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On Growth of Polynomials with Restricted Zeros

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z|<k$, $k\geq 1$. It is known that for each $0\leq s<n$ and $1\leq R\leq k$,

$$M(P^{(s)},R)\leq\Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{1+k}\Big)^n M(P,1).$$

In this paper, we obtain certain extensions and refinements of this inequality by involving binomial coefficients and some of the coefficients of the polynomial $P(z)$.

Key Words: Polynomial, maximum modulus princple, zeros.

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1 Introduction and statement of results

Let P_n be the class of polynomials

$$P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$$

of degree n, z being a complex variable and $P^{(s)}(z)$ be its s^{th} derivative. For $P\in P_n$, let $M(P,R)=\max_{|z|=R}|P(z)|$. It is well known that

$$M(P',1)\leq nM(P,1), \tag{1.1}$$

and

$$M(P,R)\leq R^nM(P,1),\quad R\geq 1. \tag{1.2}$$

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The inequality (1.1) is a famous result of S. Bernstein (for reference, see [9]) whereas the inequality (1.2) is a simple consequence of Maximum Modulus Principle (see [8]). It was shown by Ankeny and Rivlin [1] that if $P\in P_n$ and $P(z)\neq 0$ in $|z|<1$, then (1.2) can be replaced by

$$M(P,R)\leq\Big(\frac{R^n+1}{2}\Big)(P,1),\quad R\geq 1. \tag{1.3}$$

Recently, Jain [5] obtained a generalization of (1.3) by considering polynomials with no zeros in $|z|<k$, $k\geq 1$ and simultaneously have taken into consideration the s^{th} derivative of the polynomial, $(0\leq s<n)$, instead of the polynomial itself. More precisely, he proved the following result.

Theorem 1.1. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0\leq s<n$,*

$$M\big(P^{(s)},R\big)\leq\frac{1}{2}\Big\{\frac{d^{(s)}}{dR^{(s)}}(R^n+k^n)\Big\}\Big(\frac{2}{1+k}\Big)^nM(P,1)\quad for\ \ R\geq k, \tag{1.4}$$

and

$$M\big(P^{(s)},R\big)\leq\Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{1+k}\Big)^nM(P,1)\quad for\ \ 1\leq R\leq k. \tag{1.5}$$

Equality holds in (1.4) *(with $k=1$ and $s=0$) for $P(z)=z^n+1$ and equality holds in* (1.5) *(with $s=1$) for $P(z)=(z+k)^n$.*

In this paper, we obtain certain extensions and refinements of the inequality (1.5) of the above theorem by involving binomial coefficients and some of the coefficients of polynomial $P(z)$. More precisely, we prove

Theorem 1.2. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\leq s<n$ and $0<r\leq R\leq k$, we have*

$$\begin{aligned}M\big(P^{(s)},R\big)\leq&\Big\{\frac{c(n,s)R+|\frac{a_s}{a_0}|k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+|\frac{a_s}{a_0}|\big(k^{s+1}R^s+Rk^{2s}\big)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\\&\times\Big\{\exp\Big(n\int_r^R\frac{t+\frac{1}{n}|\frac{a_1}{a_0}|k^2}{t^2+k^2+\frac{2k^2}{n}|\frac{a_1}{a_0}|t}dt\Big)\Big\}M(P,r).\end{aligned} \tag{1.6}$$

The result is best possible (with $s=1$) and equality in (1.6) *holds for $P(z)=(z+k)^n$.*

Remark 1.1. Since if $P(z)\neq 0$ in $|z|<k$, $k>0$, then by Lemma 2.5 (stated in Section 2), we have for $0\leq s<n$,

$$\frac{1}{c(n,s)}\Big|\frac{a_s}{a_0}\Big|k^s\leq 1, \tag{1.7}$$

which can also be taken as equivalent to

$$\frac{c(n,s)t^{s+1}+\left|\frac{a_s}{a_0}\right|k^{s+1}t^s}{c(n,s)\left(k^{s+1}+t^{s+1}\right)+\left|\frac{a_s}{a_0}\right|\left(k^{s+1}t^s+tk^{2s}\right)}\le\frac{t^s}{t^s+k^s}\quad\text{for }\ 0<t\le k. \tag{1.8}$$

Since $R\le k$, if we take $t=R$ in (1.8), we get

$$\frac{c(n,s)R+\left|\frac{a_s}{a_0}\right|k^{s+1}}{c(n,s)\left(k^{s+1}+R^{s+1}\right)+\left|\frac{a_s}{a_0}\right|\left(k^{s+1}R^s+Rk^{2s}\right)}\le\frac{1}{R^s+k^s}. \tag{1.9}$$

