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The Fractional Maximal Operator and Marcinkiewicz Integrals Associated with Schrödinger Operators on Morrey Spaces with Variable Exponent

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Abstract. In this paper, we prove the boundedness of the fractional maximal operator, Hardy-Littlewood maximal operator and marcinkiewicz integrals associated with Schrödinger operator on Morrey spaces with variable exponent.

Key Words: Fractional maximal operator, Marcinkiewicz integrals, Schrödinger, variable exponent, Morrey space.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

In this paper, we consider the Schrödinger differential operator

$$\mathcal{L}=-\Delta+V(x) \quad \text{on } \mathbb{R}^n, \quad n\geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class B_q for $q\geq n/2$.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(q>1)$ if there exists a constant $C>0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V^q dx\Big)^{\frac{1}{q}}\leq C\Big(\frac{1}{|B|}\int_B V dx\Big)$$

holds for every ball in $\mathbb{R}^n$, see [1].

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The Marcinkiewicz integral operator μ is defined by

$$\mu f=\Big(\int_0^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\Big|^2\frac{dt}{t^3}\Big)^{\frac12}.$$

Stein [2] first introduced the operator μ and proved that μ is of type (p,p) $(1<p\le 2)$ and of weak type (1,1) in the case of $\Omega\in Lip_\gamma(S^{n-1})$ $(0<\gamma\le 1)$. Benedek, Calderón and Panzone [3] extended Stein's results, proved that if $\Omega\in C^1(S^{n-1})$, then μ is of type (p,p) $(1<p<\infty)$.

Similar to the classical marcinkiewicz function μ, one defines the Marcinkiewicz functions μ_j^L associated with the Schrödinger operator L by

$$\mu_j^L f(x)=\Big(\int_0^\infty\Big|\int_{|x-y|\le t}K_j^L(x,y)f(y)dy\Big|^2\frac{dt}{t^3}\Big)^{\frac12},$$

where $K_j^L(x,y)=\tilde{K}_j^L(x,y)|x-y|$ and $\tilde{K}_j^L(x,y)$ is the kernel of $R_j^L=(\partial/\partial x_j)L^{-1/2}$, $j=1,\cdots,n$. In particular, when $V=0$, $K_j^\Delta(x,y)=\tilde{K}_\Delta^L(x,y)|x-y|=(|x_j-y_j|/|x-y|)/|x-y|^{n-1}$ and $\tilde{K}_\Delta^L(x,y)$ is the kernel of $R_j=(\partial/\partial x_j)\Delta^{-1/2}$, $j=1,\cdots,n$. In this paper, we write $K_j^\Delta(x,y)=K_j(x,y)$ and

$$\mu_j f(x)=\Big(\int_0^\infty\Big|\int_{|x-y|\le t}K_j(x,y)f(y)dy\Big|^2\frac{dt}{t^3}\Big)^{\frac12}.$$

Obviously, μ_j are classical marcinkiewicz functions. Gao and Tang [4] considered the boundedness of marcinkiewicz integral μ_j^L on $L^p(\mathbb{R}^n)$. Chen and Zou [5] also proved that the marcinkiewicz integral μ_j^L has the same boundedness. The paper [6] by Tang and Dong proved the boundedness of some schrödinger type operators on Morrey spaces related to certain nonnegative potentials. Recently, Chen and Jin [7] have showed that marcinkiewicz integrals associated with Schrödinger Operator is bounded on Morrey Spaces.

It is well known that function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with non-standard growth conditions and so on, we mention e.g., (see [8, 9]). A great deal of work has been done to extend the theory of maximal,potential, singular and marcinkiewicz integrals operators on the classical spaces to the variable exponent case, (see [10–14]). Recently, the author in [15] introduces a new function space that is Morrey space with variable exponents related to certain nonnegative potentials and considers the boundedness of some Schrödinger type operators on Morrey with variable exponent. Hence, it will be an interesting problem whether we can establish the boundedness of the fractional maximal operator, maximal operator and Marcinkiewicz integrals associated with Schrödinger operators on Morrey spaces with variable exponent related to certain nonnegative potentials. The main purpose of this paper is to answer the problem.

