

Positive Ground State Solutions for Schrödinger-Poisson System with General Nonlinearity and Critical Exponent

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Abstract. In this paper, we consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \eta \phi u = f(x, u) + u^5, & x \in \Omega, \\ -\Delta \phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in R^3 , $\eta = \pm 1$ and the continuous function f satisfies some suitable conditions. Based on the Mountain pass theorem, we prove the existence of positive ground state solutions.

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1 Introduction

In this paper, we study the following Schrödinger-Poisson system with general nonlinearity and critical exponent on bounded domain

$$\begin{cases} -\Delta u + \eta \phi u = f(x, u) + u^5, & x \in \Omega, \\ -\Delta \phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

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where Ω is a smooth bounded domain in R^3 , $\eta = \pm 1$, and the continuous function f satisfies some suitable conditions.

The Schrödinger equation, which is the first equation in system (1.1), describes quantum particles interacting with the electromagnetic field generated by the motion. An interesting class of Schrödinger equation is the case where the potential $\phi(x)$ is determined by the charge of the wave function itself, that is, when the second equation in system (1.1) holds, see [1]. From [2-5] and the references therein, we can learn more information about the physical relevance of the Schrödinger-Poisson system.

To the best of our knowledge, researchers only obtained a few results about the Schrödinger-Poisson system with critical exponent on bounded domain, see for instance [1], [6-11].

In [6], assuming that $\eta = \lambda$, $f(x, u) = \lambda u^{q-1}$, $\lambda > 0$ and $1 < q < 2$, via using the variational method, the authors proved that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda \in (0, \lambda_*)$, where λ_* is a positive constant. In [7], let $\eta = -1$, $f(x, u) = \lambda f_\lambda(x) u^{q-1}$, $f_\lambda = \lambda f^+ + f^-$, $\lambda > 0$ and $1 < q < 2$, by using the variational method and analytic techniques, they got that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda \in (0, \lambda_*)$, where λ_* is a positive constant. In [8], when $\eta = -1$, $f(x, u) = \lambda u^{q-1}$, $\lambda > 0$ and $2 < q < 6$, by the Mountain pass theorem and the concentration compactness principle, they obtained that if $2 < q \leq 4$, system (1.1) has at least one positive ground state solution for all $\lambda > \lambda_*$, where λ_* is a positive constant; if $4 < q < 6$, system (1.1) has at least one positive ground state solution for all $\lambda > 0$.

In [9], let $\eta = \lambda$, $f(x, u) = \frac{\lambda}{u^r}$, $\lambda > 0$ and $0 < r < 1$, the author got that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda \in (0, \lambda_*)$, where λ_* is a positive constant. In [10], assuming that $\eta = -1$, $f(x, u) = \frac{\lambda}{u^r}$, $\lambda > 0$ and $0 < r < 1$, the authors proved that system (1.1) has at least two positive solutions for all $\lambda \in (0, \lambda_*)$, where λ_* is a positive constant. In [11], let $\eta = 1$, $f(x, u) = \frac{\lambda}{|x|^\beta u^r}$, $\lambda > 0$, $0 < r < 1$ and $0 \leq \beta < \frac{5+r}{2}$, combining with the variational method and Nehari manifold method, two positive solutions of system (1.1) are obtained.

In [1], assuming that $\eta = 1$, the nonlinear term $f: \Omega \times R \rightarrow R$ and its primitive F satisfies the following conditions:

- (f₁) $|f(x, s)| \leq C(1 + |s|^{p-1})$ for some $p \in (2, 6)$, where C is positive constant;
- (f₂) $f(x, s) = o(|s|)$ uniformly in x as $|s| \rightarrow 0$;
- (f₃) $s \mapsto \frac{f(x, s)}{s^3}$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$;
- (f₄) $\frac{F(x, s)}{|s|^4} \rightarrow +\infty$ uniformly in x as $s \rightarrow +\infty$.

Via the variational methods, the authors got that system (1.1) has at least one nontrivial ground state solution.

