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A New Blowup Criterion for Ideal Viscoelastic Flow

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Abstract. In this paper, we give a new blowup criteria of strong solutions to the Oldroyd model for ideal viscoelastic flow.

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1 Introduction and main results

In this short note, we will give the Beale-Kato-Majda type criteria for the breakdown of smooth solutions to the incompressible ideal viscoelastic flow of the following system

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \nabla\cdot FF^T, \\ \partial_t F + (u\cdot\nabla)F = \nabla u F, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^3, t\geq 0, u$ is the flow velocity, $F=F(x,t)$ represents the local deformation gradient of the fluid. This system of partial differential equations arises in the Oldroyd model for ideal viscoelastic flow i.e. a viscoelastic fluid whose elastic properties dominate its behavior (see [1]). Global existence for solutions near equilibrium of the viscous analog of (1.1) has been verified by Lin et al. [1] and Lei et al. [2]; A further discussion of these topics can be found in the work Lin and Zhang [3]. Noting the second equation of system

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(1.1), if $\nabla F_k = 0$ initially, it will remain so for later times. In what follows, we will make this assumption and also note that it implies the equality

$$\nabla \cdot FF^T = \sum_{k=1}^{3} (F_k \cdot \nabla) F_k.$$

Under the assumed properties of the deformation gradient in above references, the system (1.1) can be rewritten as eqnarray

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \kappa \nabla \cdot \tau, \\ \partial_t \tau + u \cdot \nabla \tau - \mu \Delta \tau - b\tau = Q(\nabla u, \tau) + aDu, \\ \nabla \cdot u = 0, \\ t = 0: \ \ u = u_0(x), \ \ \tau = \tau_0(x). \end{cases} \tag{1.2}$$

For the vanishing viscosity case, Sideris and Thomases [4] also established the global existence of smooth solutions near the equilibrium using the incompressible limit by imposing a null condition on the elastic stress. Using the standard energy method [5], it is known that for $(u_0, F_{k0}) \in H^s$, $s \geq 3$, there exists $T > 0$ such that the Cauchy problem (1.1) has a unique smooth solution $(u(t,x), F(t,x))$ on $[0,T]$ satisfying

$$(u,F) \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}). \tag{1.3}$$

Recently, Hu and Hynd [6] get an analog of the Beale-Kato-Majda criterion [7] for singularities of smooth solutions of the system of PDE arising in the Oldroyd model for ideal viscoelastic flow. More precisely, they showed that if the smooth solution (u,F) satisfies the following condition:

$$\int_0^T \|\nabla \times u\|_{L^\infty} \mathrm{d}t < \infty \ \text{ and } \ \int_0^T \|\nabla \times F\|_{L^\infty} \mathrm{d}t < \infty, \tag{1.4}$$

then the solution (u,F) can be extended beyond $t = T$, namely, for some $T < T^\star, (u,F) \in C([0,T^\star); H^s(\mathbb{R}^3)) \cap C^1([0,T^\star); H^{s-1}(\mathbb{R}^3))$.

More recently, for the following incompressible Euler equations

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ t = 0: \ u = u_0. \end{cases} \tag{1.5}$$

H. Kozono et al. [8] refined the result of Beale et al. [7] and showed that if the solution u to (1.3) satisfies

$$\int_0^T \|\nabla \times u\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t \leq \infty, \tag{1.6}$$

then the solution u can be extended beyond $t=T$. Here and thereafter, $\dot{B}^s_{p,q}$ denotes the homogenous Besov space, see Section 2 for definition. We remark that $L^\infty \subset \dot{B}^0_{\infty,\infty}$.

