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# $L^1$ Existence and Uniqueness of Entropy Solutions to Nonlinear Multivalued Elliptic Equations with Homogeneous Neumann Boundary Condition and Variable Exponent

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**Abstract.** In this work, we study the following nonlinear homogeneous Neumann boundary value problem $\beta(u)-\mathrm{div}a(x,\nabla u)\ni f$ in $\Omega$, $a(x,\nabla u)\cdot\eta=0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded open domain in $\mathbb{R}^N$, $N\geq 3$ with smooth boundary $\partial\Omega$ and $\eta$ the outer unit normal vector on $\partial\Omega$. We prove the existence and uniqueness of an entropy solution for $L^1$-data $f$. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

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## 1 Introduction

The paper is motivated by phenomena which are described by the homogeneous Neumann boundary value problem of the form

$$\begin{cases} \beta(u)-\mathrm{div}a(x,\nabla u)\ni f, & \text{in } \Omega,\\ a(x,\nabla u)\cdot\eta=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\eta$ is the unit outward normal vector on $\partial\Omega$, $\Omega$ is a smooth bounded open domain in $\mathbb{R}^N, N\geq 3, \beta=\partial j$ is a maximal monotone graph in $\mathbb{R}^2$ with $\mathrm{dom}(\beta)$ bounded on $\mathbb{R}$ and $0\in\beta(0)$, $f\in L^1(\Omega)$ and $a$ is a Leray-Lions operator which involves variable exponents.

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Note that $j$ is a nonnegative, convex and l.s.c. function on $\mathbb{R}$ and, $\partial j$ is the subdifferential of $j$. We set

$$\overline{\mathrm{dom}(\beta)}=[m,M]\subset\mathbb{R} \text{ with } m\leq 0\leq M.$$

Recall that a Leray-Lions operator which involves variable exponents is a Carathéodory function $a(x,\xi):\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ (i.e. $a(x,\xi)$ is continuous in $\xi$ for a.e. $x\in\Omega$ and measurable in $x$ for every $\xi\in\mathbb{R}^N$) such that:

• There exists a positive constant $C_1$ such that

$$|a(x,\xi)|\leq C_1(j(x)+|\xi|^{p(x)-1}), \tag{1.2}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$ where $j$ is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $1/p(x)+1/p'(x)=1$.

• The following inequalities hold

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)>0, \tag{1.3}$$

for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^N$, with $\xi\neq\eta$, and

$$\frac{1}{C}|\xi|^{p(x)}\leq a(x,\xi)\cdot\xi, \tag{1.4}$$

for almost every $x\in\Omega, C>0$ and for every $\xi\in\mathbb{R}^N$.

In this paper, we make the following assumption on the variable exponent:

$$p(\cdot):\overline{\Omega}\to\mathbb{R} \text{ is a continuous function such that } 1<p_-\leq p_+<+\infty, \tag{1.5}$$

where $p_-:=\mathrm{essinf}_{x\in\Omega}\,p(x)$ and $p_+:=\mathrm{esssup}_{x\in\Omega}\,p(x)$.

As the exponent $p(\cdot)$ appearing in (1.2) and (1.4) depends on the variable $x$, the functional setting for the study of problem (1.1) involves Lebesgue and Sobolev spaces with variable exponents $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. In the next section, we will make a brief presentation of the variable exponent spaces.

Many results are known as regards to elliptic problems in the variational setting for Dirichlet or Dirichlet-Neumann problems (cf. [1–9]).

Problem (1.1) can be viewed as an extension of the following

$$\begin{cases} b(u)-\mathrm{div}a(x,\nabla u)=f, & \text{in } \Omega,\\ a(x,\nabla u)\cdot\eta=0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega$ is a smooth bounded open domain in $\mathbb{R}^N, N\geq 3$ and $\eta$ the outer unit normal vector on $\partial\Omega$. $b:\mathbb{R}\to\mathbb{R}$ is a continuous, nondecreasing function, surjective such that $b(0)=0$, $f\in L^1(\Omega)$ and $a$ is a Lerray-Lions operator which involves variable exponents.

Problem (1.6) was studied by Bonzi, Nyanquini and Ouaro (cf. [2]) where they proved the existence and uniqueness of an entropy solution. An equivalent notion of solution is called renormalized solution. The concept of renormalized solutions was introduced by Diperna and Lions in [10]. This notion was then adapted to the study of various problems of PDEs. In [9], Wittbold and Zimmermann adapted this notion of solution to a new and interesting problem of the form

$$\begin{cases} \beta(u) - \mathrm{div} a(x,\nabla u) - \mathrm{div} F(u) \ni f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

They proved for $F$ locally Lipschitz continuous, $\beta$ maximal monotone mapping with $0 \in \beta(0), f \in L^1(\Omega)$ and a continuous exponent $p(\cdot)$, the existence and uniqueness of a renormalized solution for problem (1.7).

In this work, we consider an homogeneous Neumann boundary condition instead of the Dirichlet boundary condition considered in [9]. One of the main difficulty which appears in this case is the famous Poincaré inequality which dœsn't apply and even the Poincaré-Wirtinger inequality also. An other difficulty is that, since we assume that the domain of $\beta$ is bounded, it appears in the definition of the solution, a bounded Radon diffuse measure in order to take into account the border of the domain. The techniques used in this work are close to those used in [5,7].

We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure in $\Omega$, equipped with its standard norm $||\cdot||_{\mathcal{M}_b(\Omega)}$. Given $\nu \in \mathcal{M}_b(\Omega)$, we say that $\nu$ is diffuse with respect to the capacity $W^{1,p(\cdot)}(\Omega)(p(\cdot)-$capacity for short) if $\nu(E) = 0$ for every set $E$ such that $\mathrm{Cap}_{p(\cdot)}(E,\Omega) = 0$, where the Sobolev $p(\cdot)$-capacity of $E$ is defined by

$$\mathrm{Cap}_{p(\cdot)}(E,\Omega) = \inf_{u \in S_{p(\cdot)}(E)} \int_\Omega \left( |u|^{p(x)} + |\nabla u|^{p(x)} \right) \mathrm{d}x,$$

with

$$S_{p(\cdot)}(E) = \{ u \in W^{1,p(\cdot)}(\Omega) : u \geq 1 \text{ in an open set containing } E \text{ and } \ u \geq 0 \text{ in } \Omega \}.$$

In the case $S_{p(\cdot)}(E) = \emptyset$, we set $\mathrm{Cap}_{p(\cdot)}(E,\Omega) = +\infty$. The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(\Omega)$.

The remaining part of the paper is the following: in Section 2, we introduce some notations and functional spaces. In Section 3, we prove the existence of entropy solution to the problem (1.1) and in Section 4, we prove the uniqueness of entropy solution.

## 2 Assumptions and preliminary

As the exponent $p(\cdot)$ appearing in (1.2) and (1.4) depends on the variable $x$, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} \mathrm{d}x$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Finally, we have the Hölder type inequality:

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p_+}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

which is a Banach space equipped with the following norm

$$||u||_{1,p(\cdot)} = |u|_{p(\cdot)} + |(|\nabla u|)|_{1,p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), ||u||_{1,p(\cdot)})$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [11] (see also [12]).

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (cf. [13]):

**Lemma 2.1.** *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*

1. $|u|_{p(\cdot)} > 1 \implies |u|^{p_-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p_+}_{p(\cdot)}$;
2. $|u|_{p(\cdot)} < 1 \implies |u|^{p_+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p_-}_{p(\cdot)}$;
3. $|u|_{p(\cdot)} < 1$ (respectively $= 1; > 1$) $\iff \rho_{p(\cdot)}(u) < 1$ (respectively $= 1; > 1$);
4. $|u_n|_{p(\cdot)} \longrightarrow 0$ (respectively $\longrightarrow +\infty$) $\Longleftrightarrow \rho_{p(\cdot)}(u_n) \longrightarrow 0$ (respectively $\longrightarrow +\infty$);
5. $\rho_{p(\cdot)}\big(u/|u|_{p(\cdot)}\big) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the function

$$\rho_{1,p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)}\mathrm{d}x.$$

Then we have the following lemma (see [14, 15]).

**Lemma 2.2.** *If $u \in W^{1,p(\cdot)}(\Omega)$, then the following properties hold:*

1. $|u|_{1,p(\cdot)} > 1 \Longrightarrow |u|_{1,p(\cdot)}^{p_-} \leq \rho_{1,p(\cdot)}(u) \leq |u|_{1,p(\cdot)}^{p_+}$;
2. $|u|_{1,p(\cdot)} < 1 \Longrightarrow |u|_{1,p(\cdot)}^{p_+} \leq \rho_{1,p(\cdot)}(u) \leq |u|_{1,p(\cdot)}^{p_-}$;
3. $|u|_{1,p(\cdot)} < 1$ (respectively $=1; >1$) $\Longleftrightarrow \rho_{1,p(\cdot)}(u) < 1$ (respectively $=1; >1$).

For any given $l,k>0$, we define the function $h_l$ by $h_l(r) = \min\big((l+1-|r|)^+, 1\big)$ and the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \max\{-k, \min(k,s)\}$.

For any $l_0$, we consider the function $h_0 = h_{l_0}$ defined by

$$\begin{cases} h_0 \in C_c^1(\mathbb{R}),\ h_0(r) \geq 0,\ \forall r \in \mathbb{R}, \\ h_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } h_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases}$$

Let $\gamma$ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section $\gamma_0$ of $\gamma$:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s), & \text{if } \gamma(s) \neq \emptyset, \\ +\infty, & \text{if } [s,+\infty) \cap D(\gamma) = \emptyset, \\ -\infty, & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We write for any $u : \Omega \to \mathbb{R}$ and $k \geq 0, \{|u| \leq k (<k, >k, \geq k, =k)\}$ for the set $\{x \in \Omega / |u(x)| \leq k (<k, >k, \geq k, =k)\}$.