On the basis of the above literature, especially [1], we continues to study system (1.1) with general nonlinearity and critical exponent on bounded domain. We assume that $\eta = \pm 1$ and the nonlinear term $f: \Omega \times R \rightarrow R$ satisfies the following assumptions

- (F₁) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, s) \geq 0$ if $s \geq 0$ and $f(x, s) = 0$ if $s \leq 0$ for all $x \in \Omega$;
 (F₂) $\lim_{s \rightarrow 0^+} \frac{f(x, s)}{s} = 0$ and $\lim_{s \rightarrow +\infty} \frac{f(x, s)}{s^5} = 0$ uniformly for all $x \in \Omega$;
 (F₃) there exists $0 < a < \lambda_1$ such that $f(x, s)s - 4F(x, s) \geq -as^2$ for all $x \in \Omega$ and $s \geq 0$,
 where $F(x, s) = \int_0^s f(x, t) dt$ and λ_1 is first eigenvalue of the operator $-\Delta$;
 (F₄) there exists a nonempty open set $\omega \subset \Omega$ with $0 \in \omega$ such that $\lim_{s \rightarrow +\infty} \frac{f(x, s)}{s^3} = +\infty$ uniformly for $x \in \omega$.

Now, we give our main result.

Theorem 1.1. *Assume that $\eta = \pm 1$ and (F₁)-(F₄) hold, then system (1.1) has at least one positive ground state solution.*

Remark 1.1. Compared with [1], on one hand, our assumptions (F₃)-(F₄) for the nonlinear term f are weaker than assumptions (f₃)-(f₄) in [1]. On the other hand, we also consider the case of $\eta = -1$.

This paper is organized as follows. In Section 2, we give some necessary notations and important preliminaries. The proof of Theorem 1.1 is given in Section 3.

2 Notations and preliminaries

Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$\langle u, v \rangle = \int_{\Omega} (\nabla u, \nabla v) dx, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}}.$$

$L^p(\Omega)$ ($1 \leq p \leq +\infty$) denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by

$$\|u\|_p = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

C_i ($i = 1, 2, 3, \dots$) denote various positive constants, which may vary from line to line. Let S be the best Sobolev constant, namely,

$$S := \inf_{u \in H \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^6 dx \right)^{\frac{1}{3}}}. \quad (2.1)$$

Via Lax-Milgram theorem, for every $u \in H$, the Poisson equation $-\Delta \phi = u^2$ has a unique solution $\phi_u \in H$. We insert ϕ_u into the first equation of system (1.1), then system (1.1) is translated into the following problem

$$\begin{cases} -\Delta u + \eta \phi_u u = f(x, u) + u^5, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (2.2)$$

The energy functional I corresponding to problem (2.2) is given by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\eta}{4} \int_{\Omega} \phi_u (u^+)^2 dx - \int_{\Omega} F(x, u^+) dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx. \quad (2.3)$$

So for all $u, v \in H$, it holds

$$\langle I'(u), v \rangle = \int_{\Omega} (\nabla u, \nabla v) dx + \eta \int_{\Omega} \phi_u (u^+) v dx - \int_{\Omega} f(x, u^+) v dx - \int_{\Omega} (u^+)^5 v dx. \quad (2.4)$$

Recall that u is called weak solution of problem (2.2) if

$$\int_{\Omega} (\nabla u, \nabla v) dx + \eta \int_{\Omega} \phi_u (u^+) v dx = \int_{\Omega} f(x, u^+) v dx + \int_{\Omega} (u^+)^5 v dx, \quad \forall v \in H. \quad (2.5)$$

We need the following lemma from [12,13].

Theorem 2.1. *For every $u \in H$, there exists a unique element $\phi_u \in H$ solution of*

$$\begin{cases} -\Delta \phi = u^2, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases}$$

and

- (a) $\|\phi_{8u}\|^2 = \int_{\Omega} \phi_u u^2 dx$;
- (b) $\phi_u \geq 0, \forall u \in H$. Moreover, $\phi_u > 0$ when $u \neq 0$;
- (c) for each $t \neq 0, \phi_{tu} = t^2 \phi_u$;
- (d) there exists C_1 such that $\|\phi_u\| \leq C_1 \|u\|^4$ and

$$\int_{\Omega} |\nabla \phi_u|^2 dx = \int_{\Omega} \phi_u u^2 dx \leq C_1 \|u\|^4, \quad \forall u \in H;$$

- (e) assume that $u_n \rightharpoonup u$ in H , then $\phi_{u_n} \rightharpoonup \phi_u$ in H and

$$\int_{\Omega} \phi_{u_n} u_n v dx \rightarrow \int_{\Omega} \phi_u u v dx, \quad \text{for every } v \in H;$$

- (f) set $\mathcal{F}(u) = \int_{\Omega} \phi_u u^2 dx$, then $\mathcal{F}: H \rightarrow H$ is C^1 and

$$\langle \mathcal{F}'(u), v \rangle = 4 \int_{\Omega} \phi_u u v dx, \quad \forall v \in H.$$

3 Proof of Theorem 1.1

First of all, we claim that the functional I has the geometry of the Mountain pass theorem.

