Anal. Theory Appl.
Vol. 27, No. 3 (2011), 220–223
DOI10.1007/s10496-011-0220-6

SOME APPLICATIONS OF BP-THEOREM IN APPROXIMATION THEORY

I. Sadeqi

(*Sahand University of Technology, Iran*)

R. Zarghami

(*University of Tabriz, Iran*)

Received May 31, 2010

Abstract. In this paper we apply Bishop-Phelps property to show that if X is a Banach space and $G \subseteq X$ is the maximal subspace so that $G^{\perp} = \{x^* \in X^* | x^*(y) = 0; \quad \forall y \in G\}$ is an L-summand in X^*, then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

Key words: *Bishop-Phelps theorem, support point, proximinality, L-projection*

AMS (2010) subject classification: 46E99

1 Introduction

In the note we need some definitions and notations which are the following. Let (Ω, Σ, μ) be a measure space with non-negative complete σ-finite measure μ and σ-algebra Σ of μ-measurable sets. We denote by $L_P(\Omega, \Sigma, \mu : X) = L_p(\Omega, X)$ the Banach space of all equivalence classes of all Bochner integrable functions $f : \Omega \to X$ with the norm

$$\|f\| = (\int_{\Omega} \|f(t)\|^p \mathrm{d}\mu)^{\frac{1}{p}}; 1 \le p < \infty,$$

$$\|f\|_{\infty} = \operatorname*{ess\,sup}_{t \in \Omega} \|f(t)\|; p = \infty.$$

A set M of measurable functions $f : \Omega \to X$ is decomposable if for any two elements f , g in M and $E \in \Sigma$, $\chi_E f + \chi_{\Omega \setminus E} g \in M$, where χ_E is the characteristic function of E. Let X be a real or complex Banach space and C be a closed convex subset of X. The set of support points of C is the collection of all points $z \in C$ for which there exists nontrivial $f \in X^*$ such that $\sup\limits_{x\in C}|f(x)|=|f(z)|$. Such an f is called support functional. The support point z is said to be exposed, if Re $f(x)<\mathrm{Re}f(z)$, for $x(\neq z)\in C$. We denote by Supp C and ΣC the set of support points and support functionals, respectively. Bishop and Phelps[5] have shown that if C is a closed convex and bounded subset of X then Supp C is dense in the boundary of C and ΣC is dense in X^*. The complex case of the Bishop-Phelps theorem is also studied in [6] and some results are given.

Let X be a Banach space and G a closed subspace of X. The subspace G is called proximinal in X if for every $x\in X$ there exists at least one $y\in G$ such that

$$||x-y||=\inf\{||x-z||:z\in G\}.$$

A linear projection P is called an L-projecton if

$$\|x\|=\|Px\|+\|x-Px\|;\qquad \forall x\in X.$$

A closed subspace $Y\subset X$ is called an $L-summand$ if it is the range of an $L-projection$.

The natural question is that, whether or not $L^1(\Omega,G)$ is proximinal in $L^1(\Omega,X)$ if G is proximinal in X [3].We will show that if $G^{\perp}$ is an $L-summand$ then $L^1(\Omega,G)$ is contained in a maximal proximinal subspace of $L^1(\Omega,X)$.

2 The Main Results

Theorem 2.1[4]**.** *If X is a Banach space and $T\in X^*$, then $kerT$ is a proximinal set in X if and only if T supports some points of the unit ball of X.*

Lemma 2.2. *Let X be a Banach space and G a support set in X. Suppose $L^1(\Omega,G)$ is a decomposable set. Then each constant function of $L^1(\Omega,G)$ is a support point for $L^1(\Omega,G)$.*

Proof. Let $g_0\in L^1(\Omega,G)$ be a constant function, then there exists a point $x_0\in G$ such that $g_0(t)=x_0$. Since G is a support set, we have

$$\exists T_0\in X^*\ \text{ s. t. }\quad \inf_G T_0=T_0(x_0).$$

We define $F_0:L^1(\Omega,X)\to R$ as follows:

$$F_0(g)=\int_{\Omega}T_0(g(t))\mathrm{d}\mu.$$

It is obvious that $F_0\in L^1(\Omega,X)^*$, because if

$$g_n\to g\quad (||g_n-g||\to 0),$$

then

$$\begin{aligned} |F_0(g_n) - F_0(g)| &= |\int_\Omega T_0(g_n(t) - g(t))\mathrm{d}\mu| \\ &\leq \int_\Omega |T_0(g_n(t) - g(t))|\mathrm{d}\mu \\ &\leq \int_\Omega ||T_0|| ||(g_n(t) - g(t))||\mathrm{d}\mu \\ &= ||T_0|| ||g_n - g|| \to 0. \end{aligned} \tag{2.1}$$

hence $F_0(g_n) \to F_0(g)$ therefore $F_0 \in L^1(\Omega, X)^*$.