Before introducing the notion of entropy solution for the problem (1.1), we define the following spaces which are similar to that introduced in [16, 17]. We note

$$\mathcal{T}^{1,p(\cdot)}(\Omega) := \big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable};\ \ T_k(u) \in W^{1,p(\cdot)}(\Omega) \text{ for all } k > 0 \big\}.$$

As in [17], we can prove that for $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $w : \Omega \longrightarrow \mathbb{R}$ such that $\nabla T_k(u) = w \chi_{\{|w|<k\}}$ $\forall k > 0$. This function $w$ will be denoted $\nabla u$.

We define $\mathcal{T}_{\mathcal{H}}^{1,p(\cdot)}(\Omega)$ (see [2]) as the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_\delta)_{\delta > 0} \in W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:

(i) $u_\delta \longrightarrow u$ a.e. in $\Omega$.

(ii) $\nabla T_k(u_\delta) \longrightarrow \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$.

The symbol $\mathcal{H}$ in the notation is related to the fact that we consider here homogeneous Neumann boundary condition.

To end this section, we give some useful convergence results.

**Lemma 2.3.** *Let $(\beta_n)_{n \geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of graphs (i.e. for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n \geq 1} \subset L^1(\Omega)$ and $(w_n)_{n \geq 1} \subset L^1(\Omega)$. We suppose that:*

*$\forall n \geq 1, w_n \in \beta_n(z_n), (w_n)_{n \geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then $z \in dom(\beta)$.*

*Proof.* For the proof of Lemma 2.3, we need the "biting lemma of Chacon". Let us recall it.

**Lemma 2.4.** (The "biting lemma of Chacon") [19]. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain and $(f_n)_{n\geq 1}$ a bounded sequence in $L^1(\Omega)$. Then there exists $f \in L^1(\Omega)$, a sequence $(f_{n_k})_{k\geq 1}$ and a sequence of measurable sets $(E_j)_{j\geq 1}, E_j \subset \Omega, \forall j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim_{j\to+\infty}|E_j| = 0$, such that for any $j \in \mathbb{N}^*, f_{n_k} \rightharpoonup f$ in $L^1(\Omega \setminus E_j)$.*

Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon", there exists $w \in L^1(\Omega)$, a subsequence $(w_{n_k})_{k\geq 1}$ and a sequence of measurable sets $(E_j)_{j\in\mathbb{N}^*}$ in $\Omega$ such that $\forall j \in \mathbb{N}^*, E_{j+1} \subset E_j, \lim_{j\to+\infty}|E_j| = 0$ and $\forall j \in \mathbb{N}^*, w_{n_k} \rightharpoonup w$ in $L^1(\Omega \setminus E_j)$. Since $z_{n_k} \rightharpoonup z$ in $L^1(\Omega)$ and so in $L^1(\Omega \setminus E_j), \forall j \in \mathbb{N}$ and $\beta_{n_k} \to \beta$ in the sense of graphs, we have $w \in \beta(z)$ a.e. in $\Omega \setminus E_j$. Thus $z \in \mathrm{dom}(\beta)$ a.e. in $\Omega \setminus E_j$. Finally, we obtain $z \in \mathrm{dom}(\beta)$ a.e. in $\Omega$. □

**Lemma 2.5.** (cf. [13], Theorem 1.4) *Let $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \cdots$. Then the following statements are equivalent to each other:*

1) $\lim\limits_{n\to+\infty} |u_n - u|_{p(\cdot)} = 0$;

2) $\lim\limits_{n\to+\infty} \rho_{p(\cdot)}(u_n - u) = 0$;

3) $u_n$ *converges to* $u$ *in* $\Omega$ *in measure and*

$$\lim_{n\to+\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u).$$

**Lemma 2.6.** (Lebesgue generalized convergence theorem) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and $f$ a measurable function such that $f_n \to f$ a.e. in $\Omega$. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that for all $n \in \mathbb{N}$, $|f_n| \leq g_n$ a.e. in $\Omega$ and $g_n \to g$ in $L^1(\Omega)$. Then*

$$\int_\Omega f_n \,\mathrm{d}x \to \int_\Omega f \,\mathrm{d}x.$$

# 3 Statement of the main results

We introduce the following concepts of solution for problem (1.1).

**Definition 3.1.** *A solution of* (1.1) *is a couple $(u, b) \in \mathcal{T}_{\mathcal{H}}^{1,p(\cdot)}(\Omega) \times L^1(\Omega)$, such that*

$$\begin{cases} u \in dom(\beta)\, \mathcal{L}^N - a.e. \text{ in } \Omega, \ b \in \beta(u)\, \mathcal{L}^N - a.e. \text{ in } \Omega, \\ \text{there exists } \mu \in \mathcal{M}_b^{p(\cdot)}(\Omega) \text{ with } \mu \perp \mathcal{L}^N, \\ \mu^+ \text{ is concentrated on } \{u = M\}, \ \mu^- \text{ is concentrated on } \{u = m\}, \end{cases}$$

*and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega b\varphi \,\mathrm{d}x + \int_\Omega \varphi \,\mathrm{d}\mu = \int_\Omega f\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \tag{3.1}$$

**Remark 3.1.** If $(u,b)$ is a solution of the problem (1.1) then, it satisfy the following entropic formulation

$$\int_{\Omega} a(x,\nabla u)\cdot\nabla T_k(u-\xi)\mathrm{d}x+\int_{\Omega} bT_k(u-\xi)\mathrm{d}x\leq\int_{\Omega} fT_k(u-\xi)\mathrm{d}x, \tag{3.2}$$

for any $\xi\in W^{1,p(\cdot)}(\Omega)\cap L^{\infty}(\Omega)$ such that $\xi\in\mathrm{dom}(\beta)$ $\mathcal{L}^N-a.e.$ *in* $\Omega$.

Our main result is the following.

**Theorem 3.1.** *Assume that* (1.2)-(1.5) *hold true and* $f\in L^1(\Omega)$*, there exists a unique entropy solution to problem* (1.1)*. Moreover*

$$\lim_{m\to+\infty}\int_{[m\leq|u|\leq m+1]}|\nabla u|^{p(x)}\mathrm{d}x=0. \tag{3.3}$$

*Proof.* The proof of Theorem 3.1 is divided into several steps.

## 3.1 Regularized problem

For every $\epsilon>0$, we consider the Yosida regularisation $\beta_\epsilon$ of $\beta$ given by

$$\beta_\epsilon=\frac{1}{\epsilon}\big(I-(I+\epsilon\beta)^{-1}\big),$$

and we set

$$j_\epsilon(s)=\min_{r\in\mathbb{R}}\Big\{\frac{1}{2\epsilon}|s-r|^2+j(r)\Big\},\qquad \forall s\in\mathbb{R},\ \forall\epsilon>0.$$

According to Proposition 2.11 in [18], we have

$$\begin{cases}\mathrm{dom}(\beta)\subset\mathrm{dom}(j)\subset\overline{\mathrm{dom}(j)}\subset\overline{\mathrm{dom}(\beta)}.\\ j_\epsilon(s)=\dfrac{\epsilon}{2}\big|\beta_\epsilon(s)\big|^2+j(J_\epsilon),\ \text{where}\ J_\epsilon=(I+\epsilon\beta)^{-1},\\ j_\epsilon\ \text{is convex, Frechet-differentiable and}\ \beta_\epsilon=\partial j_\epsilon,\\ j_\epsilon\uparrow j\ \text{as}\ \epsilon\downarrow 0.\end{cases}$$

Note that $\beta_\epsilon$ is a nondecreasing and Lipschitz-continuous function. We also define the function $f_\epsilon$ by $f_\epsilon(x)=T_{\frac{1}{\epsilon}}\big(f(x)\big)$ for any $x\in\Omega$. Then $\big(f_\epsilon\big)_{\epsilon>0}$ is a sequence of bounded functions which converges strongly to $f\in L^1(\Omega)$ and such that

$$\|f_\epsilon\|_1\leq\|f\|_1,\qquad\forall\ \epsilon>0.$$

**Lemma 3.1.** *The Yosida regularisation* $\beta_\epsilon$ *is a surjective operator.*

*Proof.* Since $\operatorname{dom}(\beta)\subset[m,M]$, then $\forall\, r\in\mathbb{R}$, $J_\epsilon(r)=\big(I+\epsilon\beta\big)^{-1}(r)\in[m,M]$. Consequently

$$\lim_{r\to+\infty}\beta_\epsilon(r)=\lim_{r\to+\infty}\frac{r-J_\epsilon(r)}{\epsilon}=+\infty,$$

and

$$\lim_{r\to-\infty}\beta_\epsilon(r)=\lim_{r\to-\infty}\frac{r-J_\epsilon(r)}{\epsilon}=-\infty.$$

As $\beta_\epsilon$ is a maximal monotone graph, thanks to ([18], Corollaire 2.3), we conclude that $\beta_\epsilon$ is surjective. □

Now, we consider the approximated problem

$$\begin{cases}\beta_\epsilon\big(u_\epsilon\big)-\operatorname{div}a(x,\nabla u_\epsilon)=f_\epsilon, & \text{in } \Omega,\\ a(x,\nabla u_\epsilon)\cdot\eta=0, & \text{on } \partial\Omega,\end{cases} \tag{3.4}$$

where the notation $u_\epsilon$ is used for any solution of the problem (3.4).