Theorem 3.1. *Suppose that f satisfies (F_1) and (F_2) , then there exist some constants $\rho, \alpha > 0$ such that $I(u) \geq \alpha > 0$ whenever $\|u\| = \rho$.*

Proof. By (F_1) and (F_2) , there exists $C_2 > 0$ such that

$$|f(x, s^+)| \leq \frac{\lambda_1}{2} s^+ + 6C_2 (s^+)^5, \quad |F(x, s^+)| \leq \frac{\lambda_1}{4} (s^+)^2 + C_2 (s^+)^6, \quad \forall (x, s) \in \bar{\Omega} \times \mathbb{R}. \quad (3.1)$$

Consequently, by Lemma 2.1(d), (2.1), (3.1) and the Hölder inequality, when $\eta = -1$, we have

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{4} \int_{\Omega} \phi_u (u^+)^2 dx - \int_{\Omega} F(x, u^+) dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{C_1}{4} \|u\|^4 - \int_{\Omega} F(x, u^+) dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{C_1}{4} \|u\|^4 - \frac{\lambda_1}{4} \int_{\Omega} (u^+)^2 dx - C_2 \int_{\Omega} (u^+)^6 dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &= \frac{1}{2} \|u\|^2 - \frac{C_1}{4} \|u\|^4 - \frac{\lambda_1}{4} \int_{\Omega} (u^+)^2 dx - \frac{6C_2+1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{4} \|u\|^2 - \frac{C_1}{4} \|u\|^4 - \frac{6C_2+1}{6} S^{-3} \|u\|^6 \\ &= \|u\|^2 \left\{ \frac{1}{4} - \frac{C_1}{4} \|u\|^2 - \frac{6C_2+1}{6} S^{-3} \|u\|^4 \right\}. \end{aligned} \quad (3.2)$$

Similarly, via Lemma 2.1(a), (2.1), (3.1) and the Hölder inequality, when $\eta = 1$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\Omega} \phi_u (u^+)^2 dx - \int_{\Omega} F(x, u^+) dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x, u^+) dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\lambda_1}{4} \int_{\Omega} (u^+)^2 dx - C_2 \int_{\Omega} (u^+)^6 dx - \frac{1}{6} \int_{\Omega} (u^+)^6 dx \\ &= \frac{1}{2} \|u\|^2 - \frac{\lambda_1}{4} \int_{\Omega} (u^+)^2 dx - \frac{6C_2+1}{6} \int_{\Omega} (u^+)^6 dx \\ &\geq \frac{1}{4} \|u\|^2 - \frac{6C_2+1}{6} S^{-3} \|u\|^6 \\ &= \|u\|^2 \left\{ \frac{1}{4} - \frac{6C_2+1}{6} S^{-3} \|u\|^4 \right\}. \end{aligned} \quad (3.3)$$

Therefore, when $\eta = \pm 1$, using (3.2), (3.3) and choosing $\rho = \|u\| > 0$ sufficiently small, there exists a constant $\alpha > 0$ such that $I(u) \geq \alpha > 0$ whenever $\|u\| = \rho$. This completes the proof. \square

Theorem 3.2. Assume that (F_1) and (F_2) hold, there exists a function $e \in H$ with $\|e\| > \rho$ such that $I(e) < 0$.

Proof. By (F_1) and (F_2) , there exists $C_\varepsilon > 0$ such that

$$|F(x, s^+)| \leq \varepsilon (s^+)^6 + C_\varepsilon.$$

Let $u \in H$ and $u^+ \neq 0$, we have

$$\lim_{t \rightarrow +\infty} \frac{\left| \int_{\Omega} F(x, tu^+) dx \right|}{t^6} \leq \lim_{t \rightarrow +\infty} \frac{\varepsilon t^6 \int_{\Omega} (u^+)^6 dx + C_\varepsilon |\Omega|}{t^6} = \varepsilon \int_{\Omega} (u^+)^6 dx.$$

Using the arbitrary of ε , one gets

$$\lim_{t \rightarrow +\infty} \frac{\left| \int_{\Omega} F(x, tu^+) dx \right|}{t^6} = 0.$$

Thus, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{I(tu)}{t^6} &= \lim_{t \rightarrow +\infty} \left\{ \frac{\|u\|^2}{2t^4} + \frac{\eta \int_{\Omega} \phi_u (u^+)^2 dx}{4t^2} - \frac{\int_{\Omega} F(x, tu^+) dx}{t^6} - \frac{\int_{\Omega} (u^+)^6 dx}{6} \right\} \\ &= -\frac{1}{6} \int_{\Omega} (u^+)^6 dx < 0, \end{aligned}$$

which implies that $I(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$. Thus we can find $t_0 > 0$ large enough such that $e = \|t_0 u\| > \rho$ and $I(e) < 0$. So we complete the proof. \square

Next, we prove that the functional I satisfies $(PS)_c$ condition.

