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Existence of a Renormalised Solutions for a Class of Nonlinear Degenerated Parabolic Problems with L^1 Data

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Abstract. We study the existence of renormalized solutions for a class of nonlinear degenerated parabolic problem. The Carathéodory function satisfying the coercivity condition, the growth condition and only the large monotonicity. The data belongs to $L^1(Q)$.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2\le p<\infty$, $Q=\Omega\times]0,T[$ and $w=\{w_i(x), 0\le i\le N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω) satisfying some integrability conditions. The objective of this paper is to study the following problem in the weighted Sobolev space:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t}-\mathrm{div}(a(x,t,u,Du))+\mathrm{div}(\phi(u))=f, & \text{in } Q,\\ b(x,u)(t=0)=b(x,u_0), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\times]0,T[. \end{cases} \tag{1.1}$$

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The function b is assumed to be a strictly increasing C^1-function, the data f and $b(u_0)$ lie in $L^1(Q)$ and $L^1(\Omega)$, respectively. The functions ϕ is just assumed to be continuous of $\mathbb{R}$ with values in $\mathbb{R}^N$, and the Carathéodory function a satisfying only the large monotonicity (see assumption (H_2)).

Let us point out, the difficulties that arise in problem (1.1) are due to the following facts: the data f and u_0 only belong to L^1, a satisfies the large monotonicity that is

$$[a(x,t,s,\xi)-a(x,t,s,\eta)](\xi-\eta)\geq 0, \quad \text{for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N,$$

and the function $\phi(u)$ does not belong to $(L^1_{loc}(Q))^N$ (because the function ϕ is just assumed to be continuous on $\mathbb{R}$). To overcome this difficulty, we will apply Landes's technical (see [1,2]) and the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [3] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by L. Boccardo et al. [4] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [5] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [6] for the case of right hand side is general measure data.

For the parabolic equation (1.1) the existence of weak solution has been proved by J.-M. Rakotoson [7] with the strict monotonicity and a measure data, the existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [8] in the case where $a(x,t,s,\xi)$ is independent of s, $\phi=0$, and by D.Blanchard, F. Murat and H. Redwane [9] with the large monotonicity on a.

For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch et al. [10] in the case where a is strictly monotone, $\phi=0$ and $f\in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$. See also the existence of renormalized solution by Y.Akdim et al [11] in the case where $a(x,t,s,\xi)$ is independent of s and $\phi=0$.

Note that, this paper can be seen as a generalization of [9,10] in weighted case and as a continuation of [11].

The plan of the paper is as follows. In Section 2 we give some preliminaries and the definition of weighted Sobolev spaces. In Section 3 we make precise all the assumptions on a, ϕ, f and u_0. In Section 4 we give some technical results. In Section 5 we give the definition of a renormalized solution of (1.1) and we establish the existence of such a solution (Theorem 5.1). Section 6 is devoted to an example which illustrates our abstract result.

2 Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1<p<\infty$ and $w=\{w_i(x), 0\leq i\leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exists

$$r_0>\max(N,p) \text{ such that } w_i^{\frac{-r_0}{r_0-p}}\in L^1_{\rm loc}(\Omega), \tag{2.1}$$

$$w_i\in L^1_{\rm loc}(\Omega), \tag{2.2}$$

$$w_i^{\frac{-1}{p-1}}\in L^1(\Omega), \tag{2.3}$$

for any $0\le i\le N$. We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u\in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i}\in L^p(\Omega,w_i),\quad \text{for } i=1,\cdots,N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w}=\Big[\int_\Omega |u(x)|^p w_0(x)\,{\rm d}x+\sum_{i=1}^N\int_\Omega\Big|\frac{\partial u(x)}{\partial x_i}\Big|^p w_i(x)\,{\rm d}x\Big]^{1/p}. \tag{2.4}$$

The condition (2.2) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $V=W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4). Moreover, condition (2.3) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^*=\{w_i^*=w_i^{1-p'},\, i=0,\cdots,N\}$ and where p' is the conjugate of p i.e. $p'=p/(p-1)$, (see [12]).

3 Basic assumptions

Assumption (H1)

For $2\le p<\infty$, we assume that the expression

$$|||u|||_V=\Big(\sum_{i=1}^N\int_\Omega\Big|\frac{\partial u(x)}{\partial x_i}\Big|^p w_i(x)\,{\rm d}x\Big)^{1/p} \tag{3.1}$$

is a norm defined on V which equivalent to the norm (2.4), and there exist a weight function σ on Ω such that,

$$\sigma\in L^1(\Omega) \quad\text{and}\quad \sigma^{-1}\in L^1(\Omega).$$

We assume also the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q\sigma\,{\rm d}x\Big)^{1/q}\le c\Big(\sum_{i=1}^N\int_\Omega\Big|\frac{\partial u(x)}{\partial x_i}\Big|^p w_i(x)\,{\rm d}x\Big)^{1/p}, \tag{3.2}$$

holds for every $u\in V$ with a constant $c>0$ independent of u, and moreover, the imbedding

$$W^{1,\,p}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{3.3}$$

expressed by the inequality (3.2) is compact. Note that $(V,|||\cdot|||_V)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1. If we assume that $w_0(x)\equiv 1$ and in addition the integrability condition: There exists $\nu\in]\frac{N}{p},+\infty[\cap[\frac{1}{p-1},+\infty[$ such that

$$w_i^{-\nu}\in L^1(\Omega) \quad \text{and} \quad w_i^{\frac{N}{N-1}}\in L^1_{\text{loc}}(\Omega), \quad \text{for all } i=1,\cdots,N. \tag{3.4}$$

