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Viscoelastic Wave Equation with Logarithmic Nonlinearities in $\mathbb{R}^n$

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Abstract. In any spaces dimension, we use weighted spaces to establish a general decay rate of solution of viscoelastic wave equation with logarithmic nonlinearities. Furthermore, we establish, under convenient hypotheses on g and the initial data, the existence of weak solution associated to the equation.

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1 Introduction

It is well known that from a class of nonlinearities, the logarithmic nonlinearity is distinguished by several interesting physical properties (nuclear physics, optics, and geophysics...). We consider the following semilinear equation with logarithmic nonlinearity

$$u'' - \phi(x)\left(\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)\mathrm{d}s\right) = u\ln|u|^k, \tag{1.1}$$

where $x \in \mathbb{R}^n, t > 0, n \geq 2, k > 1$ and the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (A1). The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems.

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The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.

Eq. (1.1) is equipped by the following initial data.

$$u(0,x)=u_0(x)\in\mathcal{H}(\mathbb{R}^n),\quad u'(0,x)=u_1(x)\in L^2_\rho(\mathbb{R}^n), \tag{1.2}$$

where the weighted spaces $\mathcal{H}$ is given in Definition 2.1 and the density function $\phi(x)>0,\forall x\in\mathbb{R}^n,(\phi(x))^{-1}=\rho(x)$ satisfies

$$\rho:\mathbb{R}^n\to\mathbb{R}^*_+,\quad \rho(x)\in C^{0,\tilde{\gamma}}(\mathbb{R}^n), \tag{1.3}$$

with $\tilde{\gamma}\in(0,1)$ and $\rho\in L^s(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, where $s=2n/(2n-qn+2q)$.

First, the following initial boundary value problem

$$u''-\Delta_x u+\int_0^t g(t-s)\Delta_x u(s)\mathrm{d}s+h(u')=f(u),\quad x\in\Omega,t>0, \tag{1.4}$$

has been studied widely. For example [1–6] and [7], the authors investigated global existence, decay rate and blow-up of the solutions.

Studies in $\mathbb{R}^n$, we quote essentially the results of [8–12]. In [10], authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (1.1), (1.2) with $\rho(x)=1$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In [9], author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [9], was considered in [11], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (1.1), (1.2) without term source, for the case $\rho(x)=1$, in a bounded domain $\Omega\subset\mathbb{R}^n,(n\geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered in [12], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t)\leq -H(g(t)),\ t\geq 0,\quad H(0)=0, \tag{1.5}$$

for a positive function $H\in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0,r],1>r$. This improves the conditions considered in [8] on the relaxation functions

$$g'(t)\leq -\chi(g(t)),\quad \chi(0)=\chi'(0)=0, \tag{1.6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0,k_0]$, $k_0>0$. The goal of the present paper is to establish the existence of a weak solution to the problem (1.1)-(1.2). We obtain also, decay results.

2 Material, assumptions and technical lemmas

We omit the space variable x of $u(x,t),u'(x,t)$ and for simplicity reason denote $u(x,t)=u$ and $u'(x,t)=u'$, when no confusion arises. We denote by

$$|\nabla_x u|^2=\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2, \quad \Delta_x u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}.$$

The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u'=\mathrm{d}u(t)/\mathrm{d}t$ and $u''=\mathrm{d}^2u(t)/\mathrm{d}t^2$.

We recall and make use the following hypothesis on the function g as:

(A1) We assume that the function $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is of class C^1 satisfying:

$$1-\overline{g}=l>0, \quad g(0)=g_0>0, \tag{2.1}$$

where $\overline{g}=\int_0^\infty g(t)\mathrm{d}t$.

(A2) There exists a positive function $H\in C^1(\mathbb{R}^+)$ such that

$$g'(t)+H(g(t))\leq 0, \ \ t\geq 0, \quad H(0)=0, \tag{2.2}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0,r],1>r$.