**Definition 3.2.** *A weak solution of* (3.4) *is a measurable function* $u_\epsilon\in W^{1,p(\cdot)}(\Omega)$ *such that* $\beta_\epsilon\big(u_\epsilon\big)\in L^\infty(\Omega)$ *and*

$$\int_\Omega\beta_\epsilon\big(u_\epsilon\big)\varphi\,\mathrm{d}x+\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega f_\epsilon\varphi\,\mathrm{d}x,\qquad\forall\varphi\in W^{1,p(\cdot)}(\Omega). \tag{3.5}$$

We have the following result according to [2].

**Theorem 3.2.** *Assume that* (1.2)-(1.5) *hold and* $f_\epsilon\in L^\infty(\Omega)$*. Then, there exists a unique weak solution* $u_\epsilon$ *to problem* (3.4).

## 3.2 A priori estimates

**Lemma 3.2.** *Assume that* (1.2)-(1.5) *hold and* $f\in L^1(\Omega)$*. Let* $u_\epsilon$ *be a weak solution of* (3.4)*. Then, for all* $k>0$*,*

$$\int_\Omega|\nabla T_k(u_\epsilon)|^{p(x)}\mathrm{d}x\le Ck\|f\|_1, \tag{3.6}$$

*and*

$$\int_\Omega\beta_\epsilon(u_\epsilon)T_k(u_\epsilon)\mathrm{d}x\le Ck||f||_1, \tag{3.7}$$

*where C is a positive constant.*

*Proof.* Taking $\varphi=T_k(u_\epsilon)$ as a test function in (3.5), we get

$$\int_\Omega\beta_\epsilon\big(u_\epsilon\big)T_k\big(u_\epsilon\big)\mathrm{d}x+\int_\Omega a\big(x,\nabla u_\epsilon\big)\cdot\nabla T_k\big(u_\epsilon\big)\mathrm{d}x=\int_\Omega f_\epsilon T_k\big(u_\epsilon\big)\mathrm{d}x. \tag{3.8}$$

Using (1.4) and the fact that $\beta_\epsilon$ is nondecreasing and $\beta_\epsilon(0)=0$, we deduce from (3.8) that

$$C_1\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\mathrm{d}x \le \int_\Omega f_\epsilon T_k(u_\epsilon)\mathrm{d}x \le \int_\Omega |f_\epsilon T_k(u_\epsilon)|\mathrm{d}x \le k\|f\|_1.$$

Then,

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\mathrm{d}x \le Ck\|f\|_1.$$

Now, using (1.4), we have $a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon) \ge C_1|\nabla T_k(u_\epsilon)|^{p(x)} \ge 0$. Then, we get from (3.8) that

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)\mathrm{d}x \le \int_\Omega f_\epsilon T_k(u_\epsilon)\mathrm{d}x \le kC\|f\|_1.$$

The proof is complete. □

**Lemma 3.3.** *The sequence $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

*Proof.* According to (3.7), we get

$$\int_\Omega \beta_\epsilon(u_\epsilon)\frac{1}{k}T_k(u_\epsilon)\mathrm{d}x \le C\|f\|_1,$$

which gives, if we let $k\to 0$,

$$\int_\Omega |\beta_\epsilon(u_\epsilon)|\mathrm{d}x \le C\|f\|_1.$$

The proof is complete. □

**Lemma 3.4.** *The sequence $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

*Proof.* We have for all $k>0$,

$$\begin{aligned}\int_\Omega |\beta_\epsilon(T_k(u_\epsilon))|\mathrm{d}x &= \int_{\{|u_\epsilon|\le k\}}|\beta_\epsilon(u_\epsilon)|\mathrm{d}x + \int_{\{u_\epsilon>k\}}|\beta_\epsilon(k)|\mathrm{d}x + \int_{\{u_\epsilon<-k\}}|\beta_\epsilon(-k)|\mathrm{d}x\\ &\le \int_{\{|u_\epsilon|\le k\}}|\beta_\epsilon(u_\epsilon)|\mathrm{d}x + \int_{\{u_\epsilon>k\}}|\beta_\epsilon(u_\epsilon)|\mathrm{d}x + \int_{\{u_\epsilon<-k\}}|\beta_\epsilon(u_\epsilon)|\mathrm{d}x\\ &= \int_\Omega |\beta_\epsilon(u_\epsilon)|\mathrm{d}x.\end{aligned}$$

Then, according to Lemma 3.3, $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$. □

**Lemma 3.5.** *Assume that* (1.2)-(1.5) *hold true and $f\in L^1(\Omega)$. Let $u_\epsilon$ be a weak solution of* (3.4), *then*

$$\mathrm{meas}\{|u_\epsilon|>k\} \le \frac{C\|f\|_1}{\min\Big(\beta_\epsilon(k),|\beta_\epsilon(-k)|\Big)}, \text{ for } k>0 \text{ large enough}, \tag{3.9}$$

*and*

$$\text{meas}\{|\nabla u_\epsilon|>k\}\le \frac{C(k+1)}{k^{p_-}}+\frac{\text{const}\big(\|f\|_1\big)}{\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)},\ \textit{for}\ \ k>0\ \textit{large enough}, \tag{3.10}$$

*where C is a positive constant.*

*Proof.* We start by the proof of (3.9). By Lemma 3.3, we have

$$\int_\Omega |\beta_\epsilon(u_\epsilon)|\mathrm{d}x\le C\|f\|_1,$$

which implies that

$$\int_{\{|u_\epsilon|>k\}}|\beta_\epsilon(u_\epsilon)|\mathrm{d}x\le C\|f\|_1. \tag{3.11}$$

Since $\beta_\epsilon$ is nondecreasing and $\beta_\epsilon(0)=0$, we have

$$u_\epsilon>k\ \Rightarrow\ \beta_\epsilon(k)\le\beta_\epsilon(u_\epsilon)\ \Rightarrow\ \beta_\epsilon(k)\le\big|\beta_\epsilon(u_\epsilon)\big|,$$

and

$$u_\epsilon<-k\ \Rightarrow\ \beta_\epsilon(-k)\ge\beta_\epsilon(u_\epsilon)\ \Rightarrow\ |\beta_\epsilon(-k)|\le\big|\beta_\epsilon(u_\epsilon)\big|.$$

Then, we deduce from (3.11) that

$$\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)\int_{\{|u_\epsilon|>k\}}dx\ \le\ C\|f\|_1,$$

i.e.

$$\text{meas}\{|u_\epsilon|>k\}\le\frac{C\|f\|_1}{\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)}.$$

Now, we prove (3.10). For $k,\lambda\ge 0$, set

$$\Phi(k,\lambda)=\text{meas}\{|\nabla u_\epsilon|^{p_-}>\lambda,\ |u_\epsilon|>k\}.$$

According to (3.9), we have

$$\Phi(k,0)\le\frac{C\|f\|_1}{\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)},\quad \text{for } k>0 \text{ large enough}. \tag{3.12}$$

Using the fact that the function $\lambda\mapsto\Phi(k,\lambda)$ is nonincreasing, we get for $k>0$ and $\lambda>0$ that

$$\begin{aligned}\Phi(0,\lambda)=&\frac{1}{\lambda}\int_0^\lambda\Phi(0,\lambda)\mathrm{d}s\le\frac{1}{\lambda}\int_0^\lambda\Phi(0,s)\mathrm{d}s\\ \le&\frac{1}{\lambda}\int_0^\lambda\big[\Phi(0,s)+\big(\Phi(k,0)-\Phi(k,s)\big)\big]\mathrm{d}s\\ \le&\Phi(k,0)+\frac{1}{\lambda}\int_0^\lambda\big(\Phi(0,s)-\Phi(k,s)\big)\mathrm{d}s.\end{aligned}$$

Observe that since

$$\Phi(0,s)-\Phi(k,s)=\text{meas}\{|u_\epsilon|\le k,|\nabla u_\epsilon|^{p_-}>s\},$$

we obtain

$$\int_0^\infty \big(\Phi(0,s)-\Phi(k,s)\big)\,\mathrm{d}s=\int_{\{|u_\epsilon|\le k\}}|\nabla u_\epsilon|^{p_-}\mathrm{d}x. \tag{3.13}$$

Note that

$$\begin{aligned}\int_{\{|u_\epsilon|\le k\}}|\nabla u_\epsilon|^{p_-}\mathrm{d}x&=\int_{\{|u_\epsilon|\le k,|\nabla u_\epsilon|>1\}}|\nabla u_\epsilon|^{p_-}\mathrm{d}x+\int_{\{|u_\epsilon|\le k,|\nabla u_\epsilon|\le 1\}}|\nabla u_\epsilon|^{p_-}\mathrm{d}x\\&\le\int_{\{|u_\epsilon|\le k,|\nabla u_\epsilon|>1\}}|\nabla u_\epsilon|^{p(x)}\mathrm{d}x+\text{meas}(\Omega)\\&\le\int_{\{|u_\epsilon|\le k\}}|\nabla u_\epsilon|^{p(x)}\mathrm{d}x+\text{meas}(\Omega).\end{aligned}$$

By the inequalities above, thanks to (3.6), we obtain

$$\int_{\{|u_\epsilon|\le k\}}|\nabla u_\epsilon|^{p_-}\mathrm{d}x\le C(k+1). \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\int_0^\infty \big(\Phi(0,s)-\Phi(k,s)\big)\,\mathrm{d}s\le C(k+1). \tag{3.15}$$