Notice that the assumptions (2.2) and (3.4) imply

$$|||u|||=\Big(\sum_{i=1}^{N}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^p w_i(x)\,\mathrm{d}x\Big)^{1/p}, \tag{3.5}$$

which is a norm defined on $W_0^{1,p}(\Omega,w)$ and its equivalent to (2.4) and that, the imbedding

$$W_0^{1,p}(\Omega,w)\hookrightarrow L^p(\Omega), \tag{3.6}$$

is compact for all $1\le q\le p_1^*$ if $p\nu<N(\nu+1)$ and for all $q\ge 1$ if $p\nu\ge N(\nu+1)$ where $p_1=p\nu/(\nu+1)$ and p_1^* is the Sobolev conjugate of p_1; see [13, pp. 30-31].

Assumption (H2)

We assume that

$$b:\Omega\times\mathbb{R}\to\mathbb{R} \ \text{ is a strictly increasing } C^1-\text{function with } b(0)=0, \tag{3.7}$$

$$|a_i(x,t,s,\xi)|\le\beta w_i^{\frac{1}{p}}(x)[k(x,t)+\sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}}+\sum_{j=1}^{N}w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \quad \text{for } i=1,\cdots,N, \tag{3.8}$$

for a.e. $(x,t)\in Q$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, some function $k(x,t)\in L^{p'}(Q)$ and $\beta>0$. Here σ and q are as in (H1),

$$[a(x,t,s,\xi)-a(x,t,s,\eta)](\xi-\eta)\ge 0, \quad \text{for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N, \tag{3.9}$$

$$a(x,t,s,\xi)\cdot\xi\ge\alpha\sum_{i=1}^{N}w_i|\xi_i|^p, \tag{3.10}$$

$$\phi:\ \mathbb{R}\to\mathbb{R}^N \ \text{ is a continuous function}, \tag{3.11}$$

$$f \ \text{ is an element of } \ L^1(Q), \tag{3.12}$$

$$u_0 \text{ is an element of } \ L^1(\Omega) \ \text{ such that } \ b(u_0)\in L^1(\Omega). \tag{3.13}$$

Where α is strictly positive constant. We recall that, for $k>1$ and s in $\mathbb{R}$, the truncation is defined as,

$$T_k(s)=\begin{cases} s, & \text{if } |s|\leq k,\\ k\frac{s}{|s|}, & \text{if } |s|>k. \end{cases}$$

4 Some technical results

Characterization of the time mollification of a function u

In order to deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. Thus we define for all $\mu\geq 0$ and all $(x,t)\in Q$,

$$u_\mu=\mu\int_\infty^t \tilde{u}(x,s)\exp(\mu(s-t))\mathrm{d}s, \qquad \text{where} \quad \tilde{u}(x,s)=u(x,s)\chi_{(0,T)}(s).$$

Proposition 4.1 ([10])**.** 1) *if $u\in L^p(Q,w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t}=\mu(u-u_\mu)$ and,*

$$\|u_\mu\|_{L^p(Q,w_i)}\leq \|u\|_{L^p(Q,w_i)}.$$

2) *If $u\in W_0^{1,p}(Q,w)$, then $u_\mu\to u$ in $W_0^{1,p}(Q,w)$ as $\mu\to\infty$.*
3) *If $u_n\to u$ in $W_0^{1,p}(Q,w)$, then $(u_n)_\mu\to u_\mu$ in $W_0^{1,p}(Q,w)$.*

Some weighted embedding and compactness results

In this section we establish some embedding and compactness results in weighted Sobolev spaces, some trace results, Aubin's and Simon's results [14]. Let

$V=W_0^{1,\,p}(\Omega,w)$, $H=L^2(\Omega,\sigma)$, and $V^*=W^{-1,p'}$, with $(2\leq p<\infty)$, $X=L^p(0,T;W_0^{1,\,p}(\Omega,w))$.

The dual space of X is $X^*=L^{p'}(0,T,V^*)$ where $1/p+1/p'=1$ and denoting the space

$$W_p^1(0,T,V,H)=\{v\in X:\ v'\in X^*\},$$

endowed with the norm

$$\|u\|_{W_p^1}=\|u\|_X+\|u'\|_{X^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)\mathrm{d}t=-\int_0^T u(t)\varphi'(t)\mathrm{d}t, \qquad \text{for all } \varphi\in C_0^\infty(0,T).$$

Lemma 4.1 ([15]). 1) *The evolution triple* $V\subseteq H\subseteq V^*$ *is verified.*
2) *The imbedding* $W_p^1(0,T,V,H)\subseteq C(0,T,H)$ *is continuous.*
3) *The imbedding* $W_p^1(0,T,V,H)\subseteq L^p(Q,\sigma)$ *is compact.*

Lemma 4.2 ([10]). *Let* $g\in L^r(Q,\gamma)$ *and let* $g_n\in L^r(Q,\gamma)$, *with* $\|g_n\|_{L^r(Q,\gamma)}\leq C$, $1<r<\infty$. *If* $g_n(x)\to g(x)$ *a.e. in* Q, *then* $g_n\rightharpoonup g$ *in* $L^r(Q,\gamma)$.

