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SOME RESULTS ON TOPICAL FUNCTIONS AND UPWARD SETS

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Abstract. The purpose of this paper is to introduce and discuss the concept of topical functions on upward sets. We give characterizations of topical functions in terms of upward sets.

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1 Introduction

If X is a partially ordered vector space X, then the set $X^+ = \{x \in X : \; x \geq 0\}$ is called the positive cone of X, and its members are called positive elements of X.

A partially ordered vector space X is called a vector lattice if for every pair of points x, y in X both $\sup\{x,y\}$ and $\inf\{x,y\}$ exist. As usual, $\sup\{x,y\}$ is denoted by $x \vee y$ and $\inf\{x,y\}$ by $x \wedge y$. That is, $\sup\{x,y\} = x \vee y$ and $\inf\{x,y\} = x \wedge y$. In a vector lattice, the positive part, the negative part and the absolute value of an element x are defined by

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and} \quad |x| = x \vee (-x),$$

respectively. Also we have

$$x = x^+ - x^-, \quad |x| = x^+ + x^-, \quad \text{and} \quad |x^+ - y^+| \leq |x - y|.$$

A norm $\|.\|$ on a vector lattice X is said to be a lattice norm, whenever $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$. A normed vector lattice is a vector lattice equipped with a lattice norm. If a normed vector lattice X is complete, then X is referred to a Banach lattice.

Recall that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbf{R}$ such that $x \leq \lambda \mathbf{1}$. Using a strong unit $\mathbf{1}$ we can prove that

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\}, \qquad \forall x \in X$$

is a norm lattice on X. We have also

$$|x| \leq \|x\| \mathbf{1}, \qquad \forall x \in X.$$

Well-know examples of the Banach lattice with strong units are the lattice of all bounded functions defined on a set X and the lattice $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions on a space S with a σ-algebra of measurable sets Σ and a measure μ.

A function $f : X \to \overline{R} = [-\infty, +\infty]$ is called topical if it is increasing $(x \leq y \Longrightarrow f(x) \leq f(y))$ and plus-homogeneous if $f(x + \lambda \mathbf{1}) = f(x) + \lambda$ for all $x \in X$ and all $\lambda \in \mathbf{R}$, and they are studied in [4-5]. The reader may find many applications in applied mathematics (see [3]).

Recall (see [3]) that a subset U of X is said to be upward, if $u \in U$ and $x \in X$ with $u \leq x$, then $x \in U$.

For any subset U of X, we shall denote by int$U$, cl$U$, and bd$U$ the interior, the closure and the boundary of U, respectively. We have

$$N(x, r) := \{y \in X : \ \|x - y\| \leq r\} = \{y \in X : \ x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}.$$

At first we stste the following lemma which is needed in the proof of the main results.

Lemma 1.1[4]**.** *Let $f : X \longrightarrow \bar{R}$ be a topical function. Then the following statements are true:*

(a) *If $x \in X$ and $f(x) = +\infty$ then $f \equiv +\infty$.*

(b) *If $x \in X$ and $f(x) = -\infty$ then $f \equiv -\infty$.*

2 Upward Sets

Note that if $U \subseteq X$, then U is an upward set if and only if for all $u \in U$ and all $x \in X$, $\max\{x,u\} \in U$.

Example 2.1. Suppose $x \in X$ and $U = \{y \in X:\ x \leq y\}$. Then U is an upward set of X.

Definition 2.2. Suppose $f : X \to \overline{R}$ is an arbitrary function. Set for $\lambda \in R$,

$$B_\lambda(f) = \{x \in X:\ f(x) \geq \lambda\},$$

then $B_\lambda(f)$ is called upper level set.

Corollary 2.3. *The function $f:\ X \longrightarrow R$ is increasing if and only if for every $\lambda \in \mathbf{R}$, $B_\lambda(f)$ is upward.*

Theorem 2.4.

(a) *The collection τ_u of upward sets is a topology in X.*

(b) *If $U \in \tau_u$, then $\bar{U} \in \tau_u$.*

(c) *If $U \in \tau_u$, and $x \in U$ then for every $\varepsilon > 0$, $x + \varepsilon\mathbf{1} \in int\ U$.*

(d) *If $U \in \tau_u$, then* $\operatorname{int} U = \{x \in X:\ x - \varepsilon\mathbf{1} \in \operatorname{int} U\ for\ some\ \varepsilon\}$.