Motivated by [9], the purpose of this paper is to obtain a similar extension criterion for the 3D ideal viscoelastic flow. More precisely, we can get the following theorem:

Theorem 1.1. *Let (u,F) be a solution to* (1.1) *in the class* (1.3) *for $s\geq 3$. Assume that*

$$\int_0^T \|\nabla\times u\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t<\infty \text{ and } \int_0^T \|\nabla\times F\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t<\infty, \tag{1.7}$$

Then, the solution (u,F) can be extended beyond $t=T$. In other words, if the solution blows up in $t=T$, then

$$\int_0^T \|\nabla\times u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla\times F\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t=\infty. \tag{1.8}$$

The central result of this work suggests that even if the incompressibility of the deformation gradient is imposed, smooth solutions of (1.1) may become singular in finite time. From this standpoint, the property that the curl of the deformation gradient is higher order is perhaps the most important when considering global existence.

2 Preliminary

We begin with the Littlewood-Paley decomposition(see also [10]). Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function. Given $f\in\mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f=\hat{f}$. We take a couple of smooth functions (χ,ψ) supported on $\{\xi: |\xi|\leq 1\}$ with values in [0,1] such that for all $\xi\in\mathbb{R}^3$,

$$\chi(\xi)+\sum_{j=0}^{\infty}\psi(2^{-j}\xi)=1, \ \psi(\xi)=\chi(\frac{\xi}{2})-\chi(\xi),$$

and we denote $\chi(2^{-j}\xi)$ by $\chi_j(\xi)$. The homogeneous dyadic blocks and lower frequency cut-off functions are defined by

$$\dot{\triangle}_j f=2^{3j}\int_{\mathbb{R}^3} h(2^j y)f(x-y)\mathrm{d}y, \ \dot{S}_j f=2^{3j}\int_{\mathbb{R}^3}\tilde{h}(2^j y)f(x-y)\mathrm{d}y, \tag{2.1}$$

with $h=\mathcal{F}^{-1}\psi$ and $\tilde{h}=\mathcal{F}^{-1}\chi$. Then, we can define the homogeneous Littlewood-Paley decomposition by

$$f=\sum_{j\in\mathbb{Z}}\dot{\triangle}_j f. \tag{2.2}$$

Using this decomposition, for any $f\in\mathcal{S}'(\mathbb{R}^3)$, we define the homogeneous Besov space as follows:

$$\dot{B}^s_{p,q}=\{f:\|f\|_{\dot{B}^s_{p,q}}=(\sum_{j\in\mathbb{Z}}2^{jsq}\|\dot{\triangle}_j u\|^q_{L^p})^{\frac{1}{q}}<\infty\}, \tag{2.3}$$

for $s\in\mathbb{R},q<\infty$.

$$\dot{B}^s_{p,\infty}=\{f\,:\|f\|_{\dot{B}^s_{p,\infty}}=\sup_{j\in\mathbb{Z}}2^{js}\|\dot{\triangle}_j u\|_{L^p}<\infty\}. \tag{2.4}$$

for $q=\infty$.

Finally, for completeness, we give a logarithmic Sobolev inequalities in terms of Besov space, which plays an important role in the proof of Theorem 1.1 .

Proposition 2.1. *Let $m>3/2$. Assume that $f\in\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\cap H^m(\mathbb{R}^3)$. Then there holds the inequality*

$$\|f\|_{L^\infty}\le C(1+\|f\|_{\dot{B}^0_{\infty,\infty}}(1+\log^+\|f\|_{H^m})), \tag{2.5}$$

where $\log^+ t=\log t$, for $t>1$, $\log^+ t=0$, for $t\le 1$ and C is an absolute constant independent of f.