Now by Theorem 2.2[2], we have

$$\begin{aligned} \inf_{L^1(\Omega,G)} F_0 &= \inf_{L^1(\Omega,G)} \int_\Omega T_0(g(t))\mathrm{d}\mu \\ &= \int_\Omega T_0(x_0)\mathrm{d}\mu = T_0(x_0). \end{aligned} \tag{2.2}$$

By letting $g_0(t) = x_0$ we get that $g_0 \in L^1(\Omega, G)$, and the required result follows:

$$\inf_{L^1(\Omega,G)} F_0 = F_0(g_0) = T_0(x_0) = \inf_G T_0.$$

Therefore, $g_0 \in L^1(\Omega, G)$ is a support point for $L^1(\Omega, G)$.

In [1] it is shown that if G is a subspace of a Banach space X such that

$$G^\perp = \{x^* \in X^* | x^*(y) = 0;\ \forall y \in G\}$$

is an L-summand in X^*, then G is proximinal in X. By applying this and the above results we will have the following theorem.

Theorem 2.3. *Let X be a Banach space and $G \subset X$ be subspace such that*

$$G^\perp = \{x^* \in X^* | x^*(y) = 0;\ \forall y \in G\}$$

is an L-summand in X^*, then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

Proof. Let $G^\perp$ be an $L-summand$ in Banach space X^* then by the above theorem and Theorem 2.1, G is proximinal in X. So there exists $T \in X^*$ such that $kerT = G$. Applying Theorem 2.1 , there exists a point x_0 in the closed unit ball of X such that T supports x_0. It is trivial that

$$F(g) = \int_\Omega T(g(t))\mathrm{d}\mu$$

is a continuous linear functional on $L^1(\Omega, X)$. Since T is a support functional, By Lemma 2.2, F is also a support functional for the closed unit ball of $L^1(\Omega, X)$ (by choosing $g_0(t) = x_0$).

Therefore, $kerF$ is proximinal in $L^1(\Omega, X)$. It is obvious that $L^1(\Omega, G) \subseteq kerF$ and $kerF$ is a maximal subspace, so $L^1(\Omega, G)$ is contained in the maximal proximinal subspace of $L^1(\Omega, X)$.

Remark 2.4. It is easy to see that if x_0 is a support point for a closed convex subset C of a Banach space $(X, \|.\|_1)$ then it may not be a support point for $C \subseteq (X, \|.\|_2)$ so that $\|.\|_2$ is a equivalent norm to $\|.\|_1$. Now from the above results we conclude that the proximinality of a subset of a Banach space does not hold with two equivalent norms in general.

References

[1] Harmand, P., Werner, D. and Werner, W., M-ideals in Banach Spaces and Banach Algebras, Lecture Notes in Math., 1574, Springer, Berlin, Heidelberg, New York, 1993.

[2] Hiai, F. and Umegaki, H., Integrals, Conditional Expection, and Martingales Multivalued Functions, J. Multivariate Anal., 7(1977), 149-182.

[3] Khalil, R. and Said, F., Best Approximation in $L^1(\Omega, X)$, Proceeding of the Amer., Math., Soc., 1(1999),183-189.

[4] Mani, P., A Characterization of Convex Set, Handbook of Convex Geometry, 1993.

[5] Phelps, R. R., The Bishop-Phelps Theorem in Complex Spaces: an Open Problem, Pure App., Math., 131(1991), 337-340.

[6] Sadeqi, I., Support Functionals and Their Relation to the RNP, IJMMS, 16(2004), 827-83.

I. Sadeqi
Department of Mathematics
Sahand University of Technology
Tabriz, Iran

E-mail: esadeqi@sut.ac.ir

R. Zarghami
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, Iran

E-mail: zarghami@tabrizu.ac.ir