Theorem 3.3. Assume that (F_1) - (F_3) hold, then the functional I satisfies the $(PS)_c$ condition for all $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$.

Proof. Let $\{u_n\} \subset H$ be a $(PS)_c$ sequence of I , that is,

$$I(u_n) \rightarrow c \quad \text{and} \quad I'(u_n) \rightarrow 0, n \rightarrow \infty. \quad (3.4)$$

Then the sequence $\{u_n\}$ is bounded in H . In fact, since $\frac{1}{4}(1 - \frac{a}{\lambda_1}) > 0$, by (F_3) , (2.3), (2.4) and (3.4), we can get

$$\begin{aligned} c + 1 + o(\|u\|) &\geq I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \\ &= \frac{1}{4} \|u_n\|^2 - \int_{\Omega} (F(x, u_n^+) - \frac{1}{4} f(x, u_n^+)) dx + \frac{1}{12} \int_{\Omega} (u_n^+)^6 dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{4} \|u_n\|^2 - \frac{a}{4} \int_{\Omega} (u_n^+)^2 dx + \frac{1}{12} \int_{\Omega} (u_n^+)^6 dx \\
&\geq \frac{1}{4} \left(1 - \frac{a}{\lambda_1}\right) \|u_n\|^2,
\end{aligned} \tag{3.5}$$

which implies that $\{u_n\}$ is bounded in H . Up to a subsequence, still denoted by $\{u_n\}$, there exists $u_* \in H$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } H, \\ u_n \rightarrow u_* & \text{strongly in } L^s(\Omega), s \in [1,6), \\ u_n(x) \rightarrow u_*(x) & \text{a.e. on } \Omega. \end{cases} \tag{3.6}$$

Since $\{u_n\}$ is a $(PS)_c$ sequence of I and $u_n \rightharpoonup u_*$ in H , we can infer

$$\langle I'(u_*), u_* \rangle = 0. \tag{3.7}$$

Consequently, by (F_3) , one has

$$\begin{aligned}
I(u_*) &\geq I(u_*) - \frac{1}{4} \langle I'(u_*), u_* \rangle \\
&= \frac{1}{4} \|u_*\|^2 - \int_{\Omega} (F(x, u_*^+) - \frac{1}{4} f(x, u_*^+)) dx + \frac{1}{12} \int_{\Omega} (u_*^+)^6 dx \\
&= \frac{1}{4} \|u_*\|^2 - \frac{a}{4} \int_{\Omega} (u_*^+)^2 dx + \frac{1}{12} \int_{\Omega} (u_*^+)^6 dx \\
&\geq \frac{1}{4} \left(1 - \frac{a}{\lambda_1}\right) \|u_*\|^2 \geq 0.
\end{aligned} \tag{3.8}$$

Using (3.6) and Lemma 2.1(e), we get that

$$\int_{\Omega} \phi_{u_n}(u_n^+)^2 dx = \int_{\Omega} \phi_{u_*}(u_*^+)^2 dx + o(1), \tag{3.9}$$

where $o(1) \rightarrow 0$ as $n \rightarrow +\infty$. Set $v_n = u_n - u_*$, the Brézis-Lieb lemma (see [15]) means that

$$\begin{cases} \|u_n\|^2 = \|v_n\|^2 + \|u_*\|^2 + o(1). \\ \int_{\Omega} (u_n^+)^6 dx = \int_{\Omega} (v_n^+)^6 dx + \int_{\Omega} (u_*^+)^6 dx + o(1). \end{cases} \tag{3.10}$$

By (F_2) and (3.6), we get that

$$\begin{cases} \lim_{n \rightarrow +\infty} \int_{\Omega} F(x, u_n^+) dx = \int_{\Omega} F(x, u_*^+) dx \\ \lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n^+) u_n dx = \int_{\Omega} f(x, u_*^+) u_* dx. \end{cases} \tag{3.11}$$