(A3) According to results in [12], we have

1- We can deduce that there exists $t_1>0$ large enough such that:

$\underline{1)\ \forall t\geq t_1}$: We have $\lim\limits_{s\to+\infty}g(s)=0$, which implies that $\lim\limits_{s\to+\infty}-g'(s)$ cannot be positive, so $\lim\limits_{s\to+\infty}-g'(s)=0$. Then $g(t_1)>0$ and

$$\max\{g(s),-g'(s)\}<\min\{r,H(r),H_0(r)\}, \tag{2.3}$$

where $H_0(t)=H(D(t))$ provided that D is a positive C^1 function, with $D(0)=0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0,r]$ and

$$\int_0^{+\infty}g(s)H_0(-g'(s))\mathrm{d}s<+\infty.$$

$\underline{2)\ \forall t\in[0,t_1]}$: As g is nonincreasing, $g(0)>0$ and $g(t_1)>0$ then $g(t)>0$ and

$$g(0)\geq g(t)\geq g(t_1)>0.$$

Therefore, since H is a positive continuous function, then

$$a\leq H(g(t))\leq b,$$

for some positive constants a and b. Consequently,

$$g'(t) \leq -H(g(t)) \leq -kg(t), \quad k>0,$$

which gives

$$g'(t) \leq -kg(t), \qquad k>0. \tag{2.4}$$

2- Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [13], pages 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0,H_0'(r)),$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0,H_0'(r)),\ B \in (0,r]. \tag{2.5}$$

The space $\mathcal{H}(\mathbb{R}^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the norm $\|u\|^2_{\mathcal{H}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_x u|^2 \mathrm{d}x$. It is defined in the next definition

Definition 2.1 ([4]). *We define the function spaces of our problem and its norm as follows:*

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\} \tag{2.6}$$

and that $\mathcal{H}$ is embedded continuously in $L^{2n/(n-2)}$.

The space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h \mathrm{d}x.$$

For $1<q<\infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q \mathrm{d}x \right)^{1/q}. \tag{2.7}$$

Remark 2.1. The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

The following technical Lemmas will play an important role in the sequel.

Lemma 2.1. ([14, Lemma 1.1]) *For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$v'(t) \int_0^t g(t-s) v(s) \mathrm{d}s = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(t-s) |v(t)-v(s)|^2 \mathrm{d}s$$

$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(s)\mathrm{d}s\right)|v(t)|^2+\frac{1}{2}\int_0^t g'(t-s)|v(t)-v(s)|^2\mathrm{d}s-\frac{1}{2}g(t)|v(t)|^2, \tag{2.8}$$

$$\begin{aligned}&\left|\int_0^t g(t-s)(v(t)-v(s))\mathrm{d}s\right|^2\\ &\quad\le\left(\int_0^t |g(s)|^{2(1-\theta)}\mathrm{d}s\right)\left(\int_0^t |g(t-s)|^{2\theta}|v(t)-v(s)|^2\mathrm{d}s\right).\end{aligned} \tag{2.9}$$

The next Lemma can be easily shown (see [2,15]).

Lemma 2.2. *If ρ satisfies* (1.3)*, then for any $u\in\mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)}\le\|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{2.10}$$

with $s=\frac{2n}{2n-qn+2q}$, $2\le q\le\frac{2n}{n-2}$.

Now, using Lemma 2.2, we give the following Lemma concerning Logarithmic Sobolev inequality.

Lemma 2.3. (see [16–18]) *Let $u\in\mathcal{H}(\mathbb{R}^n)$ be any function and $c_1,c_2>0$ be any numbers. Then*

$$2\int_{\mathbb{R}^n}\rho(x)|u|^2\ln\left(\frac{|u|}{\|u\|^2_{L^2_\rho}}\right)\mathrm{d}x+n(1+c_1)\|u\|^2_{L^2_\rho}\le c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla_x u\|_2^2. \tag{2.11}$$

Definition 2.2. *By the weak solution of* (1.1) *over $[0,T]$ we mean a function*

$$u\in C([0,T],\mathcal{H}(\mathbb{R}^n))\cap C^1([0,T],L^2_\rho(\mathbb{R}^n))\cap C^2([0,T],\mathcal{H}^{-1}(\mathbb{R}^n))$$

with $u'\in L^2([0,T],\mathcal{H}(\mathbb{R}^n))$, such that $u(0)=u_0, u'(0)=u_1$ and for all $v\in\mathcal{H}, t\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k v\mathrm{d}x&=\int_{\mathbb{R}^n}\rho(x)u''v\mathrm{d}x+\int_{\mathbb{R}^n}\nabla_x u\nabla_x v\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^n}\int_0^t g(t-s)\nabla_x u(s)\mathrm{d}s\nabla_x v\mathrm{d}x.\end{aligned} \tag{2.12}$$