Lemma 4.3 ([10]). *Assume that,*

$$\frac{\partial v_n}{\partial t}=\alpha_n+\beta_n,\quad \text{in } D'(Q),$$

where α_n *and* β_n *are bounded respectively in* X^* *and in* $L^1(Q)$. *If*

$$v_n \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega,w)),$$

then $v_n\to v$ *in* $L^p_{loc}(Q,\sigma)$. *Further* $v_n\to v$ *strongly in* $L^1(Q)$.

5 Main results

Definition 5.1. *Let* $f\in L^1(Q)$ *and* $b(u_0)\in L^1(\Omega)$. *A real-valued function* u *defined on* $\Omega\times]0,T[$ *is a renormalized solution of problem* (1.1) *if*

$$T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega,w)),\ \text{for all}\,(k\geq 0)\ \text{ and }\ b(u)\in L^\infty(0,T;L^1(\Omega)); \tag{5.1}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)\mathrm{D}u\mathrm{d}x\mathrm{d}t\to 0,\quad \text{as } m\to +\infty; \tag{5.2}$$

$$\begin{aligned}\frac{\partial B_S(u)}{\partial t}&-\mathrm{div}\left(S'(u)a(u,Du)\right)+S''(u)a(u,Du)Du\\ &+\mathrm{div}(S'(u)\phi(u))-S''(u)\phi(u)Du=fS'(u),\quad \text{in } D'(Q);\end{aligned} \tag{5.3}$$

for all functions $S\in W^{2,\infty}(\mathbb{R})$ *which compact support in* $\mathbb{R}$, *where* $B_S(z)=\int_0^z b'(r)S'(r)\mathrm{d}r$ *and*

$$B_S(u)(t=0)=B_S(u_0),\ \text{ in }\ \Omega. \tag{5.4}$$

Remark 5.1. Eq. (5.3) is formally obtained through pointwise multiplication of Eq. (1.1) by $S'(u)$. However, while $a(u,Du)$ and $\phi(u)$ does not in general make sense in (1.1), all the terms in (5.3) have a meaning in $D'(Q)$.

Indeed, if M is such that supp$S'\subset[-M,M]$, the following identifications are made in (5.3):

- $B_S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function and

$$DB_S(u)=S'(u)b'(T_M(u))DT_M(u).$$

• $S'(u)a(u,Du)$ identifies with $S'(u)a(T_M(u),DT_M(u))$ a.e. in Q. Since $|T_M(u)|\le M$ a.e. in Q and $S'(u)\in L^\infty(Q)$, we obtain from (3.8) and (5.1) that

$$S'(u)a(T_M(u),DT_M(u))\in\prod_{i=1}^N L^{p'}(Q,w_i^*).$$

• $S''(u)a(u,Du)Du$ identifies with $S''(u)a(T_M(u),DT_M(u))DT_M(u)$ and

$$S''(u)a(T_M(u),DT_M(u))DT_M(u)\in L^1(Q).$$

• $S''(u)\phi(u)Du$ and $S'(u)\phi(u)$ respectively identify with

$$S''(u)\phi(T_M(u))DT_M(u) \quad\text{and}\quad S'(u)\phi(T_M(u)).$$

Due to the properties of S' and to (3.11), the functions S', S'' and $\phi o T_M$ are bounded on $\mathbb{R}$ so that (5.1) implies that

$$S'(u)\phi(T_M(u))\in(L^\infty(Q))^N,$$

and

$$S''(u)\phi(T_M(u))DT_M(u)\in L^p(Q,w).$$

• $S'(u)f$ belongs to $L^1(Q)$.

The above considerations show that Eq. (5.3) holds in $D'(Q)$, $\partial B_S(u)/\partial t$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega,w_i^*))+L^1(Q)$ and $B_S(u)\in L^p(0,T;W_0^{1,p}(\Omega,w))\cap L^\infty(Q)$. It follows that $B_S(u)$ belongs to $C^0([0,T];L^1(\Omega))$ so that the initial condition (5.4) makes sense.

Theorem 5.1. *Let $f\in L^1(Q)$ and $u_0\in L^1(\Omega)$. Assume that* (H1) *and* (H2)*, there exists at least a renormalized solution u* (*in the sense of Definition* 5.1).

Proof. **Step 1: The approximate problem.**
For $n>0$, let us define the following approximation of b, a, ϕ, f and u_0;

$$b_n(r)=T_n(b(r))+\frac{1}{n}r, \qquad \text{for } n>0, \tag{5.5}$$

$$a_n(x,t,s,d)=a(x,t,T_n(s),d), \qquad \text{a.e. in } Q,\ \forall s\in\mathbb{R},\ \forall d\in\mathbb{R}^N. \tag{5.6}$$

In view of (5.6), a_n satisfy (3.10) and (3.8), there exists $k_n\in L^{p'}(Q)$ and $\beta_n>0$ such that

$$|a_i^n(x,t,s,\xi)|\le\beta_n w_i^{\frac{1}{p}}(x)\left[k_n(x,t)+\sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}}+\sum_{j=1}^N w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}\right],\quad \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{5.7}$$

ϕ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$, (5.8) such that ϕ_n uniformly converges to ϕ on any compact subset of $\mathbb{R}$ as n tends to $+\infty$,

$$f_n \in L^{p'}(Q) \text{ and } f_n \to f, \text{ a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to +\infty, \tag{5.9}$$

$$u_{0n} \in D(\Omega): \ \|b_n(u_{0n})\|_{L^1} \leq \|b(u_0)\|_{L^1},$$

$$b_n(u_{0n}) \to b(u_0), \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{5.10}$$

Let us now consider the approximate problem:

$$\begin{cases} \dfrac{\partial b_n(u_n)}{\partial t} - \operatorname{div}(a_n(x,t,u_n,Du_n)) + \operatorname{div}(\phi_n(u_n)) = f_n, & \text{in } D'(Q), \\ u_n = 0, & \text{in } (0,T)\times\partial\Omega, \\ b_n(u_n(t=0)) = b_n(u_{0n}), & \text{in } \Omega. \end{cases} \tag{5.11}$$

As a consequence, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,\,p}(\Omega,w))$ of (5.11) is an easy task (see e.g. [16,17]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (5.11).