Proof. The part *(a)* is trivial.

(b) Suppose $U \in \tau_u$ and $x \in \bar{U}$, if $x \leq y$, we show that $y \in \bar{U}$. Consider the sequence $\{x_\alpha\}_{\alpha\geq 1}$ such that $\|x_\alpha - x\| \to 0$. Put $\varepsilon_\alpha = \|x_\alpha - x\|$. Then for every $\alpha \geq 1$, $x_\alpha \leq \varepsilon_\alpha \mathbf{1} + x$. Therefore for every $\alpha \geq 1$, $\varepsilon_\alpha \mathbf{1} + x \in U$. For every $\alpha \geq 1$, put $y_\alpha = \varepsilon_\alpha \mathbf{1} + y$ then $y_\alpha \in U$, also $\varepsilon_\alpha \to 0$ hence $y \in \bar{U}$.

(c) Suppose $U \in \tau_u$, $x \in U$ and $\varepsilon > 0$. Consider the neighborhood of $(x + \varepsilon\mathbf{1})$.

$$V = \{y \in X:\ \|y - (x + \varepsilon\mathbf{1})\| < \varepsilon\}.$$

Then $V = \{y \in X:\ x < y < x + 2\varepsilon\mathbf{1}\}$. Since U is upward and $x \in X$, $V \subseteq U$. Therefore $x + \varepsilon\mathbf{1} \in$ *int* U. *(d)* Suppose $U \in \tau_u$ and for some $\varepsilon > 0$, $x - \varepsilon\mathbf{1} \in U$. Then by (c) we have $x = (x - \varepsilon\mathbf{1}) + \varepsilon\mathbf{1} \in \operatorname{int} U$. If $x \in \operatorname{int} U$ The there exists a neighborhood $N(x,\ \varepsilon)$ of x such that $N(x,\ \varepsilon) \subseteq U$. Also $x - \varepsilon\mathbf{1} \in N(x,\ \varepsilon)$, therefore $x - \varepsilon\mathbf{1} \in U$.

Corollary 2.5. *Let $U \in \tau_u$ be closed and $u \in U$. Then $u \in$* bd *U if and only if for every $\lambda > 0$, $u - \lambda\mathbf{1} \notin U$.*

Let X be a normed linear space and U a nonempty subset of X. Then a point $g_0 \in U$ is said to be a best approximation for $x \in X$, if

$$\|x - g_0\| = d(x,U) = \inf\{\|x - g\|:\ g \in U\}.$$

If each $x \in X$ has at least one best approximation in U, then U is called a proximinal subset of X. Let U be a subset of a normed linear space X, then for $x \in X$ we put

$$P_U(x) = \{g_0 \in U \ : \ \|x - g_0\| = d(x,U)\},$$

the set of all best approximations for $x \in X$.

Theorem 2.6. *Let $U \in \tau_u$ be closed in X. Then U is proximinal.*

Proof. Suppose $x_0 \in X \backslash \overline{U}$ and $r = d(x_0,U) = \inf_{u \in U} \|x_0 - u\|$. Since U is closed, for $\varepsilon > 0$, there exists $u_\varepsilon \in U$ such that $\|x_0 - u_\varepsilon\| < r + \varepsilon$. Therefore

$$-(r+\varepsilon)\mathbf{1} \leq u_\varepsilon - x_0 \leq (r+\varepsilon)\mathbf{1}.$$

Put $u_0 = x_0 + r\mathbf{1}$. Then we can clearly prove that $u_0 \in P_U(x_0)$.

Corollary 2.7. *Let $U \in \tau_u$ is closed of X. Then for $x_0 \in X \backslash \overline{U}$, $u_0 = x_0 + r\mathbf{1} \in P_U(x_0)$, where*

$$r = d(x_0,U).$$

Definition 2.8. Suppose $U \in \tau_u$. Define the function $\rho_U : \ X \to \bar{R}$ for $x \in X$

$$\rho_U(x) = \sup \ \{\lambda \in \mathbf{R} : \ x \in \lambda\mathbf{1} + U\}.$$

Note that if $U = X$ then for every $x \in X$, we have

$$\{\lambda \in \mathbf{R} : x \in \lambda\mathbf{1} + U\} = \mathbf{R}.$$

Also if $U = \emptyset$ and $\lambda \in \mathbf{R}$, then for every $x \in X$, we have

$$\{\lambda \in \mathbf{R} : x \in \lambda\mathbf{1} + U\} = \emptyset.$$

Lemma 2.9. *Let U be a nonempty upward subset of X. Then for every $x \in X$*

$$\{\lambda \in \mathbf{R} : x \in \lambda\mathbf{1} + U\} \neq \emptyset.$$

and the set $\{\lambda \in \mathbf{R} : x \in \lambda\mathbf{1} + U\}$ is an interval to form $(-\infty, r)$ or $(-\infty, r]$.