Multiplying the equation (1.1) by $\rho(x)u'$, and integrating by parts over $\mathbb{R}^n$, we will obtain the energy of u at time t as

$$\begin{aligned}E(t)=&\frac{1}{2}\Bigg(\|u'\|^2_{L^2_\rho}+\Big(1-\int_0^t g(s)\mathrm{d}s\Big)\|\nabla_x u\|_2^2+(g\circ\nabla_x u)\\ &-\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k\mathrm{d}x\Bigg)+\frac{k}{4}\|u\|^2_{L^2_\rho},\end{aligned} \tag{2.13}$$

and the following energy functional law holds:

$$E'(t)=\frac{1}{2}(g'\circ\nabla_x u)(t)-\frac{1}{2}g(t)\|\nabla_x u(t)\|_2^2, \qquad \text{for all } t\geq 0, \tag{2.14}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g\circ\nabla_x u)(t)=\int_0^t g(t-\tau)\|\nabla_x u(t)-\nabla_x u(\tau)\|_2^2 \mathrm{d}\tau, \tag{2.15}$$

for $u(t)\in\mathcal{H}(\mathbb{R}^n)$, $t\geq 0$.

3 Global existence in time

According to logarithmic Sobolev inequality and by using Galerkin method combined with compact theorem, similar to the proof in ([16,18–21]), we have the following result.

Theorem 3.1. (Local existence) *Let $u_0(x)\in\mathcal{H}(\mathbb{R}^n)$,$u_1(x)\in L^2_\rho(\mathbb{R}^n)$ be given. Then, under hypothesis* (A1), (A2) *and* (1.3), *the problem* (1.1) *has a unique local solution*

$$u\in C([0,T],\mathcal{H}(\mathbb{R}^n))\cap C^1([0,T],L^2_\rho(\mathbb{R}^n)).$$

Now, we introduce two functionals

$$J(t)=\frac{1}{2}\left(\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla_x u\|_2^2+(g\circ\nabla_x u)-\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k\mathrm{d}x\right)+\frac{k}{4}\|u\|^2_{L^2_\rho}, \tag{3.1}$$

$$I(t)=\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla_x u\|_2^2+(g\circ\nabla_x u)-\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k\mathrm{d}x. \tag{3.2}$$

Then,

$$J(t)=\frac{1}{2}I(t)+\frac{k}{4}\|u\|^2_{L^2_\rho}. \tag{3.3}$$

As in ([22]) to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows:

$$W=\left\{u\in\mathcal{H}(\mathbb{R}^n):I(t)>0,J(t)<d\right\}\cup\{0\}. \tag{3.4}$$

Remark 3.1. We notice that the mountain pass level d given in (3.4) defined by

$$d=\inf\left\{\sup_{u\in\mathcal{H}(\mathbb{R}^n)\backslash\{0\}\mu\geq 0} J(\mu u)\right\}. \tag{3.5}$$

Also, by introducing the so called "Nehari manifold"

$$\mathcal{N}=\{u\in\mathcal{H}(\mathbb{R}^n)\backslash\{0\}: I(t)=0\}.$$

Similar to results in [23], it is readily seen that the potential depth d is also characterized by

$$d=\inf_{u\in\mathcal{N}} J(t). \tag{3.6}$$

This characterization of d shows that

$$\operatorname{dist}(0,\mathcal{N})=\min_{u\in\mathcal{N}}\|u\|_{\mathcal{H}(\mathbb{R}^n)}. \tag{3.7}$$

By the fact that (2.14), we will prove the invariance of the set W. That is if for some $t_0>0$ if $u(t_0)\in W$, then $u(t)\in W$, $\forall t\geq t_0$, let us beginning by giving the existence Lemma of the potential depth (See [16, Lemma 2.4]).