Using in (5.11) the test function $T_k(u_n)\chi_{(0,\tau)}$, we get, for every $\tau \in [0,T]$.

$$\begin{aligned} &\left\langle \frac{\partial b_n(u_n)}{\partial t}, T_k(u_n)\chi_{(0,\tau)} \right\rangle + \int_{Q_\tau} a(x,t,T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{Q_\tau} \phi_n(u_n)\mathrm{D}T_k(u_n)\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} f_n T_k(u_n)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{5.12}$$

which implies that,

$$\begin{aligned} &\int_\Omega B_k^n(u_n(\tau))\mathrm{d}x + \int_0^\tau \int_\Omega a(x,t,T_k(u_n),\mathrm{D}T_k(u_n))\mathrm{D}T_k(u_n)\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{Q_\tau} \phi_n(u_n)DT_k(u_n)\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} f_n T_k(u_n)\mathrm{d}x\mathrm{d}t + \int_\Omega B_k^n(u_{0n})\mathrm{d}x, \end{aligned} \tag{5.13}$$

where $B_k^n(r) = \int_0^r T_k(s)b_n'(s)ds$. The Lipschitz character of ϕ_n and stokes' formula together with the boundary condition 2 of problem (5.11) give

$$\int_0^\tau \int_\Omega \phi_n(u_n)DT_k(u_n)\mathrm{d}x\mathrm{d}t = 0. \tag{5.14}$$

Due to the definition of B_k^n we have

$$0 \leq \int_\Omega B_k^n(u_{0n})\mathrm{d}x \leq k\int_\Omega |b_n(u_{0n})|\,\mathrm{d}x \leq k\|b(u_0)\|_{L^1(\Omega)}. \tag{5.15}$$

Using (5.14), (5.15) and $B_k^n(u_n) \geq 0$, it follows from (5.13) that

$$\int_0^\tau \int_\Omega a(x,t,T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}t \leq k(\|f_n\|_{L^1(Q)} + \|b_n(u_{0n})\|_{L^1(\Omega)}) \leq Ck. \tag{5.16}$$

Thanks to (3.10) we have

$$\alpha\int_Q\sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p \mathrm{d}x\mathrm{d}t\le Ck,\qquad \forall k\ge 1. \tag{5.17}$$

We deduce from that above inequality (5.13) and (5.15) that

$$\int_\Omega B_k^n(u_n)\mathrm{d}x\le k(\|f\|_{L^1(Q)}+\|b(u_0)\|_{L^1(\Omega)})\equiv Ck. \tag{5.18}$$

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, $T_k(u_n)\rightharpoonup v_k$ in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and by the compact imbedding (3.6) gives,

$$T_k(u_n)\to v_k,\ \text{ strongly in } L^p(Q,\sigma) \text{ and a.e. in } Q.$$

Let $k>0$ large enough and B_R be a ball of Ω, we have,

$$\begin{aligned}
k\ \text{meas}(\{|u_n|>k\}\cap B_R\times[0,T]) &= \int_0^T\int_{\{|u_n|>k\}\cap B_R}|T_k(u_n)|\mathrm{d}x\mathrm{d}t\\
&\le \int_0^T\int_{B_R}|T_k(u_n)|\mathrm{d}x\mathrm{d}t\\
&\le\left(\int_Q|T_k(u_n)|^p\sigma\mathrm{d}x\mathrm{d}t\right)^{\frac1p}\left(\int_0^T\int_{B_R}\sigma^{1-p'}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\\
&\le Tc_R\left(\int_Q\sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p\mathrm{d}x\mathrm{d}t\right)^{\frac1p}\le ck^{\frac1p},
\end{aligned} \tag{5.19}$$

which implies that,

$$\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T])\le\frac{c}{k^{1-\frac1p}},\qquad \forall k\ge1.$$

So, we have

$$\lim_{k\to+\infty}(\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T]))=0.$$

Now we turn to prove the almost every convergence of u_n and $b_n(u_n)$.

Consider now a function non decreasing $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le k/2$ and $g_k(s)=k$ for $|s|\ge k$.

