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Characterizations of Function Spaces via Boundedness of Commutators

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Abstract. In this paper, we give some creative characterizations of Campanato spaces via the boundedness of commutators associated with the Calderón-Zygmund singular integral operator, fractional integrals and Hardy type operators. Furthermore, we put forward a few problems on the characterizations of Campanato type spaces via the boundedness of commutators.

Key Words: Campanato space, Morrey space, commutator.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

Let $-n/p\le\beta<1$ and $1\le p<\infty$. Then the Campanato space $C^{p,\beta}(\mathbb{R}^n)$ was defined by the norm

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)}=\sup_B\|f\|_{C^{p,\beta}(B)}:=\sup_B\frac{1}{|B|^{\frac{\beta}{n}}}\left(\frac{1}{|B|}\int_B|f-f_B|^pdx\right)^{1/p}, \tag{1.1}$$

where

$$f_B=\frac{1}{|B|}\int_B f(x)dx,$$

B denotes any ball contained in $\mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. Campanato spaces are useful tools in the regularity theory of PDE as a result of their better structures that allow to give an integral characterization of the spaces of Hölder continuous functions, which leads to a generalization of the classical Sobolev embedding theorem, see e.g., [20] and Lu's work (see [23, 24]). It is also well known that $C^{1,1/p-1}$ is the dual

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space of Hardy space $H^p(\mathbb{R}^n)$ $(0<p<1)$ (see [36]). For a recent account of the theory on $C^{p,\beta}(\mathbb{R}^n)$, we refer the reader to [11,21,28,39] and the references therein.

Many important functional spaces are special cases of Campanato space. In fact, we have

$$C^{p,\beta}(\mathbb{R}^n)=\begin{cases} BMO(\mathbb{R}^n), & \beta=0,\\ Lip_\beta(\mathbb{R}^n), & 0<\beta<1,\\ \supset M^{p,\beta}(\mathbb{R}^n), & -n/p\le\beta<0. \end{cases}$$

Here $BMO(\mathbb{R}^n)$ denote the space of Bounded Mean Oscillation functions, $Lip_\beta(\mathbb{R}^n)$, $0<\beta<1$, is the Lipschitz functional space and $M^{p,\beta}(\mathbb{R}^n)$, $-n/p\le\beta<0$, is the Morrey space with the following norm

$$\|f\|_{M^{p,\beta}(\mathbb{R}^n)}=\sup_B\frac{1}{|B|^{\frac{\beta}{n}}}\left(\frac{1}{|B|}\int_B|f(x)|^p dx\right)^{1/p}.$$

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be an integral operator. Then the commutator operator formed by T and b was denoted by

$$[b,T](f):=bTf-T(bf).$$

The function b was also called the symbol function of $[b,T]$. The investigation of the operator $[b,T]$ begin with Calderón-Zygmund pioneering study of the operator T (see [4] and [8]). They found that the theory of commutators play an important role in studying the regularity of solutions to elliptic PDEs of the second order. The well-posedness of solutions to many PDEs can be attributed to the corresponding commutator's boundedness for singular integral operators. However, this topic exceeds the scope of this paper, for more information about this, see for example [3,10,12] and [35]. Especially in [35], the authors simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly by some estimates of commutators which were obtained by Yan in his Ph.D. thesis [38] (see also Lu and Yan's work in [27]). Since $L^\infty(\mathbb{R}^n)\subsetneq BMO(\mathbb{R}^n)$, the boundedness of $[b,T]$ is worse than T (for example, the singularity, see also [30]). Therefore, many authors want to know whether $[b,T]$ shares the similar boundedness with T. Many authors are interested in the study of commutators when the symbol functions b belong to BMO spaces and Lipschitz spaces. For some of this classical works, we refer the reader to [1,18,25] and [29].

In this paper, we focus on some characterizations of Campanato spaces via the boundedness of $[b,T]$ when

T is Calderón-Zygmund singular integral operator;

T is fractional integral;

T is Hardy type operator.

Throughout this paper, for $x_0\in\mathbb{R}^n$, $r>0$ and $\lambda>0$, $B=B(x_0,r)$ denotes the ball centered at x_0 with radius r and $\lambda B=B(x_0,\lambda r)$. C is a constant which may change from line to line.