To meet the requirements in the next sections, here, basic elements of the theory of Lebsegue spaces with variable exponent are briefly presented.

Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n):=\Big\{f \text{ is measurable}: \int_{\mathbb{R}^n}\Big|\frac{f(x)}{\lambda}\Big|^{p(x)}dx<\infty \text{ for some constant } \lambda>0\Big\}.$$

The space $L^{p(\cdot)}_{loc}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{loc}(\mathbb{R}^n):=\Big\{f \text{ is measurable}: f\in L^{p(\cdot)}(K) \text{ for all compact subsets } K\subset\mathbb{R}^n\Big\}.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\Big\{\lambda>0: \int_{\mathbb{R}^n}\Big|\frac{f(x)}{\lambda}\Big|^{p(x)}dx\le 1\Big\}.$$

We denote

$$p_-:=ess\inf_{x\in\mathbb{R}^n}p(x),\quad p_+:=ess\sup_{x\in\mathbb{R}^n}p(x).$$

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $[1,\infty)$ such that $1<p_-\le p(\cdot)\le p_+<\infty$.

Given a function $f\in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x):=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy,$$

and the fractional operator function M_β is defined by

$$M_\beta f(x):=\sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}}\int_{B(x,r)}|f(y)|dy,\quad 0<\beta<n.$$

$\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

For $x\in\mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x)=\frac{1}{m_V(x)}=\sup_{r>0}\Big\{r:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\Big\},$$

and we write $\Psi_\theta(B(x,r))=(1+r/\rho(x))^\theta$, where $\theta>0$.

A variant of Hardy-Littlewood maximal operator M_V^θ (see [16]) is defined by

$$M_V^\theta f(x):=\sup_{r>0}\frac{1}{\Psi_\theta(B(x,r))|B(x,r)|}\int_{B(x,r)}|f(y)|dy,$$

and a variant of fractional maximal operator $M_{\beta,V}^\theta$ (see [17]) is defined by

$$M_{\beta,V}^\theta f(x):=\sup_{r>0}\frac{1}{(\Psi_\theta(B(x,r))|B(x,r)|)^{1-\frac{\beta}{n}}}\int_{B(x,r)}|f(y)|dy,\quad 0<\beta<n.$$

$\mathcal{B}^\theta(\mathbb{R}^n)$ is the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M_V^θ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Remark 1.1. It is easy to see that $M_V^\theta f(x) \le Mf(x)$ and $M_{\beta,V}^\theta f(x) \le M_\beta f(x)$ for a.e. $x \in \mathbb{R}^n$ and $\theta \ge 0$. So $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ implies that $p(\cdot) \in \mathcal{B}^\theta(\mathbb{R}^n)$.

For brevity, in this paper, C always means a positive constant independent of the main parameters and may change from one occurrence to another. $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, χ_{B_k} be the characteristic function of the set B_k for $k \in \mathbb{Z}$. $|S|$ denotes the Lebesgue measure of S. The exponent $p'(x)$ means the conjugate of $p(x)$, that is, $1/p'(x) + 1/p(x) = 1$.

Definition 1.1 (see [12]). For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, let $k_{p(\cdot)}$ denote the supremum of those $q > 1$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$. Let $e_{p(\cdot)}$ be the conjugate of $k_{p'(\cdot)}$.

Definition 1.2 (see [12]). Let $p(\cdot) \in L^\infty(\mathbb{R}^n)$ and $1 < p(x) < \infty$. A Lebesgue measurable function $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is said to be a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x,2^{j+1}r) < Cu(x,r). \tag{1.1}$$

We denote the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$. Next we define the Morrey spaces with variable exponent related to the nonnegative potential V.