Multiplying the approximate equation by $g_k'(b_n(u_n))$, we get

$$\begin{aligned}
&\frac{\partial g_k(b_n(u_n))}{\partial t}-\text{div}(a(x,t,u_n,Du_n)g_k'(b_n(u_n)))+a(x,t,u_n,Du_n)g_k''(b_n(u_n))b_n'(u_n)Du_n\\
&\quad-\text{div}(g_k'(b_n(u_n))\phi_n(u_n))+g_k''(b_n(u_n))b_n'(u_n)\phi_n(u_n)Du_n=f_ng_k'(b_n(u_n)),
\end{aligned} \tag{5.20}$$

in the sense of distributions, which implies that

$$g_k(b_n(u_n)) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.21}$$

and

$$\frac{\partial g_k(b_n(u_n))}{\partial t} \text{ is bounded in } X^* + L^1(Q), \tag{5.22}$$

independently of n as soon as $k < n$. Due to Definition (3.7) and (5.5) of b_n, it is clear that

$$\{|b_n(u_n)| \le k\} \subset \{|u_n| \le k^*\},$$

as soon as $k < n$ and k^* is a constant independent of n. As a first consequence we have

$$Dg_k(b_n(u_n)) = g_k'(b_n(u_n))b_n'(T_{k^*}(u_n))DT_{k^*}(u_n), \text{ a.e. in } Q, \tag{5.23}$$

as soon as $k < n$. Secondly, the following estimate holds true

$$\|g_k'(b_n(u_n))b_n'(T_{k^*}(u_n))\|_{L^\infty(Q)} \le \|g_k'\|_{L^\infty(Q)}(\max_{|r|\le k^*}(b'(r))+1).$$

As a consequence of (5.17), (5.23) we then obtain (5.21). To show that (5.22) holds true, due to (5.20) we obtain

$$\begin{aligned}\frac{\partial g_k(b_n(u_n))}{\partial t} &= \operatorname{div}(a(x,t,u_n,Du_n)g_k'(b_n(u_n))) - a(x,t,u_n,Du_n)g_k''(b_n(u_n))b_n'(u_n)Du_n \\ &\quad + \operatorname{div}(g_k'(b_n(u_n))\phi_n(u_n)) - g_k''(b_n(u_n))b_n'(u_n)\phi_n(u_n)Du_n + f_n g_k'(b_n(u_n)).\end{aligned} \tag{5.24}$$

Since $\operatorname{supp} g_k'$ and $\operatorname{supp} g_k''$ are both included in $[-k,k]$, u_n may be replaced by $T_{k^*}(u_n)$ in each of these terms. As a consequence, each term on the right-hand side of (5.24) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ or in $L^1(Q)$. Hence Lemma 4.3 allows us to conclude that $g_k(b_n(u_n))$ is compact in $L^p_{loc}(Q,\sigma)$.

Thus, for a subsequence, it also converges in measure and almost every where in Q, due to the choice of g_k, we conclude that for each k, the sequence $T_k(b_n(u_n))$ converges almost everywhere in Q (since we have, for every $\lambda > 0$,)

$$\begin{aligned}\operatorname{meas}(\{|b_n(u_n) - b_m(u_m)| > \lambda\} \cap B_R \times [0,T]) &\le \operatorname{meas}(\{|b_n(u_n)| > k\} \cap B_R \times [0,T]) \\ &\quad + \operatorname{meas}(\{|b_m(u_m)| > k\} \cap B_R \times [0,T]) + \operatorname{meas}(\{|g_k(b_n(u_n)) - g_k(b_m(u_m))| > \lambda\}).\end{aligned}$$

Let $\varepsilon > 0$, then, there exist $k(\varepsilon) > 0$ such that,

$$\operatorname{meas}(\{|b_n(u_n) - b_m(u_m)| > \lambda\} \cap B_R \times [0,T]) \le \varepsilon, \text{ for all } n,m \ge n_0(k(\varepsilon),\lambda,R).$$

This proves that $(b_n(u_n))$ is a Cauchy sequence in measure in $B_R \times [0,T]$, thus converges almost everywhere to some measurable function v. Then for a subsequence denoted again u_n,

$$u_n \to u, \text{ a.e. in } Q, \tag{5.25}$$

and

$$b_n(u_n)\to b(u), \text{ a.e. in } Q, \tag{5.26}$$

we can deduce from (5.17) that,

$$T_k(u_n)\rightharpoonup T_k(u), \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.27}$$

and then, the compact imbedding (3.3) gives,

$$T_k(u_n)\to T_k(u), \text{ strongly in } L^q(Q,\sigma) \text{ and a.e. in } Q.$$

Which implies, by using (3.8), for all $k>0$ that there exists a function $h_k\in\prod_{i=1}^N L^{p'}(Q,w_i^*)$, such that

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup h_k, \text{ weakly in } \prod_{i=1}^N L^{p'}(Q,w_i^*). \tag{5.28}$$

We now establish that $b(u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (5.25) and passing to the limit-inf in (5.18) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_k(u)(\tau)\mathrm{d}x\le[\|f\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}]\equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(s)$ and the fact that $\frac{1}{k}B_k(u)$ converges pointwise to $b(u)$, as k tends to $+\infty$, shows that $b(u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Step 3: This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ and to establish the following limits:

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup a(x,t,T_k(u),DT_k(u)), \text{ weakly in } \prod_{i=1}^N L^{p'}(Q,w_i^*), \tag{5.29}$$

as n tends to $+\infty$.

This proof is devoted to introduce for $k\ge 0$ fixed, a time regularization of the function $T_k(u)$ in order to perform the monotonicity method.

Firstly we prove the following lemma:

Lemma 5.1.

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t=0, \tag{5.30}$$

for any integer $m\ge 1$.