Proof. Consider $u \in U$ and $x \in X$, if $\lambda = -\inf\{\lambda \in \mathbf{R} : u - x \leq \lambda\mathbf{1}\}$ then $u \leq x - \lambda\mathbf{1}$. Since U is upward, $x - \lambda\mathbf{1} \in U$. Therefore

$$\{\lambda \in \mathbf{R} : x \in \lambda\mathbf{1} + U\} \neq \emptyset,$$

If $r_0 \in \mathbf{R}$ and $x \in r_0\mathbf{1}+U$. If $r \leq r_0$ put $\eta = r_0 - r$, since U is upward then

$$x - r\mathbf{1} = (x - r_0\mathbf{1}) + \eta\mathbf{1} \in U.$$

Theorem 2.10. *Let $U \in \tau_u$, then*

(a) ρ_U *is topical.*

(b) $\rho_U \equiv -\infty$ *if and only if* $U = \emptyset$.

(c) $\rho_U \equiv +\infty$ *if and only if* $U = X$.

(d) ρ_U *is finite if and only if* $\emptyset \neq U \subset X$.

(e) *If $U \in \tau_u$ is a closed upward subset of X and $u \in U$, then $\rho_U(u) = 0$ if and only if $u \in \mathrm{bd}\, U$.*

(f) *If $U \in \tau_u$ is a closed upward subset of X, then*

$$\mathrm{bd}\ u = \{u \in X :\ \rho_U(u) = 0\}.$$

Proof. (a) Since U is upward, ρ_U is increasing. Suppose $x \in X$ and $\alpha \in \mathbf{R}$, then

$$\begin{aligned}
\rho_U(x+\alpha\mathbf{1}) &= \sup\ \{\lambda \in \mathbf{R} :\ x+\alpha\mathbf{1} \in \lambda\mathbf{1}+U\} \\
&= \sup\ \{\lambda \in \mathbf{R} :\ x \in (\lambda-\alpha)\mathbf{1}+U\} \\
&= \sup\ \{(\beta+\alpha) \in \mathbf{R} :\ x \in \beta\mathbf{1}+U\} \\
&= \sup\ \{\beta \in \mathbf{R} :\ x \in \beta\mathbf{1}+U\}+\alpha \\
&= \rho_U(x)+\alpha.
\end{aligned}$$

That is ρ_U is plus-homogeneous and topical.

(b), (c) and (d) are trivial.

(e) Suppose $\rho_U = 0$ and $u \notin \mathrm{bd}\ U$, then by Corollary 2.5, for some $\lambda > 0$, $u - \lambda\mathbf{1} \in U$, it follows that $\rho_U \geq \lambda > 0$. This is a contradiction.

Conversely, suppose $u \in \mathrm{bd}\ U$. Therefore by Corollary 2.5, for every $\lambda > 0$, $u \notin \lambda\mathbf{1}+U$. Since U is closed and $u \in U = 0\mathbf{1}+U$, we have $\rho_U = 0$.

(f) is a consequence of (e).

Theorem 2.11. *Let U be an upward subset of X. Then*

(a) $\{x \in X :\ \rho_U(x) > 0\} = \mathrm{int}\ U \subseteq U$.

(b) $X \backslash \mathrm{int}U = \{x \in X :\ \rho_U(x) \leq 0\}$.

Proof. (a) Suppose $x \in \{x \in X :\ \rho_U(x) > 0\}$. Then there exists $\lambda > 0$ such that $x \in \lambda\mathbf{1}+U$. From Theorem 2.4 (c), $x = (x-\lambda\mathbf{1})+\lambda\mathbf{1} \in \mathrm{int}\ U$. That is $\{x \in X :\ \rho_U(x) > 0\} \subseteq \mathrm{int}\ U$.