2 Calderón-Zygmund singular integral operator

We will provide in this section some characterizations of Campanato spaces via the boundedness of $[b,T]$ when T is Calderón-Zygmund singular integral operator. There were many classical works about the characterizations of Campanato spaces by the boundedness of $[b,T]$ on Lebesgue spaces.

When $\beta=0$, Coifman, Rochberg and Weiss [8] gave a characterization of $BMO(\mathbb{R}^n)$ in virtue of commutator $[b,T]$ in 1976:

Theorem 2.1. *Let $1<p<\infty$. Then*

$$b\in BMO(\mathbb{R}^n)\Longleftrightarrow [b,T]:L^p(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n).$$

Ding established another characterization of $BMO(\mathbb{R}^n)$ by $(M^{p,\beta}(\mathbb{R}^n),\ M^{p,\beta}(\mathbb{R}^n))$-boundedness of $[b,T]$ in [9] as:

$$b\in BMO(\mathbb{R}^n)\Longleftrightarrow [b,T]:M^{p,\beta}(\mathbb{R}^n)\longrightarrow M^{p,\beta}(\mathbb{R}^n)\quad \text{for } 1<p<\infty,\quad -n/p\le\beta<0.$$

For $0<\beta<1$, Janson [18] gave a characterization of $Lip_\beta(\mathbb{R}^n)$ by (L^p,L^q)-boundedness of commutator $[b,T]$:

Theorem 2.2. *For $1<p<q<\infty$, $0<\beta<1$ and $1/q=1/p-\beta/n$, one has*

$$b\in Lip_\beta(\mathbb{R}^n)\Longleftrightarrow [b,T]:L^p(\mathbb{R}^n)\longrightarrow L^q(\mathbb{R}^n).$$

Paluszynski gave another characterization of $Lip_\beta(\mathbb{R}^n)$ by $(L^p,\dot{F}^\beta_{p,\infty})$-boundedness of commutator $[b,T]$ in [29] as:

$$b\in Lip_\beta(\mathbb{R}^n)\Longleftrightarrow [b,T]:L^p(\mathbb{R}^n)\longrightarrow \dot{F}^\beta_{p,\infty}(\mathbb{R}^n)\quad \text{for } 0<\beta<1,\quad 1<p<\infty,$$

where $\dot{F}^\beta_{p,\infty}(\mathbb{R}^n)$ is the homogenous Triebel-Lizorkin space with the equivalent norm

$$\|f\|_{\dot{F}^\beta_{p,\infty}(\mathbb{R}^n)}\approx\left\|\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|b-b_Q|\right\|_{L^p}.$$

In 2013, Shi and Lu [31] gave a new characterization of $Lip_\beta(\mathbb{R}^n)$ on Morrey spaces as

$$b\in Lip_\beta(\mathbb{R}^n)\Longleftrightarrow [b,T]:M^{p,\beta}(\mathbb{R}^n)\longrightarrow M^{q,\tilde{\beta}}(\mathbb{R}^n),$$

where $1<p<\infty$, $0<\alpha<1$, $-n/p\le\beta<0$, $1+p\beta/n<p/q$, $1/q=1/p-\alpha/n$ and $\tilde{\beta}=(q-p)/p+q\beta/n$.

For the case $\beta<0$, Shi and Lu [31] gave some creative characterizations of Campanato spaces via the boundedness of $[b,T]$ by some new methods instead of the sharp maximal function theorem as:

Theorem 2.3. *Let* $\max\{1,n/(1-\beta)\}<p<\infty$, $-n/p\leq\beta<0$, $1<p_i<\infty$, $p_1\in\mathbb{N}$, $-n/p_i\leq\beta_i<0$, $i=1,2$, $1/p=1/p_1+1/p_2$ *and* $\beta=\beta_1+\beta_2$. *Then*

$$b\in C^{p_1,\beta_1}(\mathbb{R}^n)\Longleftrightarrow[b,T]:M^{p_2,\beta_2}(\mathbb{R}^n)\longrightarrow C^{p,\beta}(\mathbb{R}^n),$$

where b further satisfies the well known mean value inequality

$$\sup_B|b-b_B|\leq\frac{C}{|B|}\int_B|b-b_B|. \tag{2.1}$$

Eq. (2.1) (which were also well known as the Reverse Hölder classes) was one part of mean value equalities. Besides polynomial functions, the mean value equalities also characterize harmonic functions (see [16]). Our result gains in interest if we realize that it meets the fact that both Morrey spaces and Campanato spaces are closely related to the study of PDEs since solutions to a large class of elliptic PDEs of the second order satisfy (2.1). Therefore, Theorem 2.3 can give characterizations of the space of solutions to some second order elliptic PDE. Take Laplace's equation for example, if b is a solution to the equation

$$\Delta u=0, \tag{2.2}$$

where Δ is the Laplace operator and u is a function defined on bounded domain $\Omega\subset\mathbb{R}^n$. By Theorem 2.1 of [16], b satisfies (2.1). Therefore, if the commutator $[b,T]$ associated to b is bounded from $M^{p_2,\beta_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$, then the space of solutions to (2.2) is Campanato space $C^{p_1,\beta_1}(\mathbb{R}^n)$.

We emphasize that the methods in dealing with $C^{p,\beta}$ when $\beta<0$ are quite different from that of $\beta\geq 0$ and there are essential difficulties to establish the characterizations of Campanato spaces on Morrey spaces when $\beta<0$. Therefore, we set up Theorem 2.3 under the condition that the symbol of the commutator satisfies the mean value inequality. Condition (2.1) in Theorem 2.3 was intrinsic in the proof of the converse characterizations of $C^{p_1,\beta_1}(\mathbb{R}^n)$. Of course, there are essential differences between the ideas in the proof of Theorem 2.3 and that of [18] and [29], where the sharp maximal function theorem were used. It is also worth pointing out that our paper is the first work on the problem of commutators whose symbol belongs to Morrey spaces. Our viewpoints will shed some new lights on characterizations of Campanato spaces via commutators formed by other operators on Morrey spaces.

3 Fractional integral operator

This section devoted to the characterizations of Campanato spaces via the boundedness of $[b,T]$ when T is fractional integral operator. Let $0<\alpha<n$. Then the fractional integral is defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|y-x|^{n-\alpha}}dy.$$

The problem of fractional derivation was an early impetus to study fractional integrals. In addition to their contribution to harmonic analysis, fractional integrals also play an essential role in many fields. The Hardy-Littlewood-Sobolev inequality about the fractional integral I_α is still an indispensable tool in the establishment of time-space estimates for the heat semigroup of nonlinear evolution equations. More recently, the applications to chaos and fractal have become another reason to study fractional integrals.

There are many classical works on the characterizations of BMO spaces and Lipschitz spaces via the boundedness of $[b,I_\alpha]$ on Lebesgue spaces. In 1982, Chanillo [5] gave an early characterization of BMO space by a (L^p,L^q) boundedness of $[b,I_\alpha]$ as:

Theorem 3.1. *For $1<p<q<\infty$ and $1/p-1/q=\alpha/n$, we have*

$$b\in BMO(\mathbb{R}^n)\Longleftrightarrow [b,I_\alpha]:L^p(\mathbb{R}^n)\longrightarrow L^q(\mathbb{R}^n).$$

For $0<\beta<1$, Paluszynski [29] gave two characterizations of $Lip_\beta(\mathbb{R}^n)$ involving the boundedness of $[b,I_\alpha]$ as:

Theorem 3.2. *Let $0<\beta<1$. Then*

$$b\in Lip_\beta(\mathbb{R}^n)\Longleftrightarrow [b,I_\alpha]:L^p(\mathbb{R}^n)\longrightarrow L^r(\mathbb{R}^n)\quad \text{for } 1<p<\infty \quad\text{and}\quad 1/p-1/r=(\beta+\alpha)/n$$

and

$$b\in Lip_\beta(\mathbb{R}^n)\Longleftrightarrow [b,I_\alpha]:L^p(\mathbb{R}^n)\longrightarrow \dot{F}^{\beta}_{q,\infty}(\mathbb{R}^n)\quad \text{for } 1<p<\infty \quad\text{and}\quad 1/q=1/p-\beta/n,$$

where $\dot{F}^{\beta}_{p,\infty}(\mathbb{R}^n)$ is the homogenous Triebel-Lizorkin space.