Moreover, using (3.4), (3.9)-(3.11), we can deduce that

$$\begin{aligned}
c+o(1) &= I(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{\eta}{4} \int_{\Omega} \phi_{u_n}(u_n^+)^2 dx - \int_{\Omega} F(x, u_n^+) dx - \frac{1}{6} \int_{\Omega} (u_n^+)^6 dx \\
&= \frac{1}{2} \{ \|u_*\|^2 + \|v_n\|^2 \} + \frac{\eta}{4} \int_{\Omega} \phi_{u_*}(u_*^+)^2 dx - \int_{\Omega} F(x, u_*^+) dx \\
&\quad - \frac{1}{6} \left\{ \int_{\Omega} (u_*^+)^6 dx + \int_{\Omega} (v_n^+)^6 dx \right\} \\
&= \left\{ \frac{1}{2} \|u_*\|^2 + \frac{\eta}{4} \int_{\Omega} \phi_{u_*}(u_*^+)^2 dx - \int_{\Omega} F(x, u_*^+) dx - \frac{1}{6} \int_{\Omega} (u_*^+)^6 dx \right\} \\
&\quad + \frac{1}{2} \|v_n\|^2 - \frac{1}{6} \int_{\Omega} (v_n^+)^6 dx \\
&= I(u_*) + \frac{1}{2} \|v_n\|^2 - \frac{1}{6} \int_{\Omega} (v_n^+)^6 dx. \tag{3.12}
\end{aligned}$$

From (3.12), one gets

$$\frac{1}{2} \|v_n\|^2 - \frac{1}{6} \int_{\Omega} (v_n^+)^6 dx = c - I(u_*) + o(1). \tag{3.13}$$

Similarly, by (3.4), (3.9)-(3.11), we obtain

$$\begin{aligned}
o(1) &= \langle I'(u_n), u_n \rangle \\
&= \|u_n\|^2 + \eta \int_{\Omega} \phi_{u_n}(u_n^+)^2 dx - \int_{\Omega} f(x, u_n^+) u_n dx - \int_{\Omega} (u_n^+)^6 dx \\
&= \{ \|u_*\|^2 + \|v_n\|^2 \} + \eta \int_{\Omega} \phi_{u_*}(u_*^+)^2 dx - \int_{\Omega} f(x, u_*^+) u_* dx \\
&\quad - \left\{ \int_{\Omega} (u_*^+)^6 dx + \int_{\Omega} (v_n^+)^6 dx \right\} \\
&= \left\{ \|u_*\|^2 + \eta \int_{\Omega} \phi_{u_*}(u_*^+)^2 dx - \int_{\Omega} f(x, u_*^+) u_* dx - \int_{\Omega} (u_*^+)^6 dx \right\} \\
&\quad + \|v_n\|^2 - \int_{\Omega} (v_n^+)^6 dx \\
&= \langle I'(u_*), u_* \rangle + \|v_n\|^2 - \int_{\Omega} (v_n^+)^6 dx \\
&= \|v_n\|^2 - \int_{\Omega} (v_n^+)^6 dx. \tag{3.14}
\end{aligned}$$

From (3.14), one has

$$\|v_n\|^2 - \int_{\Omega} (v_n^+)^6 dx = o(1). \tag{3.15}$$

Let

$$\|v_n\|^2 \rightarrow l^2 \quad \text{and} \quad \int_{\Omega} (v_n^+)^6 dx \rightarrow l^2. \tag{3.16}$$

Using the Sobolev inequality and (3.16), we get that $l=0$ or $l^2 \geq S^{\frac{3}{2}}$. If $l=0$, this completes the proof. Assume that $l^2 \geq S^{\frac{3}{2}}$. Then, passing to the limit in (3.13) and taking into the account (3.8) and (3.16), we obtain

$$c \geq \frac{1}{2}l^2 - \frac{1}{6}l^2 = \frac{1}{3}l^2 \geq \frac{1}{3}S^{\frac{3}{2}}, \quad (3.17)$$

which is a contradiction. Therefore, $l=0$ and we conclude that $u_n \rightarrow u_*$ in H . This completes the proof. \square

Next, we estimate the level value of functional I .

Theorem 3.4. *Assume that $\eta = \pm 1$ and (F_1) - (F_4) hold, there exists $u_\varepsilon \in H$ such that*

$$\sup_{t \geq 0} I(tu_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}}.$$