Proof. Using the Littlewood-Paley decomposition (2.2), we decompose f as follows:

$$f=\sum_{j<-N}\dot{\triangle}_j f+\sum_{j=-N}^{N}\dot{\triangle}_j f+\sum_{j>N}\dot{\triangle}_j f\equiv f_1+f_2+f_3, \tag{2.6}$$

where N is a positive integer to be determined later. We first estimate f_1. Notice that $\dot{\triangle}_j=\dot{S}_j-\dot{S}_{j-1}$, we have

$$f_1(x)=\dot{S}_{-(N+1)}f(x)\tilde{h}_{-(N+1)}*f(x).$$

Thus by the Hausdorff-Young inequality, we obtain

$$\|f_1\|_{L^\infty}\le\|\tilde{h}_{-(N+1)}\|_{L^2}\|f\|_{L^2}\le C2^{\frac{-3N}{2}}\|f\|_{L^2}. \tag{2.7}$$

Next we turn to estimate f_2. By the definition of Besov space, we have

$$\|f_2\|_{L^\infty}\le\sum_{i=-N}^{N}\|\triangle_j f\|_{L^\infty}\le(2N+1)\|f\|_{\dot{B}^0_{\infty,\infty}}. \tag{2.8}$$

Finally we estimate f_3. By Bernstein inequality, we have

$$\begin{aligned}\|f_3\|&\le\sum_{j>N}\|\triangle_j f\|_\infty\le C\sum_{j>N}2^{3j/2}\|\triangle_j f\|_{L^2}\\&\le C\sum_{j>N}2^{(-m+\frac{3}{2})j}\|f\|\dot{B}^s_{2,\infty}\le C2^{(-m+\frac{3}{2})N}\|f\|_{H^m},\end{aligned} \tag{2.9}$$

where we have use the fact that $s>3/2$ and $H^m\hookrightarrow\dot{B}^s_{2,\infty}$ in the last inequality of (2.9).

Combining (2.6)-(2.9), we obtain

$$\|f\|_\infty\le C2^{\frac{-3N}{2}}\|f\|_{L^2}+N\|f\|_{\dot{B}^0_{\infty,\infty}}+2^{(-m+\frac{3}{2})N}\|f\|_{H^m}).$$

Set $\alpha=\min(3/2,m-3/2)$, we have

$$\|f\|_\infty \le C(2^{-\alpha N}\|f\|_{H^m}+N\|f\|_{\dot{B}^0_{\infty,\infty}}). \tag{2.10}$$

Now we choose N such that $2^{-\alpha N}\|f\|_{H^m}\le 1$, i.e.,

$$N\ge \frac{\log\|f\|_{H^m}}{\alpha\log 2}.$$

Then, the desired estimate (2.5) follows from the above inequality and (2.10) □

Remark 2.1. More general result can be found in [11, 12].

3 Proof of Theorem 1.1

In the section, using the energy estimate method, we will give the proof of Theorem 1.1.

Proof. Firstly, we give the basic energy estimate of system (1.1). Multiplying u and F to the equations and integrating them in $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|u|^2+|F_k|^2\mathrm{d}x=&-\int_{\mathbb{R}^3}(u\cdot\nabla u)u\mathrm{d}x+\int_{\mathbb{R}^3}(F_k\cdot\nabla u)F_k\mathrm{d}x\\&+\int_{\mathbb{R}^3}\sum_{k=1}^3(F_k\cdot\nabla F_k)u\mathrm{d}x-\int_{\mathbb{R}^3}(u\cdot\nabla F_k)F_k\mathrm{d}x.\end{aligned}$$

Noting the divergence free of u and F_k, we can get

$$\|(u(t),F_k(t))\|_{L^2}\le C\|(u_0,F_{k0})\|_{L^2}.$$

Taking the operation ∂_x^α on both sides of (1.2) for $|\alpha|\le s$, multiplying $(\partial_x^\alpha u,\partial_x^\alpha F_k)$ to the resulting equation, and integrating over $\mathbb{R}^3$ with respect to x, then integrating by parts and noting the divergence free of u and F_k, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\partial_x^\alpha u|^2+|\partial_x^\alpha F_k|^2\mathrm{d}x\le&-\int_{\mathbb{R}^3}\partial_x^\alpha(u\cdot\nabla u)\partial_x^\alpha u\mathrm{d}x+\int_{\mathbb{R}^3}\partial_x^\alpha(F_k\cdot\nabla u)\partial_x^\alpha F_k\mathrm{d}x\\&+\int_{\mathbb{R}^3}\sum_{k=1}^3\partial_x^\alpha(F_k\cdot\nabla F_k)\partial_x^\alpha u\mathrm{d}x-\int_{\mathbb{R}^3}\partial_x^\alpha(u\cdot\nabla F_k)\partial_x^\alpha F_k\mathrm{d}x.\end{aligned} \tag{3.1}$$