Definition 1.3 (see [15]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $u(x,r) \in \mathbb{W}_{p(\cdot)}$ and $-\infty < \alpha < \infty$. For $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n)$ and $V \in B_q$ $(q > 1)$, we say the Morrey spaces with variable exponent related to the nonnegative potential V is the collection of all function f satisfying

$$\|f\|_{\mathcal{M}_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)} = \sup_{z \in \mathbb{R}^n, r>0} \frac{(1 + rm_V(z))^\alpha}{u(z,r)} \|\chi_{B(z,r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

In particular, when $\alpha = 0$ or $V = 0$, $\mathcal{M}_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)$ is the Morrey spaces with variable exponent $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ introduced in [12]. It is easy to see that $\mathcal{M}_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n) \subset \mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ for $\alpha > 0$ and $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n) \subset \mathcal{M}_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)$ for $\alpha < 0$. If $p(x)$ is a constant, $u(x,r) = r^\lambda$ and $\lambda \in [0, n/p)$, we have

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x,2^{j+1}r)}{u(x,r)} &= \sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p}(\mathbb{R}^n)}} \frac{(2^{j+1}r)^\lambda}{r^\lambda} \\
&= \sum_{j=0}^{\infty} 2^{(j+1)(\lambda - n/p)} < C.
\end{aligned}$$

In this case, $\mathcal{M}_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)$ is the Morrey spaces $L_{\alpha,V}^{p,\lambda}(\mathbb{R}^n)$ related to the nonnegative potential V, see [6].

Now it is in this position to state our results.

Theorem 1.1. *Let* $p(x),q(x)\in\mathcal{B}(\mathbb{R}^n)$, $V\in B_n$, $\theta\geq\max\{0,-\alpha(k_0+1)/(1-\beta/n)\}$, $-\infty<\alpha<\infty$. *Suppose* $p(x)$, $q(x)$ *and* β *satisfy*

$$p_+>\frac{n}{\beta}\quad and \quad \frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\beta}{n}.$$

If there exists q_0, $n/(n-\beta)<q_0<\infty$, *such that*

$$\frac{q(\cdot)}{q_0}\in\mathcal{B}(\mathbb{R}^n)\quad and \quad u\in\mathbb{W}_{q(\cdot)},$$

then

$$\|M^{\theta}_{\beta,V}f\|_{\mathcal{M}^{q(\cdot),u}_{\alpha,V}}\leq C\|f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}}.$$

Remark 1.2. $\mathbb{W}_{q(\cdot)}\subset\mathbb{W}_{p(\cdot)}$, see [15].

Theorem 1.2. *Suppose* $V\in B_n$, $\theta\geq\max\{0,-\alpha(k_0+1)\}$, $-\infty<\alpha<\infty$, $p(x)\in\mathcal{B}(\mathbb{R}^n)$. *If* $u\in\mathbb{W}_{p(\cdot)}$, *then*

$$\|M^{\theta}_{V}f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}}\leq C\|f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}}.$$

Theorem 1.3. *Suppose* $V\in B_n$, $-\infty<\alpha<\infty$, $p(x)\in\mathcal{B}(\mathbb{R}^n)$. *If* $u\in\mathbb{W}_{p(\cdot)}$, *then*

$$\|\mu^{L}_{j}f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}}\leq C\|f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}}.$$

In order to prove our result, we need some conclusions as follows.

Lemma 1.1 (see [18]). *Let* $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. *Then the following conditions are equivalent:*

(1) $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.
(2) $p'(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.
(3) $p(\cdot)/q\in\mathcal{B}(\mathbb{R}^n)$ *for some* $1<q<p_-$.
(4) $(p(\cdot)/q)'\in\mathcal{B}(\mathbb{R}^n)$ *for some* $1<q<p_-$.

Lemma 1.1 ensures that $k_{p(\cdot)}$ is well-defined and satisfies $1<k_{p(\cdot)}\leq p_-$. Moreover, $p_+\geq e_{p(\cdot)}$.

