boundary condition $(u_n)_\nu=0$, in the weak sense. That is, for any $\eta\in C_0^\infty(Q)$, we have

$$\frac{1}{sh}\int_\Omega u_n^{s-1}(u_n^s-u_{n-1}^s)\eta\mathrm{d}x+\int_\Omega(\nabla u_n,\nabla\eta)\mathrm{d}x=-\int_\Omega\psi(x)u_n\eta\mathrm{d}x.$$

By the minimality of u_n, we have

$$\frac{1}{2s^2h}\int_\Omega||u_n|^s-|u_{n-1}|^s|^2\mathrm{d}x+D(u_n)\leq D(u_{n-1}).$$

Adding them together, we have

$$\frac{1}{2}\int_0^T\int_\Omega\left|\frac{u_n^s-u_{n-1}^s}{sh}\right|^2 dx+D(u_N)\le D(u_0). \tag{3.2}$$

Then we have

$$\frac{1}{2s^2}\int_0^T\int_\Omega\left|\frac{u_n^s-u_{n-1}^s}{h}\right|^2 dx+\frac{1}{2}\int_\Omega|\nabla u_N|^2 dx\le D(u_0),$$

which implies that there is a uniform constant $C(u_0)>0$ such that

$$\int_\Omega|\nabla u_N|^2 dx\le C(u_0).$$

We now define the discrete Morse flow as before, with $u_N(t)\in H$ and $\hat{u}_N(t)\in H$ for $t\in[0,T]$ in such a way that, for $n=1,...,N$, $t_n=nh$,

$$u_N(t)=\frac{t_n-t}{h}u_{n-1}+\frac{t-t_{n-1}}{h}u_n;\quad \hat{u}_N(t)=u_n\quad \text{for } t\in(t_{n-1},t_n].$$

We may further let, for $n=1,...,N$,

$$\partial_t\hat{u}_N^s(t)=\frac{1}{h}(u_n^s-u_{n-1}^s),\quad \text{for } t\in[t_{n-1},t_n].$$

The triple $(u_N,\hat{u}_N,\partial_t\hat{u}_N^s)$ is called the discrete Morse flow for the flow (2.2). Then Eq. (3.1) can be written as

$$\hat{u}_N^{s-1}\partial_t\hat{u}_N^s=\Delta\hat{u}_N\quad \text{on } Q \tag{3.3}$$

in the weak sense. The following goes basically as in Section 2 and for completeness, we present the detail. Note again that $|u_N^s-\hat{u}_N^s|\le h|\partial_t u_N^s|$ and for t small,

$$|u_N(t,x)-g(x)|\le\frac{t}{h}|u_1(x)-g(x)|\le|\hat{u}_N(t_1,x)-g(x)|.$$

We remark that

$$\int_0^T dt\int_\Omega|\partial_t\hat{u}_N^s(t)|^2 dx=\frac{1}{h}\sum_1^N\int_\Omega|u_n^s-u_{n-1}^s|^2 dx.$$

The relation (3.2) implies that

$$\frac{1}{2}\int_0^T dt\int_\Omega|u_N(t)^s-\hat{u}_N^s(t)|^2 dx\le s^2h^2D(u_0)\to 0$$

as $h\to 0$. By (3.2) we have for some uniform constant $C(T)>0$,

$$\int_0^T|\partial_t u_N^s(t)|_{L^2}^2+J(u_N)\le C(T),$$

and in particular, $\int_\Omega |\nabla u_N(t)|^2 \mathrm{d}x \leq C(u_0,T)$ for some uniform constant $C(u_0,T)$.

Using the uniform bounds above and the standard knowledge from Sobolev spaces [13] we may assume that as $h \to 0$, for any $r>1$, $u_N \to u_\infty$ in $L^r(Q)$,

$$\nabla u_N \to \nabla u_\infty \ \text{ weakly in } L^2(Q),$$

and

$$\partial_t \hat{u}_N^s \to \partial_t u_\infty^s \ \text{ weakly in } L^2(Q).$$

Then the limit $u_\infty(t)$ which solves (1.1) weakly with the uniform estimate

$$\int_0^T |\partial_t u_\infty^s(t)|_{L^2}^2 + D(u_\infty) \leq C(T).$$

Again, since

$$|u_\infty^s(t) - g^s|_{L^2} \leq \int_0^t |\partial_t u_\infty^s(t)|_{L^2} \leq \sqrt{t} C(T) \to 0$$

as $t \to 0$, we know that the initial data of the flow $(u_\infty(t))$ is g in the weak L^2 sense. This completes the proof of Theorem 1.1. □

We remark that there are other types of numerical-analytical method to construct weak solutions to porous medium equations related to our work and one may prefer to lecture 1 part in [21] for this subject.

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