Coming back to (3.12) and using (3.15) we arrive at

$$\Phi(0,\lambda)\le\frac{C(k+1)}{\lambda}+\frac{C||f||_1}{\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)},\quad \text{for all } \lambda>0,k \text{ large enough.}$$

In particular,

$$\Phi(0,\lambda)\le\frac{C(k+1)}{\lambda}+\frac{C||f||_1}{\min\Big(\beta_\epsilon(k),\big|\beta_\epsilon(-k)\big|\Big)},\quad \text{for all } \lambda\ge 1,k \text{ large enough.} \tag{3.16}$$

Setting $\lambda=k^{p_-}$ in (3.16) gives (3.10). □

### 3.3 Convergence results

**Lemma 3.6.** (i) *For all $k>0,T_k(u_\epsilon)\to T_k(u)$ in $L^{p_-}$ and a.e. in $\Omega$, as $\epsilon\to 0$.*

(ii) *There exists a measurable function $u:\Omega\to\mathbb{R}$ such that $u\in dom(\beta)$ a. e. in $\Omega$ and $u_\epsilon\to u$ in measure and a.e. in $\Omega$, as $\epsilon\to 0$.*

*Proof.* For $k>0$, the sequence $(\nabla T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $L^{p(\cdot)}(\Omega)$; hence the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Then, up to a subsequence we can assume that for any $k>0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges weakly to $\sigma_k$ in $W^{1,p(\cdot)}(\Omega)$ and so $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to $\sigma_k$ in $L^{p_-}$. Let $s>0$ and define

$$E_n:=\{|u_n|>k\},\quad E_m:=\{|u_m|>k\}\quad \text{and}\quad E_{n,m}:=\{\big|T_k(u_n)-T_k(u_m)\big|>s\},$$

where $k>0$ is to be fixed. We note that

$$\{|u_n-u_m|>s\}\subset E_n\cup E_m\cup E_{n,m}$$

and hence

$$\text{meas}\{|u_n-u_m|>s\}\le\ \text{meas}(E_n)+\text{meas}(E_m)+\text{meas}(E_{n,m}). \tag{3.17}$$

Let $\epsilon>0$. Using Lemma 3.5, we choose $k=k(\epsilon)$ such that

$$\text{meas}(E_n)\le\frac{\epsilon}{3}\quad\text{and}\quad \text{meas}(E_m)\le\frac{\epsilon}{3}. \tag{3.18}$$

Since $\big(T_k(u_\epsilon)\big)_{\epsilon>0}$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Thus,

$$\text{meas}(E_{n,m})\le\frac{1}{s^{p_-}}\int_\Omega\big|T_k(u_n)-T_k(u_m)\big|^{p_-}\text{d}x\le\frac{\epsilon}{3}, \tag{3.19}$$

for all $n,m\ge n_0(s,\epsilon)$.

Finally, from (3.17)-(3.19), we obtain

$$\text{meas}\{|u_n-u_m|>s\}\le\ \epsilon,\qquad \text{for all } n,m\ge n_0(s,\epsilon). \tag{3.20}$$

Hence, the sequence $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure. We can extract a subsequence such that $u_\epsilon\to u$ a.e. in $\Omega$. As for $k>0$, $T_k$ is continuous, then $T_k(u_\epsilon)\to T_k(u)$ a.e. in $\Omega$ and $\sigma_k=T_k(u)$ a.e. in $\Omega$. Finally, using Lemma 2.3 we deduce that for all $k>0$, $T_k(u)\in$ dom($\beta$) a.e. in $\Omega$ and as dom($\beta$) is bounded, we deduce that $u\in$ dom($\beta$) a.e. in $\Omega$. $\square$

**Lemma 3.7.** *For all $k>0$,*

(i) $a(x,\nabla T_k(u_\epsilon))\rightharpoonup a(x,\nabla T_k(u))$ *in* $\big(L^{p'(\cdot)}(\Omega)\big)^N$,

(ii) $\nabla T_k(u_\epsilon)\to\nabla T_k(u)$ *a.e. in* $\Omega$,

(iii) $a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\to a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ *strongly in* $L^1(\Omega)$ *and a.e. in* $\Omega$,

(iv) $\nabla T_k(u_\epsilon)\to\nabla T_k(u)$ *in* $\big(L^{p(\cdot)}(\Omega)\big)^N$.

*Proof.* $(i)$ The sequence $\big(a(x,\nabla T_k(u_\epsilon))\big)_{\epsilon>0}$ is bounded in $\big(L^{p'(\cdot)}(\Omega)\big)^N$ according to (1.2). We can extract a subsequence such that $a(x,\nabla T_k(u_\epsilon))\rightharpoonup\Phi_k$ in $\big(L^{p'(\cdot)}(\Omega)\big)^N$. We have to show that $\Phi_k=a(x,\nabla T_k(u))$ a.e. $x\in\Omega$. The proof consists of four steps.

**Step 1:** We prove that for every function $h \in W^{1,+\infty}(\Omega)$, $h \geq 0$ with a compact support (supp $(h) \subset [-l,l] \subset \mathbb{R}$),

$$\limsup_{\epsilon \to 0} \int_\Omega a(x,\nabla u_\epsilon)\cdot \nabla \left[h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))\right] \mathrm{d}x \leq 0. \tag{3.21}$$

Let us take $\varphi = h(u_\epsilon)(T_k(u_\epsilon)-T_k(u))$, $k>0$ as a test function in (3.5). We have

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_\epsilon)\cdot \nabla \Big[h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\Big] \mathrm{d}x + \int_\Omega \beta_\epsilon(u_\epsilon) h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\mathrm{d}x \\
&= \int_\Omega f_\epsilon h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\mathrm{d}x.
\end{aligned} \tag{3.22}$$

For any $r>0$, sufficiently small, we consider

$$u_r = (u \wedge (M-r)) \vee (m+r).$$

For any $k>0$, $T_k(u_r) \in W^{1,p(\cdot)}(\Omega)$. Furthermore, we have

$$\begin{aligned}
&\int_\Omega \beta_\epsilon(u_\epsilon) h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\mathrm{d}x \\
&= \int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u_r)\big)\mathrm{d}x + \int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\mathrm{d}x \\
&= \int_\Omega h(u_\epsilon)\big(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(u_r)\big)\big(T_k(u_\epsilon)-T_k(u_r)\big)\mathrm{d}x + \int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)\big(T_k(u_\epsilon)-T_k(u_r)\big)\mathrm{d}x \\
&\quad + \int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\mathrm{d}x \\
&\geq \int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)\big(T_k(u_\epsilon)-T_k(u_r)\big)\mathrm{d}x + \int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\mathrm{d}x \\
&=: I_1 + I_2, \qquad \text{(since } h\geq 0 \text{ and } \beta_\epsilon \text{ is nondecreasing).}
\end{aligned}$$

Note that

$$m+r \leq u_r \leq M-r.$$

Since $\beta_\epsilon$ is nondecreasing we have

$$\beta_\epsilon(m+r) \leq \beta_\epsilon(u_r) \leq \beta_\epsilon(M-r) \Rightarrow \big|\beta_\epsilon(u_r)\big| \leq \max\big\{\big|\beta_\epsilon(m+r)\big|, \big|\beta_\epsilon(M-r)\big|\big\}.$$

We deduce that $h(u_\epsilon)\beta_\epsilon(u_r)\big(T_k(u_\epsilon)-T_k(u_r)\big) \in L^1(\Omega)$.

Since

$$h(u_\epsilon)\beta_\epsilon(u_r)\big(T_k(u_\epsilon)-T_k(u_r)\big) \;\to\; h(u)\beta_0(u_r)\big(T_k(u)-T_k(u_r)\big) \quad \text{a.e. in } \Omega \text{ as } \epsilon \to 0,$$

using Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{\epsilon\to 0} I_1 &= \lim_{\epsilon \to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon(u_r)\big(T_k(u_\epsilon)-T_k(u_r)\big)\mathrm{d}x \\
&= \int_\Omega h(u)\beta_0(u_r)\big(T_k(u)-T_k(u_r)\big)\mathrm{d}x.
\end{aligned} \tag{3.23}$$

The term $I_2$ can be written as

$$\begin{aligned} I_2 =& \int_\Omega f_\epsilon h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x - \int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\left[h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\right]\mathrm{d}x \\ =& \int_\Omega f_\epsilon h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x - \int_\Omega h(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x \\ & -\int_\Omega h'(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x. \end{aligned}$$

Note that $f_\epsilon h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\to 0$ a.e. in $\Omega$ as $r\to 0$,

$$\big|f_\epsilon h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\big| \le 2kl|f| \in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we get

$$\lim_{r\to 0}\int_\Omega f_\epsilon h(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x = 0. \tag{3.24}$$

As $h(u_\epsilon)a(x,\nabla u_\epsilon)=h(u_\epsilon)a(x,\nabla T_l(u_\epsilon))$ is uniformly bounded in $\big(L^{p'(\cdot)}(\Omega)\big)^N$ (by assumption (1.2) and relation (3.6)) and $\nabla\big(T_k(u_r)-T_k(u)\big)\rightharpoonup 0$ as $r\to 0$, then

$$\lim_{r\to 0}\int_\Omega h(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x = 0. \tag{3.25}$$