Lemma 1.2 (see [19]). *If* $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, *then for all* $f\in L^{p(\cdot)}(\mathbb{R}^n)$ *and all* $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, *we have*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\leq r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p:=1+1/p_--1/p_+$.

Lemma 1.3 (see [10]). *If* $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$,*then there exists* $C>0$ *such that for all balls* B *in* $\mathbb{R}^n$,

$$C^{-1}|B|\leq\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\leq C|B|.$$

Lemma 1.4 (see [12]). *Let $p(x)\in\mathcal{B}(\mathbb{R}^n)$ and $1<p_-\le p_+<\infty$. There exist $C_1,C_2>0$ such that for any $B\in\mathbb{B}$,*

$$C_1|B|^{\frac{1}{\bar{p}_B}}\le\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C_2|B|^{\frac{1}{\bar{p}_B}},$$

where

$$\frac{1}{\bar{p}_B}=\frac{1}{|B|}\int_B\frac{1}{p(x)}dx.$$

Lemma 1.5 (see [12]). *Let $p(x)\in\mathcal{B}(\mathbb{R}^n)$. For any $1<q<k_{p(\cdot)}$ and $1<s<k_{p'(\cdot)}$, there exist constant $C_1,C_2>0$ such that for any $x_0\in\mathbb{R}^n$ and $r>0$, we have*

$$C_2 2^{jn(1-\frac{1}{s})}\le\frac{\|\chi_{B(x_0,2^jr)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C_1 2^{\frac{jn}{q}},\quad\forall j\in\mathbb{N}.$$

Lemma 1.6 (see [10]). *Let $p(x),q(x)\in\mathcal{P}(\mathbb{R}^n)$ satisfy*

$$p_+<\frac{\beta}{n}\quad\text{and}\quad\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\beta}{n}.$$

If there exists q_0 satisfying

$$\frac{n}{n-\beta}<q_0<\infty\quad\text{and}\quad\frac{q(\cdot)}{q_0}\in\mathcal{B}(\mathbb{R}^n),$$

then

$$\|M_\beta f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for some $C>0$.

Lemma 1.7 (see [20]). *Let $\Omega\in Lip_\gamma(S^{n-1})$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, we have*

$$\|\mu f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 1.8. *Suppose $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, we have*

$$\|\mu_j^L f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. Using the Lemma 1.7 and the inequality $\mu_j^L f\le\mu_j f+CMf$ (see [4,7]), we have

$$\begin{aligned}\|\mu_j^L f\|_{L^{p(\cdot)}(\mathbb{R}^n)}&\le\|\mu_j f\|_{L^{p(\cdot)}(\mathbb{R}^n)}+C\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

So, we complete the proof. $\square$

Lemma 1.9 (see [1]). *If $V\in RH_q$ $(q>1)$, then,*

(1) for every N there exists a constant C such that

$$|K_j^L(x,z)|\le\frac{C(1+|x-z|/\rho(x))^{-N}}{|x-z|^{n-1}},$$

(2) for every N and $0<\delta<\min\{1,1-n/q_0\}$ there exists a constant C such that

$$|K_j^L(x,z)-K_j^L(y,z)|\le\frac{C|x-y|^{\delta}(1+|x-z|/\rho(x))^{-N}}{|x-z|^{n-1+\delta}},$$

where $|x-y|<2|x-z|/3$,

(3) if K denotes the $\mathbb{R}^n$ vector valued kernel of the classical Riesz operator, for every $0<\delta<2-n/q_0$, we have

$$|K_j^L(x,z)-K_j(x,z)|\le\frac{C}{|x-z|^{n-1}}\Big(\frac{|x-z|}{\rho(z)}\Big)^{\delta},$$

where $K_j(x,z)=(|x_j-z_j|/|x-z|)/|x-z|^{n-1}$.