Proof. Taking $T_1(u_n-T_m(u_n))$ as a test function in (5.11), we obtain

$$\begin{aligned}&\Big\langle\frac{\partial b_n(u_n)}{\partial t},T_1(u_n-T_m(u_n))\Big\rangle+\int_{\{m\le|u_n|\le m+1\}}a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&\quad+\int_Q\mathrm{div}\Big[\int_0^{u_n}\phi(r)T_1'(r-T_m(r))\Big]\mathrm{d}x\mathrm{d}t=\int_Q f_nT_1(u_n-T_m(u_n)).\end{aligned} \tag{5.31}$$

Using the fact that $\int_0^{u_n}\phi(r)T_1'(r-T_m(r))\mathrm{d}x\mathrm{d}t\in L^p(0,T;W_0^{1,\,p}(\Omega,w))$ and Stokes' formula, we get

$$\begin{aligned}&\int_\Omega B_n^m(u_n)(T)\mathrm{d}x+\int_{\{m\le|u_n|\le m+1\}}a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&\le\int_Q|f_nT_1(u_n-T_m(u_n))|\mathrm{d}x\mathrm{d}t+\int_\Omega B_n^m(u_{0n})\mathrm{d}x,\end{aligned}\tag{5.32}$$

where $B_n^m(r)=\int_0^r b_n'(s)T_1(s-T_m(s))\mathrm{d}s$.

In order to pass to the limit as n tends to $+\infty$ in (5.32), we use $B_n^m(u_n)(T)\ge 0$ and (5.9), (5.10), we obtain that

$$\begin{aligned}&\lim_{m\to+\infty}\int_{\{m\le|u_n|\le m+1\}}a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&\le\int_{\{|u(x)|>m\}}|f|\mathrm{d}x\mathrm{d}t+\int_{\{|u_0(x)|>m\}}|b(u_0(x))|\mathrm{d}x.\end{aligned}\tag{5.33}$$

Finally by (3.13), (3.12) and (5.33) we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t=0.\tag{5.34}$$

The proof is complete. □

The very definition of the sequence $(T_k(u))_\mu$ for $\mu>0$ (and fixed k) we establish the following lemma.

Lemma 5.2. *Let $k\ge 0$ be fixed. Let $(T_k(u))_\mu$ the mollification of $T_k(u)$. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r)=r\ \ for\ \ |r|\le k$ and supp S' is compact. Then,*

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\left\langle\frac{\partial b_n(u_n)}{\partial t},S'(u_n)(T_k(u_n)-(T_k(u))_\mu)\right\rangle\mathrm{d}x\mathrm{d}t\ge 0,\tag{5.35}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'}(\Omega,w^*)$ and $L^\infty(\Omega)\cap W_0^{1,\,p}(\Omega,w)$.

Proof. See H. Redwane [18]. □

We prove the following lemma, which is the key point in the monotonicity arguments.

Lemma 5.3. *The subsequence of u_n satisfies for any $k\ge 0$*

$$\begin{aligned}&\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t\\&\le\int_0^T\int_0^t\int_\Omega h_kDT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t,\end{aligned}\tag{5.36}$$

where h_k is defined in (5.28).

Proof. In the following we adapt the above-mentioned method to problem (1.1) and we first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that

$$S_m(r)=r \text{ if } |r|\le m, \quad \operatorname{supp}S'_m\subset[-(m+1),m+1], \quad \|S''_m\|_{L^\infty}\le 1, \text{ for any } m\ge 1.$$

We use the sequence $T_k(u)_\mu$ of approximations of $T_k(u)$, and plug the test function $S'_m(u_n)(T_k(u_n)-(T_k(u))_\mu)$ (for $n>0$ *and* $\mu>0$) in (5.11). Through setting, for fixed $k\le 0$,

$$W^n_\mu=T_k(u_n)-(T_k(u))_\mu,$$

we obtain upon integration over (0,t) and then over (0,T):

$$\begin{aligned}&\int_0^T\int_0^t\Big\langle\frac{\partial b_n(u_n)}{\partial t},S'_m(u_n)W^n_\mu\Big\rangle \mathrm{d}t\mathrm{d}s+\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n,Du_n)DW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t\\&\quad+\int_0^T\int_0^t\int_\Omega S''_m(u_n)a_n(u_n,Du_n)Du_nW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t-\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi_n(u_n)DW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t\\&\quad-\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t=\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t.\end{aligned}\tag{5.37}$$

In the following we pass the limit in (5.37) as n tends to $+\infty$, then μ tends to $+\infty$ and then m tends to $+\infty$, the real number $k\ge 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $k\ge 0$:

$$\liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\Big\langle\frac{\partial B^n_m(u_n)}{\partial t},W^n_\mu\Big\rangle\mathrm{d}t\mathrm{d}s\ge 0, \text{ for any } m\ge k, \tag{5.38}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi_n(u_n)DW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t=0, \text{ for any } m\ge 1, \tag{5.39}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t=0, \text{ for any } m\ge 1, \tag{5.40}$$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\Big|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(u_n,Du_n)Du_nW^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t\Big|=0, \ m\ge 1, \tag{5.41}$$

$$\lim_{\mu\to+\infty}\lim_{m\to+\infty}\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W^n_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t=0. \tag{5.42}$$

Proof of (5.38). The function S_m belongs to $C^\infty(\mathbb{R})$ and is increasing. We have for $m\ge k$, $S_m(r)=r$ for$|r|\le k$ while $\operatorname{supp}S'_m$ is compact.

In view of the definition of W^n_μ, Lemma 5.2 applies with $S=S_m$ for fixed $m\ge k$. As a consequence (5.38) holds true.