Now if $x \in U$, then $\rho_U(x) \geq 0$. Therefore $U \subseteq \{x \in X : \rho_U(x) > 0\}$ and $\bar{U} = \{x \in X : \rho_U(x) > 0\}$. From Theorem 2.10 (f) and the relation $\bar{U} = \text{int}\, U \bigcup \text{bd}\, U$. We have $\text{int}\, U = \{x \in X : \rho_U(x) > 0\}$.

(b) By (a) we have

$$X \backslash \text{int}\, U = \{x \in X : \rho_U(x) \leq 0\}.$$

Theorem 2.12. *Let U be a subset of X. Then the following statements are equivalent:*

(a) *ρ_U is topical and $U = B_0(\rho_U)$.*

(b) *U is upward and for any real sequence $\{\lambda_k\}$ with $x + \lambda_k \mathbf{1} \in U$ and $\lambda_k \longrightarrow \lambda$, one has $x + \lambda \mathbf{1} \in U$.*

Proof. $(a) \Rightarrow (b)$. Suppose $g_1 \in U = B_0(\rho_U)$ and $g_2 \in X$ where $g_2 \geq g_1$. Since ρ_U is topical $g_2 \in B_0(\rho_U)$. Therefore U is upward. Now suppose $x \in X$, $\lambda, \lambda_k \in R$ and $\lambda_k \longrightarrow \lambda$. Since for any k, $x + \lambda_k \mathbf{1} \in U$, it follows that $\rho_U(x + \lambda_k \mathbf{1}) \geq 0$ and since ρ_U is topical

$$\rho_U(x + \lambda_k \mathbf{1}) = \rho_U(x) + \lambda_k \longrightarrow \rho_U(x) + \lambda = \rho_U(x + \lambda \mathbf{1}).$$

Hence

$$\rho_U(x + \lambda \mathbf{1}) \geq 0.$$

It follows that $x + \lambda \mathbf{1} \in U$.

$(b) \Rightarrow (a)$. From Theorem 2.10, ρ_U is topical. Suppose $x \in X$ and $x \in B_0(\rho_U)$. Choose $\lambda_k > 0$ where $\lambda_k \longrightarrow 0$, since ρ_U is topical $\rho_U(x + \lambda_k \mathbf{1}) = \rho_U(x) + \lambda_k \geq \lambda_k > 0$. Therefore $x + \lambda_k \mathbf{1} \in U$, since $\lambda_k \longrightarrow 0$ it follows that $x \in U$ and $B_0(\rho_U) \subseteq U$. Also we know from Theorem 2.11, that $U \subseteq B_0(\rho_U)$, hence $U = B_0(\rho_U)$.

In the following we give a necessary and sufficient condition for topical function.

Theorem 2.13. *Let $f : X \rightarrow \bar{R}$ be a function. Then the following statements are equivalent:*

(a) *f is topical.*

(b) *The set $B_0(f) \in \tau_u$ and $f = \rho_{B_0(f)}$.*

Proof. $(b) \Rightarrow (a)$. If $B_0(f) \in \tau_u$, by Theorem 2.10, $f = \rho_{B_0(f)}$ is topical.

(a)$\Rightarrow$ (b). Suppose f is topical. If $f \equiv -\infty$ then $B_0(f) = \emptyset$; therefore by Theorem 2.10, $\rho_{B_0(f)} = -\infty$.

Suppose there exists $x \in X$ such that $f(x) = \lambda > -\infty$. Then $f(x - \lambda \mathbf{1}) = 0$, and $x - \lambda \mathbf{1} \in B_0(f)$. Hence $B_0(f) \neq \emptyset$. If $g_1 \in B_0(f)$, $g_2 \in X$ and $g_2 \geq g_1$. Since f is increasing, $g_2 \in B_0(f)$, so that $B_0(f)$ is upward. If $x \in X$ then

$$\begin{aligned} \rho_{B_0(f)}(x) &= \sup\ \{\lambda \in \mathbf{R} :\ x - \lambda \mathbf{1} \in B_0(f)\} \\ &= \sup\{\lambda \in \mathbf{R} :\ f(x - \lambda \mathbf{1}) \geq 0\} \\ &= \sup\ \{\lambda \in \mathbf{R} :\ f(x) - \lambda \geq 0\} \\ &= \sup\ \{\lambda \in \mathbf{R} :\ f(x) \geq \lambda\} \\ &= f(x). \end{aligned}$$