Lemma 1.10 (see [1,6]). *Suppose $V\in B_q$ with $q\ge n/2$. Then there exist positive constants C and k_0 such that*

(1) $m_V(x)\sim m_V(y)$ if $|x-y|\le\frac{C}{m_V(x)}$;

(2) $m_V(y)\le C(1+|x-y|m_V(x))^{k_0}m_V(x)$;

(3) $m_V(y)\ge\dfrac{Cm_V(x)}{(1+|x-y|m_V(x))^{k_0/(k_0+1)}}$.

2 Preparation of manuscript

We will give the proofs of the Theorems 1.1 and 1.3 below. The arguments for Theorem 1.2 is similar, we omit the details here.

Proof of Theorem 1.1. Without loss of generality, we may assume that $\alpha<0$. Let $f\in\mathcal{M}_{p(\cdot),u}$. For any $z\in\mathbb{R}^n$ and $r>0$, we write

$$f(x)=f^0(x)+f^1(x),$$

where $f^0=f\chi_{B(z,2r)}$, $f^1=f\chi_{\mathbb{R}^n\setminus B(z,2r)}$. Hence, we have

$$\|(M^{\theta}_{\beta,V}f)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le\|(M^{\theta}_{\beta,V}f^0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}+\|(M^{\theta}_{\beta,V}f^1)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 1.6 and Remark 1.1, we obtain

$$\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|(M^{\theta}_{\beta,V}f^0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Because the inequality (1.1) and Lemma 1.5 imply that $u(x,r) \geq Cu(x,2r)$. Therefore, we obtain

$$\begin{aligned}
&\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|(M^\theta_{\beta,V}f^0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\leq& C\frac{(1+rm_V(z))^\alpha}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\leq& C\frac{(1+2rm_V(z))^\alpha}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
\leq& C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.
\end{aligned}$$

For $x\in B(z,r)$, we get

$$\begin{aligned}
M^\theta_{\beta,V}f^1(x)=&\sup_{R>0}\frac{1}{((1+R/\rho(x))^\theta|B(x,R)|)^{1-\frac{\beta}{n}}}\int_{B(x,R)}|f^1(y)|dy\\
\leq& C\sup_{R>2r}\int_{(\mathbb{R}^n\setminus B(z,2r))\cap B(x,R)}\frac{1}{(1+|x-y|/\rho(x))^{\theta(1-\frac{\beta}{n})}}\frac{|f(y)|}{|x-y|^{n-\beta}}dy\\
\leq& C\int_{\mathbb{R}^n\setminus B(z,2r)}\frac{1}{(1+|x-y|/\rho(x))^{\theta(1-\frac{\beta}{n})}}\frac{|f(y)|}{|x-y|^{n-\beta}}dy\\
=&C\sum_{j=1}^\infty\int_{B(z,2^{j+1}r)\setminus B(z,2^jr)}\frac{1}{(1+|x-y|/\rho(x))^{\theta(1-\frac{\beta}{n})}}\frac{|f(y)|}{|x-y|^{n-\beta}}dy.
\end{aligned}$$

Furthermore, for any $j\geq 1$, $x\in B(z,r)$ and $y\in B(z,2^{j+1}r)\setminus B(z,2^jr)$, we note that $|x-y|\geq |y-z|-|x-z|>C2^jr$. Thus we get

$$|M^\theta_{\beta,V}f^1(x)|\leq C\sum_{j=1}^\infty(2^jr)^{\beta-n}\int_{B(z,2^{j+1}r)}\frac{1}{(1+2^jrm_V(x))^{\theta(1-\frac{\beta}{n})}}|f(y)|dy.$$

Using Lemma 1.10, we derive the estimate

$$\begin{aligned}
1+2^jrm_V(x)\geq& 1+2^jr\frac{Cm_V(z)}{(1+|x-z|m_V(z))^{k_0/k_0+1}}\\
\geq& C\frac{1+2^jrm_V(z)}{(1+rm_V(z))^{k_0/(k_0+1)}}\\
\geq& C(1+2^jrm_V(z))^{1/(k_0+1)}. \qquad (2.1)
\end{aligned}$$

Thus, we get that

$$|M^\theta_{\beta,V}f^1(x)|\leq C\sum_{j=1}^\infty(2^jr)^{\beta-n}\int_{B(z,2^{j+1}r)}\frac{1}{(1+2^jrm_V(z))^{\theta(1-\frac{\beta}{n})/(k_0+1)}}|f(y)|dy.$$

Lemma 1.2 ensures that

$$\int_{B(z,2^{j+1}r)}|f(y)|dy\le C\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

for some constant $C>0$.