Proof of (5.39). In order to avoid repetitions in the proofs of (5.42), let us summarize the properties of W^n_μ. For fixed $\mu>0$

$$\begin{aligned}&W^n_\mu\rightharpoonup T_k(u)-(T_k(u))_\mu, \text{ weakly in } L^p(0,T;W^{1,p}_0(\Omega,w)), \text{ as } n\to+\infty,\\&\big\|W^n_\mu\big\|_{L^\infty(Q)}\le 2k, \text{ for any } n>0 \text{ and for any } \mu>0,\end{aligned}$$

we deduce that for fixed $\mu > 0$

$$W_\mu^n \to T_k(u) - (T_k(u))_\mu, \text{ a.e. in } Q \text{ and in } L^\infty(Q)\text{weak}-*, \text{ as } n \to +\infty,$$

one has supp$S''_m \subset [-(m+1), -m] \cup [m, m+1]$ for any fixed $m \geq 1$, we have

$$S'_m(u_n)\phi_n(u_n)DW_\mu^n = S'_m(u_n)\phi_n(T_{m+1}(u_n))DW_\mu^n, \text{ a.e. in } Q, \tag{5.43}$$

since supp$S'_m \subset [-m-1, m+1]$.

Since S'_m is smooth and bounded, (3.11), (5.8), and $u_n \to u$ a.e. in Q lead to

$$S'_m(u_n)\phi_n(T_{m+1}(u_n)) \to S'_m(u)\phi(T_{m+1}(u)), \text{ a.e. in } Q \text{ and in } L^\infty(Q) \text{ weak}-*, \tag{5.44}$$

as n tends to $+\infty$. As a consequence of (5.46) and (5.44), we deduce that

$$\begin{aligned} &\lim_{n\to+\infty} \int_0^T \int_0^t \int_\Omega S'_m(u_n)\phi_n(u_n)DW_\mu^n \mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &= \int_0^T \int_0^t \int_\Omega S'_m(u)\phi(T_{m+1}(u))(DT_k(u) - D(T_k(u))_\mu)\mathrm{d}x\mathrm{d}s\mathrm{d}t, \end{aligned} \tag{5.45}$$

for any $\mu > 0$.

Passing to the limit as $\mu \to +\infty$ in (5.45) we conclude that (5.39) holds true.

Proof of (5.40). For fixed $m \geq 1$, and by the same arguments that those that lead to (5.46), we have

$$S''_m(u_n)\phi_n(u_n)Du_nW_\mu^n = S''_m(u_n)\phi_n(T_{m+1}(u_n))DT_{m+1}(u_n)W_\mu^n, \quad \text{a.e. in } Q. \tag{5.46}$$

From (3.11), $u_n \to u$ a.e. in Q and (5.27), it follows that for any $\mu > 0$

$$\begin{aligned} &\lim_{n\to+\infty} \int_0^T \int_0^t \int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW_\mu^n \mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &= \int_0^T \int_0^t \int_\Omega S''_m(u_n)\phi(T_{m+1}(u))(DT_k(u) - D(T_k(u))_\mu)\mathrm{d}x\mathrm{d}s\mathrm{d}t, \end{aligned}$$

for any $\mu > 0$.

Passing to the limit as $\mu \to +\infty$ in (5.45) we conclude that (5.40) holds true.

Proof of (5.41). One has supp$S''_m \subset [-(m+1), -m] \cup [m, m+1]$ for any $m \geq 1$. As a consequence

$$\begin{aligned} &\left| \int_0^T \int_0^t \int_\Omega S''_m(u_n)a(u_n, Du_n)Du_nW_\mu^n \mathrm{d}x\mathrm{d}s\mathrm{d}t \right| \\ &\leq T \left\| S''_m(u_n) \right\|_{L^\infty} \left\| W_\mu^n \right\|_{L^\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(u_n, Du_n)Du_n \mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.47}$$

for any $m\geq 1$, any $\mu>0$ and any $n\geq 1$, it is possible to obtain

$$\begin{aligned}&\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\left|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(u_n,Du_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t\right|\\ \leq& C\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t,\end{aligned}$$

for any $m\geq 1$, where C is a constant independent of m. Appealing now to (5.30) it possible to pass the limit as m tends to $+\infty$ to establish (5.41).

Proof of (5.42). Lebesgue's convergence theorem implies that for any $\mu>0$ and any $m\geq 1$

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t=\int_0^T\int_0^t\int_\Omega fS'_m(u)(T_k(u)-(T_k(u)_\mu))\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

Now, for fixed $m\geq 1$, using Lemma 4.1 and passing to the limit as $\mu\to+\infty$ in the above equality to obtain (5.42).