Proof. As is known to all, the function

$$U(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}^3,$$

is a positive solution of the problem $-\Delta u = u^5, \forall x \in \mathbb{R}^3$. Define a cut-off function $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1, |\nabla \psi| \leq C_3$. For some $\delta > 0$, we define

$$\psi(x) = \begin{cases} 1, & |x| \leq \delta, \\ 0, & |x| \geq 2\delta, \end{cases}$$

and $\{x: |x| \leq 2\delta\} \subset \omega$, where ω is defined by (F_4) . Set $u_\varepsilon(x) = \psi(x)U(x)$. As well known (see [16,17]), one has

$$\|u_\varepsilon\|^2 = \|U\|^2 + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon), \quad (3.18)$$

$$|u_\varepsilon|_6^6 = |U|_6^6 + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3), \quad (3.19)$$

and

$$\int_{\Omega} |u_\varepsilon|^s ds = \begin{cases} O(\varepsilon^{\frac{s}{2}}), & 1 \leq s < 3, \\ O(\varepsilon^{\frac{s}{2}} |\ln \varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{6-s}{2}}), & 3 < s < 6. \end{cases} \quad (3.20)$$

For any $\varepsilon > 0$ and $t \geq 0$, we define $I(tu_\varepsilon)$ by

$$I(tu_\varepsilon) = \frac{t^2}{2} \|u_\varepsilon\|^2 + \frac{t^4}{4} \eta \int_{\Omega} \phi_{u_\varepsilon}(u_\varepsilon^+)^2 dx - \int_{\Omega} F(x, tu_\varepsilon^+) dx - \frac{t^6}{6} \int_{\Omega} (u_\varepsilon^+)^6 dx.$$

Since $\lim_{t \rightarrow +\infty} I(tu_\varepsilon) = -\infty$ and $I(0) = 0$. By Lemmas 3.1 and 3.2, there exist two constants $t_1, t_2 > 0$, which independent of ε , such that $0 < t_1 \leq t_\varepsilon \leq t_2 < +\infty$ and $\max_{t \geq 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon)$. Let $I(tu_\varepsilon) = I_{\varepsilon,1}(t) + \eta I_{\varepsilon,2}(t) - I_{\varepsilon,3}(t)$, where

$$\begin{aligned} I_{\varepsilon,1}(t) &= \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^6}{6} \int_{\Omega} (u_\varepsilon^+)^6 dx, & I_{\varepsilon,2}(t) &= \frac{t^4}{4} \int_{\Omega} \phi_{u_\varepsilon} (u_\varepsilon^+)^2 dx, \\ I_{\varepsilon,3}(t) &= \int_{\Omega} F(x, tu_\varepsilon^+) dx. \end{aligned}$$

First, we estimate the value of $I_{\varepsilon,1}(t_\varepsilon)$. Since

$$I_{\varepsilon,1}(t) = \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^6}{6} \int_{\Omega} (u_\varepsilon^+)^6 dx, \quad t \geq 0,$$

we have

$$I'_{\varepsilon,1}(t) = t \|u_\varepsilon\|^2 - t^5 \int_{\Omega} (u_\varepsilon^+)^6 dx, \quad t \geq 0.$$

Let $I'_{\varepsilon,1}(t) = 0$, that is, $t \|u_\varepsilon\|^2 - t^5 \int_{\Omega} (u_\varepsilon^+)^6 dx = 0$. One obtains

$$t = \left(\frac{\|u_\varepsilon\|^2}{\int_{\Omega} (u_\varepsilon^+)^6 dx} \right)^{\frac{1}{4}} \triangleq T_\varepsilon.$$

Then $I'_{\varepsilon,1}(t) > 0$ for all $0 < t < T_\varepsilon$ and $I'_{\varepsilon,1}(t) < 0$ for all $t > T_\varepsilon$, so $I_{\varepsilon,1}(t)$ attains its maximum at T_ε . Thus from (3.18) and (3.19), one gets

$$\begin{aligned} I_{\varepsilon,1}(t_\varepsilon) &\leq I_{\varepsilon,1}(T_\varepsilon) = \frac{T_\varepsilon^2}{2} \|u_\varepsilon\|^2 - \frac{T_\varepsilon^6}{6} \int_{\Omega} u_\varepsilon^6 dx = \frac{1}{3} \left(\frac{\|u_\varepsilon\|^2}{\left(\int_{\Omega} (u_\varepsilon^+)^6 dx\right)^{\frac{1}{3}}} \right)^{\frac{3}{2}} \\ &= \frac{1}{3} \left(\frac{S^{\frac{3}{2}} + O(\varepsilon)}{\left(S^{\frac{3}{2}} + O(\varepsilon^3)\right)^{\frac{1}{3}}} \right)^{\frac{3}{2}} = \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon). \end{aligned} \quad (3.21)$$