Recall that $m+r\le u_r\le M-r$ and by Lemma 3.6 (part (ii)), $u\in\mathrm{dom}(\beta)\subset[m,M]$. Then

$$|T_k(u_r)-T_k(u)|\le r.$$

Thus, for the third term of $I_2$, we have

$$\begin{aligned} &\left|\int_\Omega h'(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\right| \\ =&\left|\int_\Omega h'(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)a(x,\nabla T_l(u_\epsilon))\cdot\nabla T_l u_\epsilon\,\mathrm{d}x\right| \\ \le& rC(h)\int_\Omega a(x,\nabla T_l(u_\epsilon))\cdot\nabla T_l(u_\epsilon)\,\mathrm{d}x \\ \le& rC(h)\left[\int_\Omega f_\epsilon T_l(u_\epsilon)\,\mathrm{d}x - \int_\Omega \beta_\epsilon(u_\epsilon)T_l(u_\epsilon)\,\mathrm{d}x\right] \\ \le& rC(h,l,\Omega,\|f\|_1), \end{aligned}$$

where $C(h,l,\Omega,\|f\|_1)$ is a constant depending on $h,l,\Omega$ and $\|f\|_1$. Then, we get

$$\lim_{r\to 0}\int_\Omega h'(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x = 0. \tag{3.26}$$

Therefore, combining (3.24)-(3.26) we obtain

$$\lim_{r\to 0} I_2 = \lim_{r\to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon(u_\epsilon)\big(T_k(u_r)-T_k(u)\big)\,\mathrm{d}x = 0. \tag{3.27}$$

Now, let us see that

$$h(u)\beta_0\big(u_r\big)\big(T_k(u)-T_k(u_r)\big)\geq 0.$$

Indeed,

$$\begin{aligned}h(u)\beta_0\big(u_r\big)\big(T_k(u)-T_k(u_r)\big)=&h(u)\beta_0(M-r)\big(T_k(u)-T_k(M-r)\big)\chi_{\{M-r\leq u\leq M\}}\\&+h(u)\beta_0(m+r)\big(T_k(u)-T_k(m+r)\big)\chi_{\{m\leq u\leq m+r\}}\geq 0,\end{aligned}$$

since $0\in\beta(0), m+r\leq 0\leq M-r$ and $T_k$ is nondecreasing. It follows that

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\beta_\epsilon\big(u_r\big)\big(T_k(u_\epsilon)-T_k(u_r)\big)\,\mathrm{d}x\geq 0. \tag{3.28}$$

We also have $f_\epsilon h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\to 0$ a.e. in $\Omega$ as $\epsilon\to 0$,

$$\big|f_\epsilon h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\big|\leq 2kl|f|\in L^1(\Omega).$$

Then, by the Lebesgue generalized convergence theorem, we get

$$\lim_{\epsilon\to 0}\int_\Omega f_\epsilon h(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\,\mathrm{d}x=0. \tag{3.29}$$

Passing to the limit in (3.22) as $\epsilon\to 0$ and combining (3.27)-(3.29), we obtain (3.21).

**Step 2:** We prove that

$$\limsup_{l\to+\infty}\limsup_{\epsilon\to 0}\int_{\{l<|u_\epsilon|<l+1\}} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\leq 0. \tag{3.30}$$

Let us take for $l>0$, $\varphi=T_1(u_\epsilon-T_l(u_\epsilon))$ as a test function in (3.5). We have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla T_1\big(u_\epsilon-T_l(u_\epsilon)\big)\,\mathrm{d}x+\int_\Omega\beta_\epsilon\big(u_\epsilon\big)T_1\big(u_\epsilon-T_l(u_\epsilon)\big)\,\mathrm{d}x\\&=\int_\Omega f_\epsilon T_1\big(u_\epsilon-T_l(u_\epsilon)\big)\,\mathrm{d}x.\end{aligned} \tag{3.31}$$

We have

$$T_1\big(u_\epsilon-T_l(u_\epsilon)\big)=\begin{cases}T_1\big(u_\epsilon+l\big), & \text{if } u_\epsilon<-l,\\ 0, & \text{if } |u_\epsilon|<l,\\ T_1\big(u_\epsilon-l\big), & \text{if } u_\epsilon>l.\end{cases}$$

Then

$$\beta_\epsilon(u_\epsilon)T_1\big(u_\epsilon-T_l(u_\epsilon)\big)=\begin{cases}\beta_\epsilon(u_\epsilon)T_1\big(u_\epsilon+l\big), & \text{if } u_\epsilon<-l,\\ 0, & \text{if } |u_\epsilon|<l,\\ \beta_\epsilon(u_\epsilon)T_1\big(u_\epsilon-l\big), & \text{if } u_\epsilon>l.\end{cases}$$

If $u_\epsilon<-l$ or $u_\epsilon>l$, then $u_\epsilon$ and $T_1\big(u_\epsilon-T_l(u_\epsilon)\big)$ have the same sign. We conclude that the second term of the left-hand side of (3.31) is nonnegative, i.e.

$$\int_\Omega\beta_\epsilon\big(u_\epsilon\big)T_1\big(u_\epsilon-T_l(u_\epsilon)\big)\,\mathrm{d}x\geq 0. \tag{3.32}$$

The first term of (3.31) is written as follows

$$\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla T_1\big(u_\epsilon - T_l(u_\epsilon)\big)\,\mathrm{d}x = \int_{\{l<|u_\epsilon|<l+1\}} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x. \tag{3.33}$$

As in step 1, we show that

$$\lim_{\epsilon\to 0}\int_\Omega f_\epsilon T_1\big(u_\epsilon - T_l(u_\epsilon)\big)\,\mathrm{d}x = \int_\Omega f T_1(u-T_l(u))\,\mathrm{d}x.$$

We also have $T_1(u-T_l(u))\to 0$ a.e. in $\Omega$ as $l\to+\infty$ and $\big|fT_1(u-T_l(u))\big|\le|f|\in L^1(\Omega)$. Then, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{l\to+\infty}\int_\Omega fT_1(u-T_l(u))\,\mathrm{d}x = 0.$$

We deduce that

$$\lim_{l\to+\infty}\lim_{\epsilon\to 0}\int_\Omega f_\epsilon T_1\big(u_\epsilon - T_l(u_\epsilon)\big)\,\mathrm{d}x = 0. \tag{3.34}$$

Passing to the limit as $\epsilon\to 0$, to the limit as $l\to+\infty$ in (3.31) and using (3.32)-(3.34), we deduce (3.30).

**Step 3:** We prove that for every $k>0$,

$$\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\,\mathrm{d}x \le 0. \tag{3.35}$$

For $\nu>k$, we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\big[h_\nu(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)\big]\,\mathrm{d}x\\
=&\int_{\{|u_\epsilon|\le k\}} h_\nu(u_\epsilon)a(x,\nabla T_k(u_\epsilon))\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\,\mathrm{d}x\\
&\quad+\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla\big[-T_k(u)\big]\,\mathrm{d}x\\
&\quad+\int_\Omega h'_\nu(u_\epsilon)\big[T_k(u_\epsilon)-T_k(u)\big]a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x.
\end{aligned} \tag{3.36}$$

Since $\nu>k$, on the set $\{|u_\epsilon|\le k\}$, it follows that $h_\nu(u_\epsilon)=1$ and we get

$$\begin{aligned}
&\int_{\{|u_\epsilon|\le k\}} h_\nu(u_\epsilon)a(x,\nabla T_k(u_\epsilon))\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\,\mathrm{d}x\\
=&\int_{\{|u_\epsilon|\le k\}} a(x,\nabla T_k(u_\epsilon))\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\,\mathrm{d}x\\
=&\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\,\mathrm{d}x.
\end{aligned} \tag{3.37}$$

The second term of the right-hand side of (3.36) can be written as

$$\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla\left[-T_k(u)\right]\mathrm{d}x = -\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla T_{\nu+1}(u_\epsilon))\cdot\nabla T_k(u)\,\mathrm{d}x.$$

Using the Lebesgue dominated convergence theorem, we deduce that

$$h_\nu(u_\epsilon)\chi_{\{|u_\epsilon|>k\}}\nabla T_k(u) \to h_\nu(u)\chi_{\{|u_\epsilon|>k\}}\nabla T_k(u) \quad \text{strongly in } \left(L^{p(\cdot)}(\Omega)\right)^N.$$

The sequence $(a(x,\nabla T_{\nu+1}(u_\epsilon)))_{\epsilon>0}$ is bounded in $\left(L^{p'(\cdot)}(\Omega)\right)^N$, then it converges weakly in $\left(L^{p'(\cdot)}(\Omega)\right)^N$ to $\Gamma_{\nu+1}$. By the Lebesgue dominated convergence theorem, we find

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left(-\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla T_{\nu+1}(u_\epsilon))\cdot\nabla T_k(u)\,\mathrm{d}x\right)\\
=&-\int_{\{|u_\epsilon|>k\}} h_\nu(u)\Gamma_{\nu+1}\cdot\nabla T_k(u)\,\mathrm{d}x = 0,
\end{aligned}$$

i.e.

$$\lim_{\epsilon\to 0}\left(-\int_{\{|u_\epsilon|>k\}} h_\nu(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla T_k(u)\,\mathrm{d}x\right) = 0. \tag{3.38}$$

Considering the third term of the right-hand side of (3.36), we have

$$\begin{aligned}
&-\int_\Omega h'_\nu(u_\epsilon)\left[T_k(u_\epsilon)-T_k(u)\right]a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\\
\leq&\left|\int_\Omega h'_\nu(u_\epsilon)\left[T_k(u_\epsilon)-T_k(u)\right]a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\right|\\
\leq&2k\int_{\{\nu<|u_\epsilon|<\nu+1\}} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x.
\end{aligned}$$

Using the result of step 2, we obtain

$$\limsup_{\nu\to+\infty}\limsup_{\epsilon\to 0}\left(-\int_\Omega h'_\nu(u_\epsilon)\left[T_k(u_\epsilon)-T_k(u)\right]a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\right)\leq 0. \tag{3.39}$$

Applying (3.21) with $h$ replaced by $h_\nu$, $\nu>k$ in (3.36) and using (3.37)-(3.39), it follows that

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla\left[T_k(u_\epsilon)-T_k(u)\right]\mathrm{d}x\\
\leq&\limsup_{\nu\to+\infty}\limsup_{\epsilon\to 0}\left(-\int_\Omega h'_\nu(u_\epsilon)\left[T_k(u_\epsilon)-T_k(u)\right]a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,\mathrm{d}x\right)\leq 0.
\end{aligned}$$

Therefore, (3.35) follows.