Noting the divergence free of u, we have

$$\int_{\mathbb{R}^3}\partial_x^\alpha(u\cdot\nabla u)\partial_x^\alpha u\mathrm{d}x=\int_{\mathbb{R}^3}(\partial_x^\alpha(u\cdot\nabla u)-(u\cdot\partial_x^\alpha\nabla u))\partial_x^\alpha u\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^3}\partial_x^\alpha(u\cdot\nabla F_k)\partial_x^\alpha F_k\mathrm{d}x=\int_{\mathbb{R}^3}(\partial_x^\alpha(u\cdot\nabla F_k)-(u\cdot\partial_x^\alpha\nabla F_k))\partial_x^\alpha F_k\mathrm{d}x.$$

Recall the following estimate

$$\|\partial_x^\alpha(fg)-f\partial_x^\alpha g\|_{L^2}\leq C(\|f\|_{H^s}\|g\|_{L^\infty}+\|\nabla f\|_{L^\infty}\|g\|_{H^s}), \tag{3.2}$$

and using the divergence free of F_k

$$\int_{\mathbb{R}^3}\partial_x^\alpha(F_k\cdot\nabla u)\partial_x^\alpha F_k\mathrm{d}x+\int_{\mathbb{R}^3}\sum_{k=1}^3\partial_x^\alpha(F_k\cdot\nabla F_k)\partial_x^\alpha u\mathrm{d}x=0.$$

Then, doing summation over $|\alpha|\leq s$, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^s}^2+\|F_k\|_{H^s}^2)\leq C(\|\nabla u\|_{L^\infty}+\|\nabla F_k\|_{L^\infty})(\|u\|_{H^s}^2+\|F_k\|_{H^s}^2). \tag{3.3}$$

Then, by Gronwall inequality, we have

$$\|(u(t),F_k(t))\|_{H^s}\leq\|(u_0,F_{k0})\|_{H^s}\exp\left(C\int_0^t(\|\nabla u(\tau)\|_{L^\infty}+\|\nabla F_k(\tau)\|_{L^\infty})\mathrm{d}\tau\right). \tag{3.4}$$

Using the Proposition 2.1 and divergence free of u and F_k, we can get

$$\|(\nabla u,\nabla F_k)\|_{L^\infty}\leq C(1+\|(\nabla\times u,\nabla\times F_k)\|_{\dot{B}^0_{\infty,\infty}}(1+\log^+\|(u,F_k)\|_{H^s})). \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\begin{aligned}\|(u(t),F_k(t))\|_{H^s}\leq\|(u_0,F_{k0})\|_{H^s}\exp\Big(C\int_0^t(C(1+\|(\nabla\times u,\nabla\times F_k)\|_{\dot{B}^0_{\infty,\infty}}\\ \times(1+\log^+\|(u,F_k)\|_{H^s})))\mathrm{d}\tau\Big).\end{aligned} \tag{3.6}$$

Applying the Gronwall inequality, we deduce

$$\|(u(t),F_k(t))\|_{H^s}\leq C\|(u_0,F_{k0})\|_{H^s}\exp\left(t+\exp\left(C\int_0^t\|(\nabla\times u(\tau),\nabla\times F_k(\tau))\|_{\dot{B}^0_{\infty,\infty}}\mathrm{d}\tau\right)\right).$$

Therefore, by the standard argument of continuation of local solutions, we complete the proof of Theorem 1.1. □

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