Secondly, we estimate the value of $I_{\varepsilon,2}(t_\varepsilon)$. Using (2.1), (3.20) and the Hölder inequality, we have

$$\begin{aligned} I_{\varepsilon,2}(t_\varepsilon) &= \frac{t_\varepsilon^4}{4} \int_{\Omega} \phi_{u_\varepsilon} (u_\varepsilon^+)^2 dx \leq \frac{t_\varepsilon^4}{4} \left(\int_{\Omega} \phi_{u_\varepsilon}^6 dx \right)^{\frac{1}{6}} \left(\int_{\Omega} (u_\varepsilon^+)^{\frac{12}{5}} dx \right)^{\frac{5}{6}} \\ &\leq C_3 \left(\int_{\Omega} u_\varepsilon^{\frac{12}{5}} dx \right)^{\frac{5}{3}} = O(\varepsilon^2). \end{aligned} \quad (3.22)$$

Thirdly, we estimate the value of $I_{\varepsilon,3}(t_\varepsilon)$. One has that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\int_{\Omega} F(x, t_\varepsilon u_\varepsilon^+) dx}{\varepsilon} = +\infty. \quad (3.23)$$

Let $m(t) = \inf_{x \in \omega} f(x, t)$, via (F_1) and (F_4) , we have

$$f(x, t) \geq m(t) \geq 0, \quad \lim_{t \rightarrow +\infty} \frac{m(t)}{t^3} = +\infty,$$

for almost $x \in \omega$ and $t > 0$. So for any $\mu > 0$, there exists $A > 0$ such that $M(t) \geq \mu t^4$ for all $t \geq A$, where $M(t) = \int_0^t m(s) ds$. Thus, one obtains

$$\begin{aligned} & \frac{\int_{\Omega} F(x, t_{\varepsilon} u_{\varepsilon}^+) dx}{\varepsilon} \geq \frac{\int_{|x| < \delta} F(x, t_{\varepsilon} u_{\varepsilon}^+) dx}{\varepsilon} \geq \frac{\int_{|x| < \delta} M(t_{\varepsilon} u_{\varepsilon}^+) dx}{\varepsilon} \\ & = \varepsilon^{-1} \int_0^{\delta} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + r^2)^{\frac{1}{2}}} \right] r^2 dr = \varepsilon^2 \int_0^{\delta \varepsilon^{-1}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr \\ & = \varepsilon^2 \int_0^{\varepsilon^{-1}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr - \varepsilon^2 \int_{\delta \varepsilon^{-1}}^{\varepsilon^{-1}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr. \end{aligned} \tag{3.24}$$

We prove that $M(t)$ is increasing for all $t > 0$ since $m(t) > 0$ for all $t > 0$. Using (F_2) , one has $M(t) \leq C_4 t^2$ for all $t > 0$ small enough. Consequently, we have

$$\left| \varepsilon^2 \int_{\delta \varepsilon^{-1}}^{\varepsilon^{-1}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr \right| \leq C_4 \varepsilon^{-1} M(t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}) \leq C_4 \varepsilon^{-1} M(T_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}) \leq C_4, \tag{3.25}$$

for all $\varepsilon > 0$ small enough. Fixed A , there exists $B > 0$ such that $\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \geq A$ for all $1 < r < B \varepsilon^{-\frac{1}{2}}$. Therefore, one obtains

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^2 \int_0^{\varepsilon^{-1}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr \geq \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^2 \int_1^{B \varepsilon^{-\frac{1}{2}}} M \left[\frac{t_{\varepsilon} 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \right] r^2 dr \\ & \geq \liminf_{\varepsilon \rightarrow 0^+} C_4 \mu \varepsilon^2 \int_1^{B \varepsilon^{-\frac{1}{2}}} \frac{\varepsilon^{-2} r^2}{(1+r^2)^2} dr = \int_1^{+\infty} \frac{r^2}{(1+r^2)^2} dr = +\infty. \end{aligned} \tag{3.26}$$

Thus, from (3.22)-(3.23), when $\eta = 1$, we have

$$\begin{aligned} \sup_{t \geq 0} I(t u_{\varepsilon}) & \leq I(t_{\varepsilon} u_{\varepsilon}) = I_{\varepsilon,1}(t_{\varepsilon}) + I_{\varepsilon,2}(t_{\varepsilon}) - I_{\varepsilon,3}(t_{\varepsilon}) \\ & \leq \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon) + O(\varepsilon^2) - I_{\varepsilon,3}(t_{\varepsilon}) < \frac{1}{3} S^{\frac{3}{2}}; \end{aligned} \tag{3.27}$$

while $\eta = -1$, one has

$$\begin{aligned} \sup_{t \geq 0} I(t u_{\varepsilon}) & \leq I(t_{\varepsilon} u_{\varepsilon}) = I_{\varepsilon,1}(t_{\varepsilon}) - I_{\varepsilon,2}(t_{\varepsilon}) - I_{\varepsilon,3}(t_{\varepsilon}) \\ & \leq I_{\varepsilon,1}(t_{\varepsilon}) - I_{\varepsilon,3}(t_{\varepsilon}) \leq \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon) - I_{\varepsilon,3}(t_{\varepsilon}) < \frac{1}{3} S^{\frac{3}{2}}. \end{aligned} \tag{3.28}$$

This completes the proof. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide two steps to prove Theorem 1.1.