Lemma 3.1. *d is positive constant.*

Lemma 3.2. *Let $u\in\mathcal{H}(\mathbb{R}^n)$ and $\beta=e^{\frac{1}{2}n(1+c_1)}$. If $0<\|u\|^2_{L^2_\rho}<\beta$, then $I(t)>0$; if $I(t)=0,\|u\|_2^2\neq 0$, then $\|u\|^2_{L^2_\rho}>\beta$.*

Proof. By (A1), (3.2) and Lemma 2.3, we have

$$\begin{aligned} I(t)=&\Big(1-\int_0^t g(s)\mathrm{d}s\Big)\|\nabla_x u\|_2^2+(g\circ\nabla_x u)-\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k\mathrm{d}x\\ \geq& \, l\|\nabla_x u\|_2^2-k\int_{\mathbb{R}^n}\rho(x)u^2\Big(\ln\frac{|u|}{\|u\|^2_{L^2_\rho}}+\ln\|u\|^2_{L^2_\rho}\Big)\mathrm{d}x\\ \geq& \left(l-\frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}\right)\|\nabla_x u\|_2^2+\frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho}-k\|u\|^2_{L^2_\rho}\ln\|u\|^2_{L^2_\rho}. \end{aligned}$$

Choosing c_2 such that $l>\frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}$, then

$$I(t)\geq k\Big(\frac{1}{2}n(1+c_1)-\ln\|u\|^2_{L^2_\rho}\Big)\|u\|^2_{L^2_\rho}.$$

Therefore, if $0<\|u\|^2_{L^2_\rho}<\beta$, then $I(t)>0$; if $I(t)=0,\|u\|_2^2\neq 0$, we have $\beta<\|u\|^2_{L^2_\rho}$ then, $\|u\|^2_{L^2_\rho}>\beta$. ☐

Theorem 3.2. (Global Existence) *Let $u_0(x)\in\mathcal{H}(\mathbb{R}^n),u_1(x)\in L^2_\rho(\mathbb{R}^n)$ and $0<E(0)<d,I(0)>0$. Then, under hypothesis* (A1), (A2) *and conditions* (1.3), *problem* (1.1) *has a global solution in time.*

Proof. From the definition of energy for the weak solution and by (2.14), we have

$$\frac{1}{2}\|u'\|^2_{L^2_\rho}+J(t)\leq\frac{1}{2}\|u_1\|^2_{L^2_\rho}+J(0),\quad \forall t\in[0,T_{\max}), \tag{3.8}$$

where $T_{\max}$ is the maximal existence time of weak solution of u. Then, by the definition of the stable set and using Lemma 3.2, we have $u\in W$, $\forall t\in[0,T_{\max})$. ☐

4 Decay estimates

We apply the multiplier techniques to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions. For this purpose, we introduce the functionals

$$\psi_1(t)=\int_{\mathbb{R}^n}\rho(x)uu'\mathrm{d}x. \tag{4.1}$$

Lemma 4.1. *Under the hypothesis* (A1) *and* (A2), *the functional* ψ_1 *satisfies, along the solution of* (1.1), (1.2)

$$\begin{aligned}\psi_1'(t)\le&\|u'\|_{L^2_\rho}^2+\frac{(1-l)}{4\sigma}(g\circ\nabla_x u)\\&+\Big[\Big(\sigma+\frac{kc_2}{2\pi}\|\rho\|^2_{L^2}-l\Big)+k\|\rho\|^2_{L^2}\Big(\ln\|u\|^2_{L^2_\rho}-\frac{1}{2}n(1+c_1)\Big)\Big]\|\nabla u\|_2^2.\end{aligned}$$

Proof. From (4.1), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}\psi_1'(t)&=\int_{\mathbb{R}^n}\rho(x)|u'|^2\mathrm{d}x+\int_{\mathbb{R}^n}\rho(x)uu''\mathrm{d}x\\&=\int_{\mathbb{R}^n}\Big(\rho(x)|u'|^2+u\Delta_x u-u\int_0^t g(t-s)\Delta_x u(s,x)\mathrm{d}s\Big)\mathrm{d}x\\&\quad+\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k\mathrm{d}x\\&\le\|u'\|^2_{L^2_\rho(\mathbb{R}^n)}-l\|\nabla_x u\|_2^2+k\int_{\mathbb{R}^n}\rho(x)u^2\Big(\ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)+\ln\|u\|^2_{L^2_\rho}\Big)\mathrm{d}x\\&\quad+\int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(s)-\nabla_x u(t))\mathrm{d}s\mathrm{d}x.\end{aligned}$$