**Step 4:** Now, we prove by standard monotonicity arguments that for all $k>0,\Phi_k=a(x,\nabla T_k(u))$ a.e. in $\Omega$. Let $\varphi\in\mathcal{D}(\Omega)$ and $\lambda\in\mathbb{R}^*$. Using (3.35), (1.3) and Lemma 3.6, we get

$$\begin{aligned}
&\lambda\lim_{\epsilon\to0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\varphi\mathrm{d}x\\
\geq&\limsup_{\epsilon\to0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\left[T_k(u_\epsilon)-T_k(u)+\lambda\varphi\right]\mathrm{d}x\\
\geq&\limsup_{\epsilon\to0}\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\cdot\nabla\left[T_k(u_\epsilon)-T_k(u)+\lambda\varphi\right]\mathrm{d}x\\
\geq&\lambda\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\cdot\nabla\varphi\mathrm{d}x.
\end{aligned}\tag{3.40}$$

Dividing the first and the last integrals of (3.40) by $\lambda>0$ and by $\lambda<0$ respectively, passing to the limit with $\lambda\to0$ it follows that

$$\lim_{\epsilon\to0}\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\varphi\mathrm{d}x=\int_\Omega a(x,\nabla T_k(u)\cdot\nabla\varphi\mathrm{d}x.$$

This means that

$$\int_\Omega\Phi_k\nabla\varphi\mathrm{d}x=\int_\Omega a(x,\nabla T_k(u)\cdot\nabla\varphi\mathrm{d}x,\qquad\forall\,k>0,$$

and so $\Phi_k=a(x,\nabla T_k(u))$ in $\mathcal{D}'(\Omega)$ for all $k>0$. Hence $\Phi_k=a(x,\nabla T_k(u))$ a.e. in $\Omega$ and we have

$$a(x,\nabla T_k(u_\epsilon))\rightharpoonup a(x,\nabla T_k(u)),\quad\text{in}\ \left(L^{p'(\cdot)}(\Omega)\right)^N.$$

(ii) From (3.35) and (1.3), we deduce that for all $k>0$

$$\lim_{\epsilon\to0}\int_\Omega\left[a(x,\nabla u_\epsilon)-a(x,\nabla u)\right]\cdot\nabla\left[T_k(u_\epsilon)-T_k(u)\right]\mathrm{d}x=0.$$

Now, set

$$g_\epsilon(x)=\left[a(x,\nabla u_\epsilon)-a(x,\nabla u)\right]\cdot\nabla\left[T_k(u_\epsilon)-T_k(u)\right]\geq0.$$

$g_\epsilon(x)\to0$ strongly in $L^1(\Omega)$ as $\epsilon\to0$. Up to a subsequence, $g_\epsilon(x)\to0$ a.e. in $\Omega$, which means that there exists $\omega\subset\Omega$ such that meas$(\omega)=0$ and $g_\epsilon(x)\to0$ in $\Omega\backslash\omega$. Let $x\in\Omega\backslash\omega$. Using assumptions (1.4) and (1.2), it follows that the sequence $\left(\nabla T_k(u_\epsilon(x))\right)_{\epsilon>0}$ is bounded in $\mathbb{R}^N$ and so we can extract a subsequence which converges to some $\theta$ in $\mathbb{R}^N$.

Passing to the limit in the expression of $g_\epsilon(x)$, it follows that

$$0=\left[a(x,\theta)-a(x,\nabla T_k(u))\right]\cdot\left[\theta-T_k(u)\right]$$

and it yields $\theta=\nabla T_k(u)$, $\forall x\in\Omega\backslash\omega$.

As the limit dœsn't depend on the subsequence, the whole sequence $\left(\nabla T_k(u_\epsilon(x))\right)_{\epsilon>0}$ converges to $\theta$ in $\mathbb{R}^N$. This means that $\nabla T_k(u_\epsilon)\to\nabla T_k(u)$ a.e. in $\Omega$.

(iii) The continuity of $a(x,\xi)$ with respect to $\xi\in\mathbb{R}^N$ gives us

$$a(x,\nabla T_k(u_\epsilon))\to a(x,\nabla T_k(u)), \quad \text{a.e. in } \Omega.$$

Therefore

$$a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\to a(x,\nabla T_k(u))\cdot\nabla T_k(u), \quad \text{a.e. in } \Omega.$$

Setting $z_\epsilon=a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)$ and $z=a(x,\nabla T_k(u))\cdot\nabla T_k(u)$, we have

$$z_\epsilon>0, \quad z_\epsilon\to z \quad \text{a.e. in } \Omega, z\in L^1(\Omega), \quad \int_\Omega z_\epsilon\,\mathrm{d}x\to\int_\Omega z_\epsilon\,\mathrm{d}x,$$

and as

$$\int_\Omega |z_\epsilon-z|\,\mathrm{d}x=2\int_\Omega (z-z_\epsilon)^+\,\mathrm{d}x+\int_\Omega (z_\epsilon-z)\,\mathrm{d}x, \quad \text{and} \quad (z-z_\epsilon)^+\le z,$$

it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{\epsilon\to 0}\int_\Omega |z_\epsilon-z|\,\mathrm{d}x=0,$$

which means that

$$a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\to a(x,\nabla T_k(u))\cdot\nabla T_k(u), \text{ strongly in } L^1(\Omega).$$

(iv) Set

$$h_\epsilon=\left|\nabla T_k(u_\epsilon)\right|^{p(x)}, \quad h=\left|\nabla T_k(u)\right|^{p(x)},$$
$$g_\epsilon=a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon), \quad g=a(x,\nabla T_k(u))\cdot\nabla T_k(u).$$

We have:

• $h_\epsilon$ is a sequence of measurable functions, $h$ is a measurable function and according to (ii), $h_\epsilon\to h$ a.e. in $\Omega$.

• Using (iii), we have $(g_\epsilon)_{\epsilon>0}\subset L^1(\Omega)$, $g_\epsilon\to g$ a.e. in $\Omega$, $g_\epsilon\to g$ in $L^1(\Omega)$ and using (1.4), we have $|h_\epsilon|\le Cg_\epsilon$.

Then, by Lemma 2.6, we have

$$\int_\Omega h_\epsilon\,\mathrm{d}x\to\int_\Omega h\,\mathrm{d}x, \quad \text{i.e.} \quad \int_\Omega \left|\nabla T_k(u_\epsilon)\right|^{p(x)}\mathrm{d}x\to\int_\Omega \left|\nabla T_k(u)\right|^{p(x)}\mathrm{d}x.$$

Note also that since $\Omega$ is bounded, we deduce from (ii) that the sequence $\left(\nabla T_k(u_\epsilon)\right)_{\epsilon>0}$ converges to $\nabla T_k(u)$ in $\Omega$ in measure. Then, by Lemma 2.5 we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega \left|\nabla T_k(u_\epsilon)-\nabla T_k(u)\right|^{p(x)}\mathrm{d}x=0,$$

i.e. $\nabla T_k(u_\epsilon)\to\nabla T_k(u)$ in $\left(L^{p(\cdot)}(\Omega)\right)^N$. $\square$

**Remark 3.2.** By Lemma 3.6 and Lemma 3.7-$(iv)$, we deduce that $u\in\mathcal{T}_{\mathcal{H}}^{1,p(\cdot)}(\Omega)$.

The following lemma is useful for the next.