Also

$$\exp\Big(n\int_r^R\frac{t+\frac{1}{n}\left|\frac{a_1}{a_0}\right|k^2}{t^2+k^2+\frac{2k^2}{n}\left|\frac{a_1}{a_0}\right|t}dt\Big)=\Big(\frac{R^2+k^2+\frac{2k^2}{n}\left|\frac{a_1}{a_0}\right|R}{r^2+k^2+\frac{2k^2}{n}\left|\frac{a_1}{a_0}\right|r}\Big)^{\frac{n}{2}}=\Big(\frac{R^2+k^2+2kR|\gamma|}{r^2+k^2+2kr|\gamma|}\Big)^{\frac{n}{2}},$$

where $\gamma=ka_1/na_0$, has absolute value ≤ 1, according to inequality (2.4) of Lemma 2.5.

Now as

$$\frac{R^2+k^2+2kR|\gamma|}{r^2+k^2+2kr|\gamma|}$$

is an increasing function of $|\gamma|$ in [0,1], hence

$$\Big(\frac{R^2+k^2+2kR|\gamma|}{r^2+k^2+2kr|\gamma|}\Big)^{\frac{n}{2}}\le\Big(\frac{R+k}{r+k}\Big)^n. \tag{1.10}$$

Combining (1.9) and (1.10), the following result immediately follows from Theorem 1.2.

Corollary 1.1. If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\le s<n$ and $0<r\le R\le k$, we have

$$M(P^{(s)},R)\le\Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{r+k}\Big)^nM(P,r). \tag{1.11}$$

The result is best possible (with $s=1$) and equality in (1.11) holds for $P(z)=(z+k)^n$.

Remark 1.2. For $r=1$, Corollary 1.1 reduces to inequality (1.5).

Next we prove the following theorem which gives an improvement of Corollary 1.1 (for $1\le s<n$), which in turn as a special case provides an improvement and extension of the inequality (1.5). In fact, we prove

Theorem 1.3. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $1\le s<n$ and $0<r\le R\le k$, we have*

$$\begin{aligned}M\big(P^{(s)},R\big)\le&\Big\{\frac{c(n,s)R+\frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)\left(k^{s+1}+R^{s+1}\right)+\frac{|a_s|}{|a_0|-m}\left(k^{s+1}R^s+Rk^{2s}\right)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\\&\times\Big\{\exp\Big(n\int_r^R\frac{t+\frac{1}{n}\frac{|a_1|}{|a_0|-m}k^2}{t^2+k^2+\frac{2k^2}{n}\frac{|a_1|}{|a_0|-m}t}dt\Big)\Big\}\big(M(P,r)-m\big),\end{aligned} \tag{1.12}$$

where $m=\min_{|z|=k}|P(z)|$.

The result is best possible (with $s=1$) and equality in (1.12) holds for $P(z)=(z+k)^n$.

Remark 1.3. Since $P(z)\neq 0$ in $|z|<k$, $k>0$, therefore, for every λ with $|\lambda|<1$, it follows by Rouche's theorem that the polynomial $P(z)-\lambda m$, has no zeros in $|z|<k$, $k>0$ and hence applying inequality (2.4) of Lemma 2.5 (stated in Section 2), we get

$$c(n,s)|a_0-\lambda m|\geq |a_s|k^s. \tag{1.13}$$

If in (1.13), we choose the argument of λ suitably and note $|a_0|>m$, from Lemma 2.3, we get

$$c(n,s)(|a_0|-|\lambda| m)\geq |a_s|k^s. \tag{1.14}$$

If we let $|\lambda|\to 1$ in (1.14), we get

$$\frac{1}{c(n,s)}\frac{|a_s|}{|a_0|-m}k^s\leq 1,$$

which further implies by using the same arguments as in Remark 1.1, that

$$\frac{c(n,s)R+\frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+\frac{|a_s|}{|a_0|-m}\big(k^{s+1}R^s+Rk^{2s}\big)}\leq \frac{1}{R^s+k^s}, \tag{1.15}$$

and

$$\exp\Big(n\int_r^R \frac{t+\frac{1}{n}\frac{|a_1|}{|a_0|-m}k^2}{t^2+k^2+\frac{2k^2}{n}\frac{|a_1|}{|a_0|-m}t}dt\Big)\leq \Big(\frac{R+k}{r+k}\Big)^n. \tag{1.16}$$

Now, using (1.15) and (1.16) in (1.12), the following improvement of Corollary 1.1 (for $1\leq s<n$) and hence of inequality (1.5) immediately follows from Theorem 1.3.

Corollary 1.2. If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $1\leq s<n$ and $0<r\leq R\leq k$, we have

$$M(P^{(s)},R)\leq \Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{r+k}\Big)^n\Big(M(P,r)-m\Big), \tag{1.17}$$

where $m=\min_{|z|=k}|P(z)|$.