We have by using the Logarithmic Sobolev inequality in Lemma 2.3 and generalized version in Lemma 2.2, we obtain

$$\begin{aligned}\psi_1'(t)\le&\|u'\|^2_{L^2_\rho}+\Big(\frac{kc_2}{2\pi}\|\rho\|^2_{L^2}-l\Big)\|\nabla_x u\|_2^2+k\|u\|^2_{L^2_\rho}\ln\|u\|^2_{L^2_\rho}+\sigma\|\nabla_x u\|_2^2\\&+\frac{1}{4\sigma}\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)|\nabla_x u(s)-\nabla_x u(t)|\mathrm{d}s\Big)^2\mathrm{d}x-\frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho}\\\le&\|u'\|^2_{L^2_\rho}+\Big(\sigma+\frac{kc_2}{2\pi}\|\rho\|^2_{L^2}-l\Big)\|\nabla_x u\|_2^2\\&+\frac{(1-l)}{4\sigma}(g\circ\nabla_x u)+k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{1}{2}n(1+c_1)\Big)\|u\|^2_{L^2_\rho}.\end{aligned}$$

Then

$$\begin{aligned}\psi_1'(t) \le & \|u'\|^2_{L^2_\rho} + \frac{(1-l)}{4\sigma}(g\circ\nabla_x u) \\ & + \left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right) + k\|\rho\|^2_{L^2}\left(\ln\|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\right]\|\nabla u\|^2_2.\end{aligned}$$

This completes the proof of the lemma. □

The existence of the memory term forces us to make second modification of the associate energy functional. Set

$$\psi_2(t) = -\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \tag{4.2}$$

Lemma 4.2. *Under the hypothesis* (A1) *and* (A2)*, the functional* ψ_2 *satisfies, along the solution of* (1.1), (1.2)*, for any* $\sigma \in (0,1)$

$$\begin{aligned}\psi_2'(t) \le & \left[\sigma + k\left(\sigma\frac{c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\right)\right]\|\nabla_x u\|^2_2 \\ & + c_\sigma\left(1 + \left(k\frac{c_2}{2\pi}+1\right)\|\rho\|^2_{L^2}\right)(g\circ\nabla_x u) - c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u) \\ & + \left(\sigma - \int_0^t g(s)\mathrm{d}s\right)\|u'\|^2_{L^2_\rho}.\end{aligned}$$

Proof. Exploiting Eqs. (1.1) and (4.2) to get

$$\begin{aligned}\psi_2'(t) = & -\int_{\mathbb{R}^n}\rho(x)u''\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ & -\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \int_0^t g(s)\mathrm{d}s\|u'\|^2_{L^2_\rho} \\ = & \int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))\mathrm{d}s\mathrm{d}x \\ & -\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ & -\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla_x u(s,x)\mathrm{d}s\right)\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))\mathrm{d}s\right)\mathrm{d}x \\ & -\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \int_0^t g(s)\mathrm{d}s\|u'\|^2_{L^2_\rho}.\end{aligned}$$

By (A1), we have

$$\psi_2'(t) = \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))\mathrm{d}s\mathrm{d}x$$

$$\begin{aligned}
&+\int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))\mathrm{d}s\Big)^2\mathrm{d}x\\
&-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
&-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
&-\int_0^t g(s)\mathrm{d}s\|u'\|^2_{L^2_\rho}+c(g\circ\nabla_x u)(t).
\end{aligned}$$

By Hölder's and Young's inequalities and Lemma 2.2, we estimate

$$\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
\le&\Big(\int_{\mathbb{R}^n}\rho(x)|u'|^2\mathrm{d}x\Big)^{1/2}\times\Big(\int_{\mathbb{R}^n}\rho(x)\Big|\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\Big|^2\Big)^{1/2}\\
\le&\sigma\|u'\|^2_{L^2_\rho}+c_\sigma\Big\|\int_0^t -g'(t-s)(u(t)-u(s))\mathrm{d}s\Big\|^2_{L^2_\rho}\\
\le&\sigma\|u'\|^2_{L^2_\rho}-c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u)(t),
\end{aligned}$$

and

$$\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\le\sigma\|u'\|^2_{L^2_\rho}+c_\sigma\|\rho\|^2_{L^2}(g\circ\nabla_x u)(t).$$