Subsequently, taking the norm $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n)}$, we have

$$\begin{aligned}\|(M^{\theta}_{\beta,V}f^1)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le& C\sum_{j=1}^{\infty}\frac{(2^jr)^{\beta-n}}{(1+2^jrm_V(z))^{\theta(1-\frac{\beta}{n})/(k_0+1)}}\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\times\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.2}$$

Applying Lemma 1.3 with $B=B(z,2^{j+1}r)$, we obtain

$$\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C\frac{(2^{j+1}r)^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Using the above inequality on (2.2), we obtain

$$\begin{aligned}&\|(M^{\theta}_{\beta,V}f^1)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\\le&C\sum_{j=1}^{\infty}\frac{(2^jr)^{\beta-n}}{(1+2^jrm_V(z))^{\theta(1-\frac{\beta}{n})/(k_0+1)}}\frac{\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}(2^{j+1}r)^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\\le&C\sum_{j=1}^{\infty}\frac{(2^jr)^{\beta}}{(1+2^jrm_V(z))^{\theta(1-\frac{\beta}{n})/(k_0+1)}}\frac{\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

In view of the fact that for any ball B, we have

$$\frac{1}{|B|}\int_B\frac{1}{p(x)}dx-\frac{1}{|B|}\int_B\frac{1}{q(x)}dx=\frac{1}{\bar{p}_B}-\frac{1}{\bar{q}_B}=\frac{\beta}{n}.$$

Lemma 1.4 implies that

$$C_2|B|^{\frac{\beta}{n}}\le\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\le C_1|B|^{\frac{\beta}{n}}\tag{2.3}$$

for some constants $C_1>C_2>0$ independent of B.

Hence, using (2.3) with $B=B(z,2^{j+1}r)$, we have

$$C_2\frac{(2^{j+1}r)^{\beta}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le\frac{1}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Thus, we arrive at the inequality

$$\begin{aligned}&\|(M^{\theta}_{\beta,V}f^1)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \le& C\sum_{j=1}^{\infty}\frac{(1+2^j rm_V(z))^{-\alpha}}{(1+2^j rm_V(z))^{\theta(1-\frac{\beta}{n})/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Since $\theta\ge -\alpha(k_0+1)/(1-\beta/n)$, we get

$$\|(M^{\theta}_{\beta,V}f^1)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.$$

As $u\in\mathbb{W}_{q(\cdot)}$ and $\alpha<0$, we have

$$\begin{aligned}&\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|(M^{\theta}_{\beta,V}f^1))\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C(1+rm_V(z))^{\alpha}\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}\le C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, we have

$$\|M^{\theta}_{\beta,V}f\|_{M^{q(\cdot),u}_{\alpha,V}}\le C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)},$$

and hence the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.3. Without loss of generality, we may assume that $\alpha<0$. Let $f\in\mathcal{M}_{p(\cdot),u}$. For any $z\in\mathbb{R}^n$ and $r>0$, we write

$$f(x)=f_0(x)+\sum_{i=1}^{\infty}f_i(x),$$

where $f_0=f\chi_{B(z,2r)}$, $f_i=f\chi_{B(z,2^{i+1}r)\setminus B(z,2^i r)}$ for $i\ge 1$. Hence, we have

$$\|(\mu^L_j f)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le\|(\mu^L_j f_0)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\sum_{i=1}^{\infty}\|(\mu^L_j f_i)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 1.8, we obtain

$$\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|(\mu^L_j f_0)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Because the inequality (1.1) and Lemma 1.5 imply that $u(x,r)\ge Cu(x,2r)$. Therefore, we obtain