The result is best possible (with $s=1$) and equality in (1.17) holds for $P(z)=(z+k)^n$.

Remark 1.4. The inequalities (1.11) and (1.17) were also recently proved by Mir (see [7]).

2 Lemmas

For the proof of these theorems, we need the following lemmas.

The first lemma is due to Aziz and Rather [2].

Lemma 2.1. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $1\leq s<n$, we have*

$$M\big(P^{(s)},1\big)\leq n(n-1)\cdots(n-s+1)\Big\{\frac{c(n,s)+\big|\frac{a_s}{a_0}\big|k^{s+1}}{c(n,s)\big(k^{s+1}+1\big)+\big|\frac{a_s}{a_0}\big|\big(k^{s+1}+k^{2s}\big)}\Big\}M(P,1), \tag{2.1}$$

where $c(n,j)$ are the binomial coefficients defined by

$$c(n,j)=\frac{n!}{j!(n-j)!},\quad 0!=1.$$

From Lemma 2.1, we easily get

Lemma 2.2. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0\leq s<n$, we have*

$$M\big(P^{(s)},1\big)\leq\Big\{\frac{c(n,s)+\big|\frac{a_s}{a_0}\big|k^{s+1}}{c(n,s)\big(k^{s+1}+1\big)+\big|\frac{a_s}{a_0}\big|\big(k^{s+1}+k^{2s}\big)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]M(P,1). \tag{2.2}$$

Lemma 2.3. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then $|P(z)|>m$ for $|z|<k$, and in particular*

$$|a_0|>m,$$

where $m=\min_{|z|=k}|P(z)|$.

The above lemma is due to Gardner, Govil and Musukula [4].

Lemma 2.4. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $1\leq s<n$ we have*

$$\begin{aligned}&M\big(P^{(s)},1\big)\\ \leq&\Big\{\frac{c(n,s)+\frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)\big(k^{s+1}+1\big)+\frac{|a_s|}{|a_0|-m}\big(k^{s+1}+k^{2s}\big)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(M(P,1)-m\Big),\end{aligned} \tag{2.3}$$

where $m=\min_{|z|=k}|P(z)|$.

The above lemma is due to Mir [7].

Lemma 2.5. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\leq s<n$, we have*

$$\frac{1}{c(n,s)}\Big|\frac{a_s}{a_0}\Big|k^s\leq 1. \tag{2.4}$$

Proof. Since

$$P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu\neq 0$$

in $|z|<k$, $k>0$. Let $z_1,z_2,\cdots,z_n$ be the zeros of $P(z)$, then $|z_\nu|\geq k$; $1\leq\nu\leq n$, and we have

$$(-1)\frac{a_{n-1}}{a_n}=\omega(n,1)=\sum z_1, \tag{2.5a}$$

$$(-1)^2\frac{a_{n-2}}{a_n}=\omega(n,2)=\sum z_1z_2,\cdots, \tag{2.5b}$$

$$(-1)^{n-s}\frac{a_s}{a_n}=\omega(n,n-s)=\sum z_1z_2\cdots z_{n-s},\cdots, \tag{2.5c}$$

$$(-1)^n\frac{a_0}{a_n}=\omega(n,n)=z_1z_2\cdots z_n, \tag{2.5d}$$

where $\omega(n,s)$ is the sum of all possible products of $z_1,z_2,\cdots,z_n$ taken s at a time. From (2.5c) and (2.5d), we get

$$\begin{aligned}\Big|\frac{a_s}{a_0}\Big|&=\Big|\frac{a_s}{a_n}\Big|\Big|\frac{a_n}{a_0}\Big|=\Big|\frac{\omega(n,n-s)}{\omega(n,n)}\Big|\\&=\Big|\frac{\sum z_1z_2\cdots z_{n-s}}{z_1z_2\cdots z_n}\Big|=\Big|\sum\frac{1}{z_1z_2\cdots z_s}\Big|\\&\leq\sum\Big|\frac{1}{z_1}\Big|\Big|\frac{1}{z_2}\Big|\cdots\Big|\frac{1}{z_s}\Big|\leq c(n,s)\frac{1}{k^s},\end{aligned}$$

which completes the proof of Lemma 2.5. □

Lemma 2.6. *If*

$$P(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu,\quad 1\leq\mu\leq n,$$

is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0<r\leq R\leq k$, we have

$$M(P,R)\leq\Big\{\exp\Big(n\int_r^R\frac{t^\mu+\frac{\mu}{n}\big|\frac{a_\mu}{a_0}\big|k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\big|\frac{a_\mu}{a_0}\big|(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\Big)\Big\}M(P,r). \tag{2.6}$$

The above result is due to Jain [6].