Moreover, by Lemma 2.2 and Lemma 2.3 and conditions in Lemma 3.2, we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)\ln|u|^k u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
\le&k\int_{\mathbb{R}^n}\rho(x)\Big(\ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)+\ln\|u\|^2_{L^2_\rho}\Big)u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\
\le&k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|u\|^2_{L^2_\rho}+k\frac{c_2}{2\pi}\Big\|u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\Big\|^2_{L^2_\rho}\\
\le&k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2}\|\nabla_x u\|^2_2\\
&\qquad+k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}\Big\|\nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\Big\|^2_{L^2_\rho}\\
\le&k\Big(\sigma\frac{c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2}\|\nabla_x u\|^2_2+c_\sigma k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}(g\circ\nabla_x u).
\end{aligned}$$

Using Young's and Poincare's inequalities and Lemma 2.1 for $\theta=1/2$, we obtain

$$\begin{aligned}\psi_2'(t)\le&\left[\sigma+k\Big(\sigma\frac{c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\right]\|\nabla_x u\|_2^2\\&+c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|^2_{L^2})(g\circ\nabla_x u)-c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u)\\&+\left(\sigma-\int_0^t g(s)\mathrm{d}s\right)\|u'\|^2_{L^2_\rho}.\end{aligned}$$

This completes the proof of the lemma. □

Now, let us define

$$L(t)=\xi_1E(t)+\psi_1(t)+\xi_2\psi_2(t), \tag{4.3}$$

for $\xi_1,\xi_2>1$. We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for $\xi_1,\xi_2>1$, we have

$$\beta_1L(t)\le E(t)\le\beta_2L(t), \tag{4.4}$$

holds for two positive constants β_1 and β_2.

Lemma 4.3. *For $\xi_1,\xi_2>1$, we have*

$$L(t)\sim E(t). \tag{4.5}$$

Proof. By (4.3) we have

$$\begin{aligned}&|L(t)-\xi_1E(t)|\le|\psi_1(t)|+\xi_2|\psi_2(t)|\\&\le\int_{\mathbb{R}^n}|\rho(x)uu'|\mathrm{d}x+\xi_2\int_{\mathbb{R}^n}\left|\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right|\mathrm{d}x.\end{aligned}$$

Thanks to Hölder and Young's inequalities, we have by using Lemma 2.2

$$\begin{aligned}\int_{\mathbb{R}^n}|\rho(x)uu'|\mathrm{d}x\le&\left(\int_{\mathbb{R}^n}\rho(x)|u|^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^n}\rho(x)|u'|^2\mathrm{d}x\right)^{1/2}\\\le&\frac12\left(\int_{\mathbb{R}^n}\rho(x)|u|^2\mathrm{d}x\right)+\frac12\left(\int_{\mathbb{R}^n}\rho(x)|u'|^2\mathrm{d}x\right)\\\le&c\|u'\|^2_{L^2_\rho}+c\|\rho\|^2_{L^2}\|\nabla_x u\|_2^2,\end{aligned}\tag{4.6}$$

$$\int_{\mathbb{R}^n}\left|\left(\rho(x)^{\frac12}u'\right)\left(\rho(x)^{\frac12}\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)\right|\mathrm{d}x$$

$$\begin{aligned}
&\le \Big(\int_{\mathbb{R}^n}\rho(x)|u'|^2\mathrm{d}x\Big)^{1/2}\times\Big(\int_{\mathbb{R}^n}\rho(x)\Big|\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\Big|^2\mathrm{d}x\Big)^{1/2}\\
&\le \frac{1}{2}\|u'\|^2_{L^2_\rho}+\frac{1}{2}\Big\|\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\Big\|^2_{L^2_\rho}\\
&\le \frac{1}{2}\|u'\|^2_{L^2_\rho}+\frac{1}{2}\|\rho\|^2_{L^2}(g\circ\nabla_x u).
\end{aligned}\tag{4.7}$$

Then,

$$|L(t)-\xi_1E(t)|\le cE(t).$$

Therefore, we can choose ξ_1 so that (4.5) is satisfied. ☐

Lemma 4.4. *For all $t\ge t_1>0$, we have*

$$\begin{aligned}
&\int_{t_1}^t(g\circ\nabla_x u)(s)\mathrm{d}s\\
\le& H_0^{-1}\Big(-\int_{t_1}^t H_0(-g'(s))g'(s)\int_{\mathbb{R}^n}g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\Big),
\end{aligned}$$

where H_0 introduced in (2.3).