For the case $\beta<0$, comparing to the rich and significant results about commutators with symbol functions belonging to BMO spaces and Lipschitz spaces, there have been only a few studies on Morrey spaces. A characterization of $C^{p,\beta}(\mathbb{R}^n)$ was provided via the boundedness of $[b,I_\alpha]$ under the assumption that b satisfies the mean value inequality (2.1) in [32].

Theorem 3.3. *Let $-n/p\le\beta<0$, $\max\{1,n/(1-\beta)\}<p<\infty$, $0<\alpha<n$, $-n/p_i\le\beta_i<0$, $1/p=\sum_{i=1}^2 1/p_i$, $\beta=\sum_{i=1}^2\beta_i$, p_1 be even number, $1/p_2=1/q-\alpha/n$, $1<q<n/\alpha$ and $\gamma=\beta_2-\alpha$. Then*

$$b\in C^{p_1,\beta_1}(\mathbb{R}^n)\Longleftrightarrow [b,I_\alpha]:M^{q,\gamma}(\mathbb{R}^n)\longrightarrow C^{p,\beta}(\mathbb{R}^n).$$

4 Hardy type operator

The topic of this section is to give some different characterizations of Campanato spaces via the boundedness of commutators formed by Hardy type operators. Assume that f is a locally integrable function on $\mathbb{R}^n$, then the $n-$dimensional Hardy operator H (cf. [13]) is defined by

$$Hf(x)=\frac{1}{|x|^n}\int_{|y|<|x|}f(y)dy,\quad x\in\mathbb{R}^n\setminus\{0\}.$$

The norm of H on $L^p(\mathbb{R}^n)$ was evaluated in [7] and was found to be equal to that of the one-dimensional Hardy operator. See [17] for more well known results for the one-dimensional Hardy operators and the Hardy type integral inequalities. The dual operator of H is H^*,

$$H^*f(x)=\int_{|y|\geq|x|}\frac{f(y)}{|y|^n}dy,\quad x\in\mathbb{R}^n\setminus\{0\}.$$

H and H^* satisfy

$$\int_{\mathbb{R}^n}g(x)Hf(x)dx=\int_{\mathbb{R}^n}f(x)H^*g(x)dx$$

for a suitable function g. The commutators of H and H^* are defined by

$$[b,H]f=b(Hf)-H(fb)$$

and

$$[b,H^*]f=b(H^*f)-H^*(fb),$$

respectively. For some known works about the boundedness of $[b,H]$ and $[b,H^*]$, see e.g., [19,22].

The fact that both H and H^* are centrosymmetric motivates some people to characterize central function spaces by the boundedness of $[b,H]$ and $[b,H^*]$. As we will see in the following, the function spaces which are characterized by the boundedness of Hardy type operators are different from that of the Calderón-Zygmund singular integral operators even under the same conditions.

Assume that $1\leq p<\infty$ and $-1/p<\lambda<1/n$, then the central Campanato space can be defined by

$$\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)=\Big\{f:\|f\|_{\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)}<\infty\Big\},$$

where

$$\|f\|_{\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)}=:\sup_{r>0}\frac{1}{|B(0,r)|^\lambda}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f-f_{B(0,r)}|^p dx\right)^{1/p}.$$

If $\lambda=0$, then $\dot{\mathfrak{C}}^{p,0}(\mathbb{R}^n)=C\dot{M}O^p(\mathbb{R}^n)$ (bounded central mean oscillation function space) which was introduced by Lu and Yang (cf. [27]) with the equivalent norm

$$\|f\|_{C\dot{M}O^p(\mathbb{R}^n)}=\sup_{r>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,r)|}|f(y)-c|^p dy\right)^{1/p}.$$

For some information as regards the space $CMO(\mathbb{R}^n)$, see [6] for example. The space $C\dot{M}O^p(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$ at the origin. That is $BMO(\mathbb{R}^n)\subset C\dot{M}O^p(\mathbb{R}^n)$ for $1\leq p<\infty$ (cf. [15]). However, they have quite different properties. There is no analysis of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for $C\dot{M}O^p(\mathbb{R}^n)$.