**Lemma 3.8.** *For any* $h\in C_c^1(\mathbb{R})$ *and* $\varphi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$,

$$\nabla[h(u_n)\varphi]\longrightarrow\nabla[h(u)\varphi],\quad\text{strongly in } (L^{p(\cdot)}(\Omega))^N \text{ as } \epsilon\to 0.$$

*Proof.* For any $h\in C_c^1(\mathbb{R})$ and $\varphi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\begin{aligned}\nabla[h(u_\epsilon)\varphi]-\nabla[h(u)\varphi]&=(h(u_\epsilon)-h(u))\nabla\varphi+h'(u_\epsilon)\varphi[\nabla u_\epsilon-\nabla u]+(h'(u_\epsilon)-h'(u))\varphi\nabla u\\&:=\psi_1^\epsilon+\psi_2^\epsilon+\psi_3^\epsilon.\end{aligned}\tag{3.41}$$

For the term $\psi_1^\epsilon$, we consider

$$\rho_{p(\cdot)}(\psi_1^\epsilon)=\int_\Omega|(h(u_\epsilon)-h(u))\nabla\varphi|^{p(x)}\mathrm{d}x.$$

Set

$$\Theta_1^\epsilon(x)=|(h(u_\epsilon)-h(u))\nabla\varphi|^{p(x)}.$$

We have $\Theta_1^\epsilon(x)\to 0$ a.e. $x\in\Omega$ as $\epsilon\to 0$ and

$$|\Theta_1^\epsilon(x)|\le C(h,p_-,p_+)|\nabla\varphi|^{p(x)}\in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we get that $\lim_{\epsilon\to 0}\rho_{p(\cdot)}(\psi_1^\epsilon)=0$. Hence,

$$\|\psi_1^\epsilon\|_{L^{p(\cdot)}(\Omega)}\to 0,\quad\text{ as } \epsilon\to 0.\tag{3.42}$$

For the term $\psi_2^\epsilon$ we consider

$$\rho_{p(\cdot)}(\psi_2^\epsilon)=\int_\Omega|h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon)-\nabla T_l(u))|^{p(x)}\mathrm{d}x$$

for some $l>0$ such that $\operatorname{supp}(h)\subset[-l,l]$. Set

$$\Theta_2^\epsilon(x)=|h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon)-\nabla T_l(u))|^{p(x)}.$$

We have $\Theta_2^\epsilon(x)\to 0$ a.e. $x\in\Omega$ as $\epsilon\to 0$ and

$$|\Theta_2^\epsilon(x)|\le C(h,p_-,p_+,\|\varphi\|_\infty)|\nabla T_l(u_\epsilon)-\nabla T_l(u)|^{p(x)}.$$

Since $\nabla T_l(u_\epsilon)\to\nabla T_l(u)$ strongly in $\big(L^{p(\cdot)}(\Omega)\big)^N$, we get

$$\rho_{p(\cdot)}(\nabla T_l(u_\epsilon)-\nabla T_l(u))\to 0,\quad\text{ as } \epsilon\to 0,$$

which is equivalent to say

$$\lim_{\epsilon\to 0}\int_\Omega|\nabla T_l(u_\epsilon)-\nabla T_l(u)|^{p(x)}\mathrm{d}x=0.$$

Then $|\nabla T_l(u_\epsilon)-\nabla T_l(u)|^{p(\cdot)}\to 0$ strongly in $L^1(\Omega)$. By the Lebesgue generalized convergence theorem, one has

$$\lim_{\epsilon\to 0}\int_\Omega \Theta_2^\epsilon(x)\,\mathrm{d}x=\lim_{\epsilon\to 0}\rho_{p(\cdot)}(\psi_2^\epsilon)=0.$$

Hence,

$$\|\psi_2^\epsilon\|_{L^{p(\cdot)}(\Omega)}\to 0,\quad \text{as } \epsilon\to 0. \tag{3.43}$$

For the term $\psi_3^\epsilon$ we consider

$$\rho_{p(\cdot)}(\psi_3^\epsilon)=\int_\Omega |(h'(u_\epsilon)-h'(u))\varphi\nabla u|^{p(x)}\mathrm{d}x.$$

Set

$$\Theta_3^\epsilon(x)=|(h'(u_\epsilon)-h'(u))\varphi\nabla u|^{p(x)}.$$

We have $\Theta_3^\epsilon(x)\to 0$ a.e. $x\in\Omega$ as $\epsilon\to 0$ and

$$|\Theta_3^\epsilon(x)|\le C(h,p_-,p_+,\|\varphi\|_\infty)|\nabla T_l(u)|^{p(x)}\in L^1(\Omega),$$

with some $l>0$ such that $\operatorname{supp}(h)\subset[-l,l]$. Then, by the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\rho_{p(\cdot)}(\psi_3^\epsilon)=0.$$

Hence,

$$\|\psi_3^\epsilon\|_{L^{p(\cdot)}(\Omega)}\to 0,\quad \text{as } \epsilon\to 0. \tag{3.44}$$

Thanks to (3.42)-(3.44), we get

$$\left\|\psi_1^\epsilon+\psi_2^\epsilon+\psi_3^\epsilon\right\|_{L^{p(\cdot)}(\Omega)}\to 0,\quad \text{as } \epsilon\to 0,$$

and the lemma is proved. □

Now, we want to pass to the limit in the first integral of (3.5). Since, for any $k>0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k\in\mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon)\overset{*}{\rightharpoonup} z_k \ \text{ in } \mathcal{M}_b(\Omega),\quad \text{as } \epsilon\to 0.$$

Moreover, for any $\varphi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \varphi\,\mathrm{d}z_k=\int_\Omega \varphi h_k(u) f\,\mathrm{d}x-\int_\Omega a(x,\nabla u)\cdot\nabla(h_k(u)\varphi)\mathrm{d}x,$$

which implies that $z_k\in\mathcal{M}_b^{p(\cdot)}(\Omega)$ and, for any $k\le l$,

$$z_k=z_l,\quad \text{on } [|T_k(u)|<k].$$

Let us consider the Radon measure $z$ defined by

$$\begin{cases} z=z_k, & \text{on } [|T_k(u)|<k] \text{ for } k\in\mathbb{N}^*,\\ z=0, & \text{on } \bigcap\limits_{k\in\mathbb{N}^*}[|T_k(u)|=k]. \end{cases} \tag{3.45}$$

For any $h\in\mathcal{C}_c^1(\mathbb{R})$, $h(u)\in L^\infty(\Omega,d|z|)$ and

$$\int_\Omega h(u)\varphi\,\mathrm{d}z=-\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)\,\mathrm{d}x+\int_\Omega h(u)\varphi f\,\mathrm{d}x,$$

for any $\varphi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$. Indeed, let $k_0>0$ be such that $\mathrm{supp}(h)\subseteq[-k_0,k_0]$,

$$\begin{aligned} &\int_\Omega h(u)\varphi\,\mathrm{d}z=\int_\Omega h(u)\varphi\,\mathrm{d}z_{k_0}\\ =&-\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla(h(u_\epsilon)\varphi)\,\mathrm{d}x+\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi f_\epsilon\,\mathrm{d}x\\ =&-\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_{k_0}(u_\epsilon))\cdot\nabla(h(u_\epsilon)\varphi)\,\mathrm{d}x+\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi f_\epsilon\,\mathrm{d}x\\ =&-\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)\,\mathrm{d}x+\int_\Omega h(u)\varphi f\,\mathrm{d}x. \end{aligned} \tag{3.46}$$

Moreover, we have

**Lemma 3.9.** *The Radon-Nikodym decomposition of the measure $z$ given by* (3.45) *with respect to $\mathcal{L}^N$,*

$$z=b\mathcal{L}^N+\mu,\quad \textit{with}\ \ \mu\perp\mathcal{L}^N,$$

*satisfies the following properties*

$$\begin{cases} b\in\beta(u)\ \mathcal{L}^N-\textit{a.e. in }\Omega,\ b\in L^1(\Omega),\ \mu\in\mathcal{M}_b^{p(\cdot)}(\Omega),\\ \mu^+\ \textit{is concentrated on } [u=M]\\ \ \textit{and } \mu^-\ \textit{is concentrated on } [u=m]. \end{cases}$$

*Proof.* Since, for any $\epsilon>0$, $z_\epsilon\in\partial j_\epsilon(u_\epsilon)$, we have

$$j(t)\geq j_\epsilon(t)\geq j_\epsilon(u_\epsilon)+(t-u_\epsilon)z_\epsilon\ \ \mathcal{L}^N-\text{a.e. in }\Omega,\quad \forall t\in\mathbb{R}.$$

Then, for any $h\in\mathcal{C}_c(\mathbb{R})$, $h\geq 0$ and $k>0$ such that $\mathrm{supp}(h)\subseteq[-k,k]$, we have

$$\xi h(u_\epsilon)j(t)\geq\xi h(u_\epsilon)j_\epsilon(u_\epsilon)+(t-u_\epsilon)\xi h(u_\epsilon)h_k(u_\epsilon)z_\epsilon.$$

In addition, for any $0<\epsilon<\tilde{\epsilon}$, we have

$$\varphi h(u_\epsilon)j(t)\geq\varphi h(u_\epsilon)j_{\tilde{\epsilon}}(u_\epsilon)+(t-u_\epsilon)\varphi h(u_\epsilon)h_k(u_\epsilon)z_\epsilon,$$

and, integrating over $\Omega$ yields

$$\int_\Omega \varphi h(u_\epsilon) j(t) \mathrm{d}x \geq \int_\Omega \varphi h(u_\epsilon) j_{\tilde{\epsilon}}(u_\epsilon) \mathrm{d}x + \int_\Omega (t-u_\epsilon)\varphi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon \mathrm{d}x.$$

As $\epsilon \to 0$, we get by using Fatou's Lemma

$$\int_\Omega \varphi h(u) j(t) \mathrm{d}x \geq \int_\Omega \varphi h(u) j_{\tilde{\epsilon}}(u) \mathrm{d}x + \liminf_{\epsilon\to 0} \int_\Omega (t-u_\epsilon)\varphi h(u_\epsilon) h_k(u_\epsilon) z_\epsilon \mathrm{d}x.$$

Now, for any $\varphi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, setting

$$\tilde{h}(r) = (t-r)h(r),$$

we have

$$\begin{aligned}\lim_{\epsilon\to 0} \int_\Omega (t-u_\epsilon) h(u_\epsilon)\varphi h_k(u_\epsilon) z_\epsilon \mathrm{d}x = \lim_{\epsilon\to 0}\int_\Omega \tilde{h}(u_\epsilon)\varphi h_k(u_\epsilon) z_\epsilon \mathrm{d}x = \int_\Omega (t-u)h(u)\varphi \mathrm{d}z_k \\ = \int_\Omega (t-u)h(u)\varphi \mathrm{d}z.\end{aligned}$$