Proof. By (2.14) and (A3), we have for all $t\ge t_1$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{t_1}g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2\mathrm{d}s\mathrm{d}x\\
\le& -\frac{1}{k}\int_{\mathbb{R}^n}\int_0^{t_1}g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2\mathrm{d}s\mathrm{d}x\le -cE'(t).
\end{aligned}$$

Now, we define

$$I(t)=\int_{t_1}^t H_0(-g'(s))(g\circ\nabla_x u)(t)\mathrm{d}s.\tag{4.8}$$

Since $\int_0^{+\infty}H_0(-g'(s))g(s)\mathrm{d}s<+\infty$, from (2.14) we have

$$\begin{aligned}
I(t)=&\int_{t_1}^t H_0(-g'(s))\int_{\mathbb{R}^n}g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\
\le& 2\int_{t_1}^t H_0(-g'(s))g(s)\int_{\mathbb{R}^n}|\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\
\le& cE(0)\int_{t_1}^t H_0(-g'(s))g(s)\mathrm{d}s<1.
\end{aligned}\tag{4.9}$$

We define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t)=-\int_{t_1}^{t} H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 \mathrm{d}x\mathrm{d}s. \tag{4.10}$$

From (A1)-(A3), we get

$$H_0(-g'(s))g(s)\le H_0(H(g(s)))g(s)=D(g(s))g(s)\le k_0,$$

for some positive constant k_0. Then, for all $t\ge t_1$

$$\begin{aligned}\lambda(t)&\le -k_0\int_{t_1}^{t} g'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\ &\le -k_0\int_{t_1}^{t} g'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\ &\le -cE(0)\int_{t_1}^{t} g'(s)\mathrm{d}s\le cE(0)g(t_1)<\min\{r,H(r),H_0(r)\}.\end{aligned} \tag{4.11}$$

Using the properties of H_0 (strictly convex in $(0,r],H_0(0)=0$), then for $x\in(0,r]$, $\theta\in[0,1]$,

$$H_0(\theta x)\le \theta H_0(x).$$

Using hypothesis in (A3), (4.9), (4.11) and Jensen's inequality leads to

$$\begin{aligned}\lambda(t)&=\frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0[H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\ &\ge\frac{1}{I(t)}\int_{t_1}^{t} H_0[I(t)H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\\ &\ge H_0\left(\frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0^{-1}(-g'(s))H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\right)\\ &\ge H_0\left(\int_{t_1}^{t}\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\right),\end{aligned}$$

which implies

$$\int_{t_1}^{t}\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2\mathrm{d}x\mathrm{d}s\le H_0^{-1}(\lambda(t)).$$

This completes the proof of the lemma. □

Our next main result reads as follows.

Theorem 4.1. *Let* $(u_0,u_1)\in\mathcal{H}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ *and suppose that* (A1)-(A2) *hold. Then there exist positive constants* $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ *such that the energy of solution given by* (1.1), (1.2) *satisfies,*

$$E(t)\le \alpha_3 H_1^{-1}(\alpha_1 t+\alpha_2),\qquad \textit{for all}\ \ t\ge 0,$$

where

$$H_1(t)=\int_t^1 (sH_0'(\alpha_0 s))^{-1}\mathrm{d}s.$$

Proof. From (2.14), results of Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} L'(t)&=\xi_1E'(t)+\psi_1'(t)+\xi_2\psi_2'(t)\\ &\le\left(\frac{1}{2}\xi_1-c_\sigma\|\rho\|_{L^2}^2\xi_2\right)(g'\circ\nabla_x u)+M_0(g\circ\nabla_x u)-M_1\|u'\|_{L^2_\rho}^2-M_2\|\nabla_x u\|_2^2, \end{aligned}$$