Lemma 2.7. *If*

$$P(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu,\quad 1\leq\mu\leq n,$$

is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, *then for* $0<r\leq R\leq k$, *we have*

$$\begin{aligned} M(P,R)\leq&\Big\{\exp\Big(n\int_r^R \frac{t^{\mu}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^{\mu}+k^{2\mu}t)}dt\Big)\Big\}M(P,r)\\ &-\Big[\Big\{\exp\Big(n\int_r^R \frac{t^{\mu}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^{\mu}+k^{2\mu}t)}dt\Big)\Big\}-1\Big]m, \end{aligned} \tag{2.7}$$

where $m=\min_{|z|=k}|P(z)|$.

The above lemma is due to Chanam and Dewan [3].

3 Proofs of theorems

Proof of Theorem 1.2. Since $P(z)\neq 0$ in $|z|<k$, $k>0$, the polynomial $P(Rz)$ has no zero in $|z|<k/R$, $k/R\geq 1$. Hence using Lemma 2.2, we have for $0\leq s<n$,

$$\begin{aligned} R^sM(P^{(s)},R)\leq&\Big\{\frac{c(n,s)+\big|\frac{a_s}{a_0}\big|R^s\big(\frac{k}{R}\big)^{s+1}}{c(n,s)\big(1+\big(\frac{k}{R}\big)^{s+1}\big)+\big|\frac{a_s}{a_0}\big|R^s\big(\big(\frac{k}{R}\big)^{s+1}+\big(\frac{k}{R}\big)^{s}\big)}\Big\}\\ &\times\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]M(P,R), \end{aligned}$$

which gives

$$\begin{aligned} M(P^{(s)},R)\leq&\Big\{\frac{c(n,s)R+\Big|\frac{a_s}{a_0}\Big|k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+\Big|\frac{a_s}{a_0}\Big|\big(k^{s+1}R^s+Rk^{2s}\big)}\Big\}\\ &\times\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]M(P,R). \end{aligned} \tag{3.1}$$

Now, if $0<r\leq R\leq k$, then by Lemma 2.6, we obtain for $\mu=1$,

$$M(P,R)\leq\Big\{\exp\Big(n\int_r^R\frac{t+\frac{1}{n}\big|\frac{a_1}{a_0}\big|k^2}{t^2+k^2+\frac{2}{n}\big|\frac{a_1}{a_0}\big|k^2t}dt\Big)\Big\}M(P,r). \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\begin{aligned} M(P^{(s)},R)\leq&\Big\{\frac{c(n,s)R+\big|\frac{a_s}{a_0}\big|k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+\big|\frac{a_s}{a_0}\big|\big(k^{s+1}R^s+Rk^{2s}\big)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\\ &\times\Big\{\exp\Big(n\int_r^R\frac{t+\frac{1}{n}\big|\frac{a_1}{a_0}\big|k^2}{t^2+k^2+\frac{2k^2}{n}\big|\frac{a_1}{a_0}\big|t}dt\Big)\Big\}M(P,r), \end{aligned}$$

which proves Theorem 1.2. □

Proof of Theorem 1.3. Since $P(z)$ has no zero in $|z|<k$, $k>0$, the polynomial $P(Rz)$ has no zero in $|z|<k/R$, $k/R\geq 1$. Hence using Lemma 2.4, we have for $1\leq s<n$,

$$R^sM\big(P^{(s)},R\big)\leq\Bigg\{\frac{c(n,s)+\frac{|a_s|}{|a_0|-m'}R^s\big(\frac{k}{R}\big)^{s+1}}{c(n,s)\big(1+\big(\frac{k}{R}\big)^{s+1}\big)+\frac{|a_s|}{|a_0|-m'}R^s\big(\big(\frac{k}{R}\big)^{s+1}+\big(\frac{k}{R}\big)^{2s}\big)}\Bigg\}$$
$$\times\bigg[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\bigg]\big(M(P,R)-m'\big), \tag{3.3}$$

where $m'=\min_{|z|=\frac{k}{R}}|P(Rz)|=\min_{|z|=k}|P(z)|=m$. This gives

$$M\big(P^{(s)},R\big)\leq\Bigg\{\frac{c(n,s)R+\frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+\frac{|a_s|}{|a_0|-m}\big(k^{s+1}R^s+Rk^{2s}\big)}\Bigg\}$$
$$\times\bigg[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\bigg]\big(M(P,R)-m\big). \tag{3.4}$$

The above inequality when combined with Lemma 2.7 (for $\mu=1$) gives inequality (1.12) and this completes the proof of Theorem 1.3. □

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