Theorem 2.14. *Let U is a closed upward subset of X and $x \in X$. Put*

$$V = \{x + \lambda \mathbf{1} :\ \lambda \in \mathbf{R}\},$$

$W = \{x + \lambda \mathbf{1} :\ \lambda > 0\}$ *and* $P = \{x + \lambda \mathbf{1} :\ \lambda \leq 0\}$. *Then the following ststments are ture:*

(a) card $(V \bigcap \text{bd } U) = 1$,

(b) card $(W \bigcap \text{bd } U) \leq 1$ *and* card $(P \bigcap \text{bd } U) \leq 1$,

(c) *If $x \notin U$, then* card $(P \bigcap \text{bd } U) = 0$ *and* card $(W \bigcap \text{bd } U) = 1$,

(d) *If $x \in U$, then* card $(P \bigcap \text{bd } U) = 1$ *and* card $(W \bigcap \text{bd } U) = 0$.

(Cardinal number of a finite set A is the number of elements in that set A and denote by *card A*).

Proof. If $V \bigcap \text{bd } U = \emptyset$, then by Theorem 2.108 (f), for every $\lambda \in \mathbf{R}$, $\rho_U(x + \lambda \mathbf{1}) \neq 0$. Since ρ_U is topical, for every $\lambda \in \mathbf{R}$, $\rho_U(x) \neq -\lambda$. Therefore $\rho_U(x)$ is not finite and ρ_U is not finite. From Theorem 2.10 (d) $U = \emptyset$ or $U = X$. It follows that card $(V \bigcap \text{bd } U) > 0$. If for every $i = 1, 2$ there exists λ_i such that $x + \lambda_i \mathbf{1} \in V \bigcap \text{bd } U$. From Theorem 2.10 (e), for every i, $\rho_U(x + \lambda_i \mathbf{1}) = 0$. Since ρ_U is topical, it follows that $\lambda_1 = \lambda_2$. Thus card $(V \bigcap \text{bd } U) \leq 1$.

(b) It is similar to (a).

(c) Suppose $x \notin U$, then $\rho_U(x) < 0$. Therefore for every $\lambda \leq 0$, $\rho_U(x) + \lambda < \lambda$. Since ρ_U is topical, then for every $\lambda \leq 0$, $\rho_U(x + \lambda \mathbf{1}) < 0$. It follows that card $(P \bigcap \text{bd } U) = 0$.

Also if put $\lambda = -\rho_U(x)$, we have $\rho_U(x + \lambda \mathbf{1}) = 0$. Hence card $(W \bigcap \text{bd } U) = 1$.

(d) Suppose $x \in U$, then $\rho_U(x) \geq 0$. Therefore for every $\lambda > 0$, $\rho_U(x) + \lambda \geq \lambda$. Since ρ_U is topical, for every $\lambda > 0$, $\rho_U(x + \lambda \mathbf{1}) > 0$. Hence card $(W \bigcap \text{bd } U) = 0$.

Also if put

$$\lambda = -\rho_U(x)$$

then $\rho_U(x + \lambda \mathbf{1}) = 0$. It follows that card $(P \bigcap \text{bd } U) = 1$.

References

[1] Birkhoff, G., Lattice Theory, Colloquium Publications, Vol. 25, American Mathematical Society, 1967.

[2] Gierz, G., Hoffmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S., Continuous Lattices and Domains, Cambridge University Press, 2003.

[3] Martinez, J. E., Legaz, A. M. Rubinov, and I. Singer, Downward Sets and Their Separation and Approximation Properties, Journal of Global Optimization, 23 (2002), 111-137.

[4] Rubinov, A. M. and Glover, B. M., Duality for Increasing Positively Homogeneous Functions and Normal Sets, Recherche Operationnelle/Operations Research, 32 (1998), 105-123.

[5] Rubinov, A. M. and Singer, I., Topical and Sub-topical Functions, Downward Sets and Abstract Convexity, Optimization, 50 (2001), 307-351.

[6] Rubinov, A. M. and Zaslavski, A. J., Two Porosity Results in Monotonic Analysis, Numerical Functional Analysis and Optimization, 23:5,5(2002), 651- 668.

[7] Singer, I., Abstract Convex Analysis, Wiley-Interscience, New York, 1987.

[8] Vulikh, B. Z., Introduction to the Theory of Partially Ordered Vector Spaces, Wolters-Noordhoff, Groningen, 1967.

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