where

$$\begin{aligned} M_0&=\left(\xi_2c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|_{L^2}^2)+\frac{(1-l)}{4\sigma}\right)>0,\\ M_1&=\left(\xi_2\left(\int_0^{t_1}g(s)\mathrm{d}s-\sigma\right)-1\right),\\ M_2&=\frac{1}{2}\xi_1 g(t_1)-\left[\left(\sigma+\frac{kc_2}{2\pi}\|\rho\|_{L^2}^2-l\right)+k\|\rho\|_{L^2}^2\left(\ln\|u\|_{L^2_\rho}^2-\frac{1}{2}n(1+c_1)\right)\right]\\ &\quad-\xi_2\left[\sigma+k\left(\sigma\frac{c_2}{2\pi}+\ln\|u\|_{L^2_\rho}^2-\frac{n(1+c_1)}{2}\right)\right], \end{aligned}$$

and t_1 was introduced in (A3). Moreover, we choose σ so small that

$$\xi_1>2c_\sigma\|\rho\|_{L^2}^2\xi_2.$$

Whence σ is fixed, we can choose

$$\xi_2>\left(\int_0^{t_1}g(s)\mathrm{d}s-\sigma\right)^{-1},$$

and ξ_1 large enough so that $M_2>0$, which yields

$$L'(t)\le M_0(g\circ\nabla_x u)-cE'(t),\qquad \forall t\ge t_1.$$

Now we set $F(t)=L(t)+cE(t)$, which is equivalent to $E(t)$. Then,

$$\begin{aligned} F'(t)=&L'(t)+cE'(t)\\ \le&-cE(t)+c\int_{\mathbb{R}^n}\int_{t_1}^t g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2\mathrm{d}s\mathrm{d}x,\quad \text{for all } t\ge t_1. \end{aligned}\tag{4.12}$$

Using Lemma (4.4), we obtain

$$F'(t)\le -cE(t)+cH_0^{-1}(\lambda(t)),\qquad \text{for all } t\ge t_1.$$

Now, we will following the steps in ([12]) and using the fact that $E'\leq 0$, $0<H_0'$, $0<H_0''$ on $(0,r]$ to define the functional

$$F_1(t)=H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)F(t)+cE(t), \qquad \alpha_0<r,\ 0<c,$$

where $F_1(t)\sim E(t)$ and

$$\begin{aligned}F_1'(t)&=\alpha_0\frac{E'(t)}{E(0)}H_0''\left(\alpha_0\frac{E(t)}{E(0)}\right)F(t)+H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)F'(t)+cE'(t)\\&\leq -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)+cH_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t))+cE'(t).\end{aligned}$$

Let H_0^* given in (A3) and using Young's inequality (2.5) with $A=H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)$, $B=H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned}F_1'(t)&\leq -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)+cH_0^*\left(H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)\right)+c\lambda(t)+cE'(t)\\&\leq -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)+c\alpha_0\frac{E(t)}{E(0)}H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)-c'E'(t)+cE'(t).\end{aligned}$$

Choosing α_0,c,c', such that for all $t\geq t_1$ we have

$$F_1'(t)\leq -k\frac{E(t)}{E(0)}H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)=-kH_2\left(\frac{E(t)}{E(0)}\right),$$

where $H_2(t)=tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0,r]$, to find that H_2',H_2 are strict positives on $(0,1]$, then

$$\begin{aligned}&R(t)=\tau\frac{k_1F_1(t)}{E(0)}\sim E(t), \qquad \tau\in(0,1),\\&R'(t)\leq -\tau k_0H_2(R(t)), \qquad k_0\in(0,+\infty),\ t\geq t_1.\end{aligned} \tag{4.13}$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t)\leq H_1^{-1}(\alpha_1 t+\alpha_2), \qquad \alpha_1,\alpha_2\in(0,+\infty),\ t\geq t_1,$$

here $H_1(t)=\int_t^1 H_2^{-1}(s)\mathrm{d}s$. From (4.13), for a positive constant α_3, we have

$$E(t)\leq \alpha_3H_1^{-1}(\alpha_1t+\alpha_2), \qquad \alpha_1,\alpha_2\in(0,+\infty),\ t\geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0,1]$ and due to properties of H_2, we have

$$\lim_{t\to 0}H_1(t)=+\infty.$$

Then

$$E(t) \le \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \text{for all } t \ge 0.$$

This completes the proof of Theorem 4.1. □

Remark 4.1. Noting that, we have obtained all results without any conditions on the exponent k in the logarithmic nonlinearities.

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