When $0<\lambda<1/p$, $\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)$ is the $\lambda-$central bounded mean oscillation space $C\dot{M}O^{p,\lambda}(\mathbb{R}^n)$ which was first introduced and studied by Alvarez, Partida and Lakey in [2]. For the case $-1/p<\lambda<0$, $\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)\supset \dot{M}^{p,\lambda}(\mathbb{R}^n)$. Here $\dot{M}^{p,\lambda}(\mathbb{R}^n)$ denote the central Morrey space with the following norm

$$\|f\|_{\dot{M}^{p,\lambda}(\mathbb{R}^n)}=\sup_{r>0}\frac{1}{|B(0,r)|^{\lambda}}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f|^p dx\right)^{1/p}.$$

In [14], Fu, Liu, Lu and Wang gave some characterizations of $C\dot{M}O^p(\mathbb{R}^n)$ for $1<p<\infty$, via the boundedness of $[b,H]$ and $[b,H^*]$, on Lebesgue spaces as:

Theorem 4.1. *Let $1<p<\infty$. Then*

$$b\in C\dot{M}O^{\max\{p,p'\}}(\mathbb{R}^n)\Longleftrightarrow [b,H]([b,H^*]):L^p(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n).$$

By different ideas comparing to that of [14], Zhao and Lu [40] characterized the space $C\dot{M}O^{p,\lambda}(\mathbb{R}^n)$ via the boundedness of $[b,H]$ and $[b,H^*]$ on Lebesgue spaces under some more conditions on λ:

Theorem 4.2. *Let $0\le\lambda=\frac{1}{p}-\frac{1}{q}$ and $1<p<\infty$. Then*

$$b\in C\dot{M}O^{\max\{p,p'\},\lambda}(\mathbb{R}^n)\Longleftrightarrow [b,H]([b,H^*]):L^p(\mathbb{R}^n)\longrightarrow L^q(\mathbb{R}^n).$$

However, for the case $-1/p<\lambda<0$, as a concept of highly independent interest, has received nearly zero attention for the characterizations of $\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)$ by the boundedness of the commutator operators of Hardy type, to the best of our knowledge. In 2015, Shi and Lu [33] settled this problem under the assumption that b satisfies the mean value inequality (2.1).

Theorem 4.3. *Let $1<p<\infty$, $-1/p<\lambda<0$, $-1/p_i<\lambda_i<0$, $i=1,2$, $1/p=\sum_{i=1}^2 1/p_i$ and $\lambda=\sum_{i=1}^2\lambda_i$. Then*

$$b\in\dot{\mathfrak{C}}^{p_1,\lambda_1}(\mathbb{R}^n)\Longleftrightarrow [b,H]([b,H^*]):\dot{M}^{p_2,\lambda_2}(\mathbb{R}^n)\longrightarrow \dot{M}^{p,\lambda}(\mathbb{R}^n).$$

Unlike the case $\lambda\ge 0$, more difficulties are caused for the case $\lambda<0$. The methods currently available for $\lambda<0$ depend heavily on the structures of the Hardy type operators and the space $\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)$. Therefore, we need the condition (2.1) to set up Theorem 4.3. This condition is essential to our proof and can not be weakened in the proof of the converse characterization of Theorem 4.3. Under some stronger conditions on λ and p, the following result can be deduced if we drop the assumption that b satisfies the condition (2.1).