So,

$$\int_\Omega \varphi h(u) j(t) \mathrm{d}x \geq \int_\Omega \varphi h(u) j_{\tilde{\epsilon}}(u) \mathrm{d}x + \int_\Omega \varphi (t-u) h(u) \mathrm{d}z.$$

As $\tilde{\epsilon} \to 0$, we get by using again Fatou's Lemma

$$\int_\Omega \varphi h(u) j(t) \mathrm{d}x \geq \int_\Omega \varphi h(u) j(u) \mathrm{d}x + \int_\Omega \varphi (t-u) h(u) \mathrm{d}z.$$

From the inequality above, we have

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)z, \quad \text{in } \mathcal{M}_b(\Omega), \forall t \in \mathbb{R}. \tag{3.47}$$

Using the Radon-Nikodym decomposition of $z$ we have $z = b\mathcal{L}^N + \mu$ with $\mu \perp \mathcal{L}^N$, $b \in L^1(\Omega)$, then comparing the regular part and the singular part of (3.47), for any $h \in \mathcal{C}_c(\mathbb{R})$, we obtain

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)b \;\; \mathcal{L}^N - \text{a.e. in } \Omega, \quad \forall t \in \mathbb{R} \tag{3.48}$$

and

$$(t-u)h(u)\mu \leq 0 \;\; \text{ in } \mathcal{M}_b(\Omega), \quad \forall\, t \in \overline{\mathrm{dom}(j)}. \tag{3.49}$$

From (3.48) we get

$$j(t) \geq j(u) + (t-u)b \;\; \mathcal{L}^N - \text{a.e. in } \Omega, \quad \forall t \in \mathbb{R},$$

so that $b \in \partial j(u)$ $\mathcal{L}^N -$a.e in $\Omega$. As to (3.49), this implies that for any $t \in \overline{\mathrm{dom}(j)}$,

$$\mu \geq 0 \;\; \text{ in } [u \in (t,\infty) \cap \mathrm{supp}(h)] \tag{3.50}$$

and

$$\mu \leq 0 \quad \text{in } [u \in (-\infty, t) \cap \operatorname{supp}(h)]. \tag{3.51}$$

In particular, this implies that

$$\mu([m < u < M]) = 0$$

and so that

$$\mu^- \text{ is concentrated on } [u = m] \quad (\text{resp. } \mu^+ \text{ is concentrated on } [u = M])$$

and the proof of the Lemma 3.9 is finished. □

To finish the proof of Theorem 3.1, we consider $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi$ as test function in (3.5). We get

$$\int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla [h(u_\epsilon)\varphi] \mathrm{d}x + \int_\Omega \beta_\epsilon(u_\epsilon) h(u_\epsilon) \varphi \mathrm{d}x = \int_\Omega h(u_\epsilon) \varphi f_\epsilon \mathrm{d}x. \tag{3.52}$$

By the Lebesgue dominated convergence theorem, we have for the term in the right hand side of (3.52),

$$\lim_{\epsilon \to 0} \int_\Omega h(u_\epsilon) \varphi f_\epsilon \mathrm{d}x = \int_\Omega h(u) \varphi f \mathrm{d}x.$$

The first term of (3.52) can be written as

$$\int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla [h(u_\epsilon)\varphi] \mathrm{d}x = \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)) \cdot \nabla [h_0(u_\epsilon)\varphi] \mathrm{d}x,$$

for some $l_0 > 0$ so that, by Lemma 3.7-$(i)$ and Lemma 3.9, we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla [h(u_\epsilon)\varphi] \mathrm{d}x &= \lim_{\epsilon \to 0} \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)) \cdot \nabla [h_0(u_\epsilon)\varphi] \mathrm{d}x \\
&= \int_\Omega a(x, \nabla T_{l_0+1}(u)) \cdot \nabla [h_0(u)\varphi] \mathrm{d}x \\
&= \int_\Omega a(x, \nabla u) \cdot \nabla [h(u)\varphi] \mathrm{d}x.
\end{aligned}$$

Thanks to the convergence of Lemma 3.9 and Lemma 3.7-$(i)$ we have from (3.52)

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_\Omega \beta_\epsilon(u_\epsilon) h(u_\epsilon) \varphi \mathrm{d}x &= \int_\Omega h(u) \varphi f \mathrm{d}x - \int_\Omega a(x, \nabla u) \cdot \nabla [h(u)\varphi] \mathrm{d}x \\
&= \int_\Omega h(u) \varphi \mathrm{d}z = \int_\Omega h(u) b \varphi \mathrm{d}x + \int_\Omega h(u) \varphi \mathrm{d}\mu \cdot
\end{aligned}$$

Letting $\epsilon$ goes to 0 in (3.52), we obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla [h(u)\varphi] \mathrm{d}x + \int_\Omega h(u) b \varphi \mathrm{d}x + \int_\Omega h(u) \varphi \mathrm{d}\mu = \int_\Omega h(u) \varphi f \mathrm{d}x. \tag{3.53}$$

In (3.53), we take $h\in C_c^1(\mathbb{R})$ such that $[m,M]\subset \operatorname{supp}(h)\subset[-l,l]$ and $h(s)=1$ for all $s\in[-l,l]$. As $u\in \operatorname{dom}(\beta)$, then $h(u)=1$ and it yields that $(u,b)$ is a solution of the problem (1.1). Now, let us prove the uniqueness of the solution for the problem (1.1). Suppose that $(u_1,w_1),(u_2,w_2)$ are two solutions of the problem (1.1). For $u_1$, we choose $\xi=u_2$ as test function in (3.2) to get

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-u_2)\mathrm{d}x+\int_\Omega w_1T_k(u_1-u_2)\mathrm{d}x\le\int_\Omega fT_k(u_1-u_2)\mathrm{d}x.$$

Similarly we get for $u_2$

$$\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-u_1)\mathrm{d}x+\int_\Omega w_2T_k(u_2-u_1)\mathrm{d}x\le\int_\Omega fT_k(u_2-u_1)\mathrm{d}x.$$

Adding these two last inequalities yields

$$\int_\Omega\Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big)\cdot\nabla T_k(u_1-u_2)\mathrm{d}x+\int_\Omega(w_1-w_2)T_k(u_1-u_2)\mathrm{d}x\le 0. \tag{3.54}$$

For any $k>0$, from (3.54) it yields

$$\int_\Omega\Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big)\cdot\nabla T_k(u_1-u_2)\mathrm{d}x=0. \tag{3.55}$$

From (3.55), it follows that there exists a constant $c$ such that $u_1-u_2=c$ a.e. in $\Omega$. At last, let us see that $w_1=w_2$ a.e. in $\Omega$ and $\nu_1=\nu_2$. Indeed for any $\varphi\in\mathcal{D}(\Omega)$, taking $\varphi$ as test function in (3.1) for the solutions $(u_1,w_1)$ and $(u_1,w_2)$, after substraction, we get

$$\int_\Omega(w_1-w_2)\varphi\mathrm{d}x+\int_\Omega\varphi\mathrm{d}(\nu_1-\nu_2)=0.$$

Hence

$$\int_\Omega w_1\varphi\mathrm{d}x+\int_\Omega\varphi\mathrm{d}\nu_1=\int_\Omega w_2\varphi\mathrm{d}x+\int_\Omega\varphi\mathrm{d}\nu_2.$$

Therefore

$$w_1\mathcal{L}^N+\nu_1=w_2\mathcal{L}^N+\nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get

$$w_1=w_2,\quad \text{a.e. in }\ \Omega,\quad \text{and } \nu_1=\nu_2.$$

To end the proof of Theorem 3.1, we prove (3.3). We take $\varphi=T_1(u_\epsilon-T_n(u_\epsilon))$ as test function in (3.1) to get

$$\begin{aligned}&\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla[T_1(u_\epsilon-T_n(u_\epsilon))]\mathrm{d}x+\int_\Omega\beta_\epsilon(u_\epsilon)T_1(u_\epsilon-T_n(u_\epsilon))\mathrm{d}x\\ =&\int_\Omega fT_1(u_\epsilon-T_n(u_\epsilon))\mathrm{d}x.\end{aligned} \tag{3.56}$$

Since

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))\mathrm{d}x \geq 0, \quad \text{and} \quad \nabla[T_1(u_\epsilon - T_n(u_\epsilon))] = \nabla u_\epsilon \chi_{[n<|u_\epsilon|<n+1]},$$

we have from equality (3.56),

$$\int_{[n<|u_\epsilon|<n+1]} a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon \mathrm{d}x \leq \int_\Omega f T_1(u_\epsilon - T_n(u_\epsilon))\mathrm{d}x. \tag{3.57}$$

Thanks to (3.34), we have

$$\lim_{n\to+\infty}\lim_{\epsilon\to 0}\int_\Omega f T_1(u_\epsilon - T_n(u_\epsilon))\mathrm{d}x = 0.$$

Using assumption (1.3), it follows if we let $\epsilon \to 0$ and $n \to +\infty$ respectively in (3.57),

$$\lim_{n\to+\infty}\lim_{\epsilon\to 0}\frac{1}{C}\int_{[n\leq|u_\epsilon|\leq n+1]} |\nabla u_\epsilon|^{p(x)}\mathrm{d}x = \lim_{n\to+\infty}\frac{1}{C}\int_{[n\leq|u|\leq n+1]} |\nabla u|^{p(x)}\mathrm{d}x \leq 0. \tag{3.58}$$

Therefore, we get (3.3). □

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