$$\begin{aligned}&\frac{(1+rm_V(z))^{\alpha}}{u(z,r)}\|(\mu^L_j f_0)\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C\frac{(1+rm_V(z))^{\alpha}}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C\frac{(1+2rm_V(z))^{\alpha}}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Furthermore, for any $i\geq 1$, $x\in B(z,r)$ and $y\in B(z,2^{i+1}r)\backslash B(z,2^i r)$, we note that $|x-y|\geq |y-z|-|x-z|>C2^i r$. By Lemma 1.9 and Minkowski's inequality, we have

$$\begin{aligned}|(\mu_j^L f_i)(x)|\leq& C\Big(\int_0^\infty\Big|\int_{|x-y|\leq t}K_j^L(x,y)f_i(y)dy\Big|^2\frac{dt}{t^3}\Big)^{\frac{1}{2}}\\ \leq& C\int_{\mathbb{R}^n}\frac{|f_i(y)|}{(1+|x-y|m_V(x))^N|x-y|^{n-1}}\Big(\int_{|x-y|\leq t}\frac{dt}{t^3}\Big)^{\frac{1}{2}}dy\\ \leq& C\int_{\mathbb{R}^n}\frac{|f_i(y)|}{(1+|x-y|m_V(x))^N|x-y|^n}dy\\ \leq& C(2^i r)^{-n}\int_{B(z,2^{i+1}r)}\frac{1}{(1+2^i r m_V(x))^N}|f(y)|dy.\end{aligned}$$

Using Lemma 1.2 and the inequality (2.1), we get that

$$\begin{aligned}|(\mu_j^L f_i)(x)|\leq& C(2^i r)^{-n}\int_{B(z,2^{j+1}r)}\frac{1}{(1+2^i r m_V(z))^{N/(k_0+1)}}|f(y)|dy\\ \leq& C\frac{(2^i r)^{-n}}{(1+2^i r m_V(z))^{N/(k_0+1)}}\|f\chi_{B(z,2^{i+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{i+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Subsequently, taking the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, we have

$$\begin{aligned}\|(\mu_j^L f_i)\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq& C\frac{(2^i r)^{-n}}{(1+2^i r m_V(z))^{N/(k_0+1)}}\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\times\|\chi_{B(z,2^{i+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{i+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.4}$$

Using Lemma 1.3 on (2.4), we obtain

$$\begin{aligned}&\|(\mu_j^L f_i)\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \leq& C\frac{1}{(1+2^j r m_V(z))^{N/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{i+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{i+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ =& C\frac{1}{(1+2^i r m_V(z))^{N/(k_0+1)}}\frac{u(z,2^{i+1}r)}{u(z,2^{i+1}r)}\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{i+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{i+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus, we arrive at the inequality

$$\begin{aligned}&\|(\mu_j^L f_i)\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \leq& C\frac{(1+2^i r m_V(z))^{-\alpha}}{(1+2^i r m_V(z))^{N/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{i+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(z,2^{i+1}r)\|f\|_{M_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned}$$

Taking $N=([-\alpha]+1)(k_0+1)$, we obtain

$$\|(\mu_i^L f_i)\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{i+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(z,2^{i+1}r)\|f\|_{M_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)}.$$

As $u \in \mathbb{W}_{p(\cdot)}$ and $\alpha < 0$, we have

$$\begin{aligned}&\frac{(1+rm_V(z))^\alpha}{u(z,r)}\sum_{i=1}^{\infty}\|(\mu_j^L f_i)\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le& C(1+rm_V(z))^\alpha\|f\|_{M_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)} \le C\|f\|_{M_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, we have

$$\|\mu_j^L f\|_{M_{\alpha,V}^{p(\cdot),u}} \le C\|f\|_{M_{\alpha,V}^{p(\cdot),u}(\mathbb{R}^n)},$$

and hence the proof of Theorem 1.3 is complete.

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