Theorem 4.4. *Let $2<p<\infty$ and $-1/(2p)<\lambda<0$. Then*

$$b\in\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)\Leftrightarrow [b,H]([b,H^*]):\dot{M}^{p,\lambda}(\mathbb{R}^n)\longrightarrow \dot{M}^{p,2\lambda}(\mathbb{R}^n).$$

Let f be a locally integrable function in $\mathbb{R}^n$ and $0<\alpha<n$. The n-dimensional fractional Hardy operator can be defined by

$$H_\alpha f(x)=\frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|} f(y)dy, \quad x\in\mathbb{R}^n\setminus\{0\}.$$

The dual operator of H_α is H_α^*,

$$H_\alpha^* f(x)=\int_{|y|\geq|x|}\frac{f(y)}{|y|^{n-\alpha}}dy, \quad x\in\mathbb{R}^n\setminus\{0\}.$$

In [14], the authors gave some characterizations of $C\dot{M}O^p(\mathbb{R}^n)$ $(1<p<\infty)$ via the boundedness of $[b,H_\alpha]$ on both Lebesgue spaces and Herz spaces by the dual method. In [33], we give some characterizations of $\dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)$ with $\lambda<0$ via the boundedness of $[b,H_\alpha]$ and $[b,H_\alpha^*]$ on $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.

Theorem 4.5. *Let p,λ, p_i, λ_i, $i=1,2$, b as in Theorem 4.3, $0<\alpha<\min\{n(1-1/p),n(\lambda_2+1/p_2)\}$ and let $\beta=\lambda_2-\alpha/n$. Then*

$$b\in\dot{\mathcal{C}}^{p_1,\lambda_1}(\mathbb{R}^n)\Leftrightarrow[b,H_\alpha]([b,H_\alpha^*]):\dot{M}^{p_2,\beta}(\mathbb{R}^n)\longrightarrow\dot{M}^{p,\lambda}(\mathbb{R}^n).$$

Theorem 4.6. *Let $2<p<\infty$, $-1/(2p)<\lambda<0$, $0<\alpha<\min\{n(1-1/p),n(\lambda+1/p)\}$ and let $\beta=\lambda-\alpha/n$. Then*

$$b\in\dot{\mathcal{C}}^{p,\lambda}(\mathbb{R}^n)\Leftrightarrow[b,H_\alpha]([b,H_\alpha^*]):\dot{M}^{p,\beta}(\mathbb{R}^n)\longrightarrow\dot{M}^{p,2\lambda}(\mathbb{R}^n).$$

We emphasize that the methods used in [33] are quite different from that of [31] and [32], which depend heavily on the smoothness of the kernel functions of the corresponding integral operators. It is also worth pointing out that the ideas used in [14] and [40], which deal with the case $\lambda\geq 0$, can not be adopted directly to the case $\lambda<0$, especially in the estimates of $[b,H^*]$ and $[b,H_\alpha^*]$. Our theorems provide a natural and intrinsic characterization of central Campanato space via the boundedness of commutator operators on central Morrey space. This work is intended as an attempt to some further characterizations of central function spaces.

5 Further problems

It is well known that the regularity of solutions to some elliptic PDEs with smooth boundary can attribute to the boundedness of corresponding commutators with smooth kernel in some sense. One question arose naturally: What happens when the boundary condition be weakened? The answer to the question need to study the boundedness of commutators with rough kernels, i.e., integrals with homogeneous kernels, which were defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy, \quad 0\leq\alpha<n,$$

where $\Omega(x)$ is homogeneous of degree 0 and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x')=0.$$

Here $d\sigma$ is the normalized Lebesgue measure and $x'=x/|x|$. When $\Omega=1$, $T_{\Omega,\alpha}$ is the same as the fractional integral I_α. If $\alpha=0$, then $T_{\Omega,\alpha}$ becomes a Calderón-Zygmund singular integral operator in the sense of the principal value Cauchy integral.

In 1978, Uchiyama [37] gave a kind of characterizations of $BMO(\mathbb{R}^n)$ space as

Theorem 5.1. *Let $\Omega\in Lip^1(S^{n-1})$, $1<p<\infty$ and $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$.*
(1) If $b\in BMO(\mathbb{R}^n)$, then $[b,T_\Omega]: L^p(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n)$.
(2) Assume that $1<p_0<\infty$, $b\in\bigcup_{q>1}L^q_{loc}(\mathbb{R}^n)$ and $[b,T_\Omega]: L^{p_0}(\mathbb{R}^n)\longrightarrow L^{p_0}(\mathbb{R}^n)$. Then $b\in BMO(\mathbb{R}^n)$.