Step 1. We prove that there exists a positive solution for problem (2.2).

Via Lemma 3.1 and Lemma 3.2, we obtain that the energy functional I has the geometry of Mountain pass theorem. By Mountain pass theorem (see [14]), we get that there exists a sequence $\{u_n\} \subset H$ satisfying $I(u_n) \rightarrow c, I'(u_n) \rightarrow 0$. Moreover, combining with Lemmas 3.3 and 3.4, we not only have $0 < \alpha \leq c < \frac{1}{3}S^{\frac{3}{2}}$, but also get that there exists a sequence $\{u_n\} \subset H$ possesses a strongly convergent subsequence (still denoted by $\{u_n\}$). Let $u^* \in H$, satisfying $u_n \rightarrow u^*$ and $I(u^*) = c \geq \alpha > 0$. So u^* is a nontrivial solution for problem (2.2).

We define $(u^*)^- = \max\{-u^*, 0\}$. By $\langle I'(u^*), (u^*)^- \rangle = 0$, one gets $(u^*)^- = 0$. Thus, $u^* \geq 0$. By the strong maximum principle, we have $u^* > 0$ in H . Thus u^* is a positive solution of problem (2.2).

Step 2. We obtain that there exists a positive ground state solution for problem (2.2).

Let

$$m = \inf\{I(u) : u \in H, u \neq 0, I'(u) = 0\}.$$

By the definition of m , there exists $\{u_n\} \subset H$ such that $u_n \neq 0$, and

$$I(u_n) \rightarrow m, \quad I'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.29)$$

From (3.5), one can easily get that $\{u_n\}$ is bounded in H . Then there exists a nonnegative subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u \in H$ such that $u_n \rightharpoonup u$ weakly in H .

We claim that $u \neq 0$. Arguing by contradiction, $u_n \rightharpoonup 0$ weakly in H , it follows that

$$u_n \rightarrow 0, \quad \text{in } L^p(\Omega) (1 \leq p < 6).$$

In particular,

$$\int_{\Omega} \phi_{u_n} (u_n^+)^2 = o(1), \quad \int_{\Omega} F(x, u_n^+) = o(1), \quad \int_{\Omega} f(x, u_n^+) u_n = o(1). \quad (3.30)$$

Therefore, by (3.29) and (3.30), one gets

$$\|u_n\|^2 - \int_{\Omega} (u_n^+)^6 dx = o(1). \quad (3.31)$$

Let $\lim_{n \rightarrow \infty} \|u_n\| = l$, this together with (3.31), we obtain

$$l^2 \geq S^{\frac{3}{2}} \quad \text{or} \quad l = 0.$$

Using (3.29) again, it follows that

$$\begin{aligned} m &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \|u_n\|^2 + \frac{\eta}{4} \int_{\Omega} \phi_{u_n} (u_n^+)^2 dx - \int_{\Omega} F(x, u_n^+) dx - \frac{1}{6} \int_{\Omega} (u_n^+)^6 dx \right\} \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} \|u_n\|^2 = \frac{1}{3} l^2. \end{aligned}$$

If $l = 0$, then $m = 0$, this contradicts to the definition of m . Consequently, $m = \frac{1}{3}l^2 \geq \frac{1}{3}S^{\frac{3}{2}}$. According to Lemma 3.4, one gets $\frac{1}{3}S^{\frac{3}{2}} \leq m < \frac{1}{3}S^{\frac{3}{2}}$, this is a contradiction. So one gets $u_n \rightharpoonup u \neq 0$ weakly in H .

Moreover, using Lemmas 3.3 and 3.4, we can get that $u_n \rightarrow u$ in H and $I(u) = m$. So u is a nontrivial solution of problem (2.2). Similar to u^* in Step 1, by the strong maximum principle, we obtain that u is a positive solution of problem (2.2), and $I(u) = m$. Thus, u is a positive ground state solution of problem (2.2). Then (u, ϕ_u) is a positive ground state solution of system (1.1). Therefore, we finish the proof of Theorem 1.1. \square

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