In [32], we obtained

Theorem 5.2. *Let $\Omega(x)\in C^\infty(S^{n-1})$ is homogeneous of degree 0, $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$ and p, q, γ, α, β, b, p_i, β_i, $i=1,2$, be as in Theorem 3.3. Then the following conditions are equivalent:*
(1) $b\in C^{p_1,\beta_1}(\mathbb{R}^n)$.
(2) $[b,T_{\Omega,\alpha}]$ is a bounded operator from $M^{q,\gamma}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$.
(3) For any ball $B\subset\mathbb{R}^n$ and $m\in\mathbb{N}^+$, there exists constant $C>0$ such that

$$\|[b,T_{\Omega,\alpha}]^m\|_{C^{p,\beta}(B)}\le C|B|^{\beta_1(m-1)/n}\|f\|_{M^{q,\gamma}(B)}.$$

We say a function $\Omega(x')$ on S^{n-1} satisfies a version of L^q-Dini condition if

$$\Omega\in L^q(S^{n-1}),\quad 1\le q<\infty, \tag{5.1a}$$

$$\int_0^1 \frac{w_q(\delta)}{\delta^2}d\delta<\infty, \tag{5.1b}$$

where $w_q(\delta)$ is called the integral continuous modulus of Ω with degree q:

$$w_q(\delta)=\sup_{\|\rho\|<\delta}\left(\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|^q d\sigma(x')\right)^{1/q},$$

here ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\|=\sup|\rho x'-x'|: x'\in S^{n-1}$. When Ω satisfies some size conditions, the kernel of the operator T_Ω has no regularity, and so the operator T_Ω is called rough singular integral operator. In [34], we set up some boundedness of $[b,T_\Omega]$ with b belongs to Morrey space under the L^q-Dini condition.

Theorem 5.3. *Let $-n/p\le\beta<0$, $1<\max\{q',n/(1-\beta)\}<p<\infty$, $1<p_i<\infty$, $-n/p_i\le\beta_i<0$, $1/p=\sum_{i=1}^2 1/p_i$ and $\beta=\sum_{i=1}^2\beta_i$, $i=1,2$. If $b\in M^{p_1,\beta_1}(\mathbb{R}^n)$ and Ω satisfies the L^q-Dini condition, then $[b,T_\Omega]$ is a bounded operator from $M^{p_2,\beta_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$.*

Theorem 5.4. *Let $q'<p<\infty$, $0<\alpha<1$, $-n/p\le\beta<0$, $1/s=1/p-\alpha/n$, $b\in Lip_\alpha$ and Ω satisfies the L^q-Dini condition. Then $[b,T_\Omega]$ is a bounded operator from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{s,\alpha+\beta}(\mathbb{R}^n)$.*

Theorem 5.3 and Theorem 5.4 can be seen as extensions of [31] to the rough kernel case. Unlike $\beta\ge 0$, there are essential difficulties to deal with the case $\beta<0$. Therefore, we have been working under the assumption that Ω satisfies the L^q-Dini conditions (5.1a)-(5.1b) instead of the classical L^q-conditions (with (5.1b) replaced by $\int_0^1\frac{w_q(\delta)}{\delta}d\delta<\infty$ or $\int_0^1\log(\frac{1}{\delta})\frac{w_q(\delta)}{\delta}d\delta<\infty$). In our judgement, this condition cannot be weaken since in the case of $\beta<0$, there need a 1 factor contribution to guarantee the series's convergence instead of a small $\varepsilon(\varepsilon>0)$ factor for the case of $\beta\ge 0$.

Inspired by the result in Section 2-Section 4, some further problems can be considered as follows:

1. Whether the condition on Ω in Theorem 5.1-Theorem 5.4 be weakened?

2. Whether the reverse of Theorem 5.3-Theorem 5.4 is true?

3. Can someone give any characterizations of $\dot{\mathfrak{C}}^{p,\lambda}(\mathbb{R}^n)$ via the following Hardy type operator
$$H_\Omega f(x)=\frac{1}{|x|^n}\int_{|t|<|x|}\Omega(x-t)f(t)dt?$$
Here $\Omega(x)$ is homogeneous of degree 0 and
$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0.$$

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