We now turn back to the proof of Lemma 5.3. Due to (5.38)-(5.42), we are in a position to pass the limit-sup when n tends to $+\infty$, then to the limit-sup when μ tends $+\infty$ and then to the limit as m tends to $+\infty$ in (5.37). We obtain by using the definition of W^n_μ that for any $k\geq 0$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n,Du_n)(DT_k(u_n)-D(T_k(u))_\mu)\mathrm{d}x\mathrm{d}s\mathrm{d}t\leq 0.$$

Since $S'_m(u_n)a_n(u_n,Du_n)DT_k(u_n)=a(u_n,Du_n)DT_k(u_n)$ for $k\leq n$ and $k\leq m$, the above inequality implies that for $k\leq m$,

$$\begin{aligned}&\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a_n(u_n,Du_n)DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t\\ \leq&\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n,Du_n)D(T_k(u))_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t.\end{aligned}\tag{5.48}$$

The right-hand side of (5.48) is computed as follows. We have for $n\geq m+1$:

$$S'_m(u_n)a_n(u_n,Du_n)=S'_m(u_n)a(T_{m+1}(u_n),DT_{m+1}(u_n))\ \text{ a.e. in } Q.$$

Due to the weak convergence of $a(DT_{m+1}(u_n))$ it follows that for fixed $m\geq 1$

$$S'_m(u_n)a_n(u_n,Du_n)\rightharpoonup S'_m(u)h_{m+1},\ \text{ weakly in } \prod_{i=1}^N L^{p'}(Q,w_i^*),$$

when n tends to $+\infty$. The strong convergence of $(T_k(u))_\mu$ to $T_k(u)$ in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ as μ tends to $+\infty$, then we conclude that

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n,Du_n)D(T_k(u))_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$=\int_0^T\int_0^t\int_\Omega S'_m(u)h_{m+1}DT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t, \tag{5.49}$$

as soon as $k\le m$, $S'_m(r)=1\ \ for\ \ |r|\le m$. Now for $k\le m$ we have,

$$a(T_{m+1}(u_n),DT_{m+1}(u_n))\chi_{\{|u_n|<k\}}=a(T_k(u_n),DT_k(u_n))\chi_{\{|u_n|<k\}},\quad \text{a.e. in } Q,$$

which implies that, passing to the limit as $n\to+\infty$,

$$h_{m+1}\chi_{\{|u_n|<k\}}=h_k\chi_{\{|u|<k\}},\ \text{a.e. in } Q-\{|u|=k\},\ \text{for } k\le m. \tag{5.50}$$

As a consequence of (5.50) we have for $k\le m$,

$$h_{m+1}DT_k(u)=h_kDT_k(u),\ \text{a.e. in } Q. \tag{5.51}$$

Recalling (5.48), (5.49), (5.51) we conclude that (5.36) holds true and the proof of Lemma 5.3 is complete. □

In this lemma we prove the following monotonicity estimate:

Lemma 5.4. *The subsequence of u_n satisfies for any $k\ge 0$*

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega [a(T_k(u_n),DT_k(u_n))-a(T_k(u_n),DT_k(u))]$$
$$\times[DT_k(u_n)-DT_k(u)]\mathrm{d}x\mathrm{d}s\mathrm{d}t=0. \tag{5.52}$$

Proof. Let $k\ge 0$ be fixed. The character (3.9) of $a(x,t,s,d)$ with respect to d implies that

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega [a(T_k(u_n),DT_k(u_n))-a(T_k(u_n),DT_k(u))]$$
$$\times[DT_k(u_n)-DT_k(u)]\mathrm{d}x\mathrm{d}s\mathrm{d}t\ge 0. \tag{5.53}$$

To pass to the limit-sup as n tends to $+\infty$ in (5.53) imply that

$$a(T_k(u_n),DT_k(u))\to a(T_k(u),DT_k(u)),\quad \text{a.e. in } Q,$$

and that,

$$|a_i(T_k(u_n),DT_k(u))|$$
$$\le\beta w_i^{\frac{1}{p}}(x)\left(k(x,t)+\sigma^{\frac{1}{p'}}|T_k(u_n)|^{\frac{q}{p'}}+\sum_{j=1}^N w_j^{\frac{1}{p'}}(x)\left|\frac{\partial T_k(u)}{\partial x_j}\right|^{p-1}\right),\quad \text{a.e. in } Q,$$

uniformly with respect to n.

It follows that when n tends to $+\infty$

$$a(T_k(u_n),DT_k(u))\to a(T_k(u),DT_k(u)), \quad \text{strongly in} \quad \prod_{i=1}^{N} L^{p'}(Q,w_i^*). \tag{5.54}$$

Lemma 5.3, weak convergence of $DT_k(u_n)$, $a(T_k(u_n),\ DT_k(u_n))$ and (5.54) make it possible to pass to the limit-sup as $n\to+\infty$ in (5.53) and to obtain the result. $\square$

In this lemma we identify the weak limit h_k and we prove the weak-L^1 convergence of the "truncated" energy $a(T(u_n),DT_k(u_n))DT(u_n)$ as n tends to $+\infty$.

Lemma 5.5. *For fixed $k\geq 0$, we have*

$$h_k=a(T(u),DT_k(u)), \text{ a.e. in } Q, \tag{5.55}$$

$$a(T(u_n),DT_k(u_n))DT(u_n)\rightharpoonup a(T(u),DT_k(u))DT_k(u), \text{ weakly in } L^1(Q). \tag{5.56}$$

Proof. The proof is standard once we remark that for any $k\geq 0$, any $n>k$ and any $d\in\mathbb{R}^N$

$$a_n(T_k(u_n),d)=a(T_k(u_n),d), \text{ a.e. in } Q,$$

which together with weak convergence of $(T_k(u_n))$, $a(DT_k(u_n))$ and (5.54) we obtain from (5.52)

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t=\int_0^T\int_0^t\int_\Omega h_kDT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t. \tag{5.57}$$

The usual Minty's argument applies in view of weak convergence of $(T_k(u_n))$, $a(DT_k(u_n))$ and (5.57). It follows that (5.55) hold true.

In order to prove (5.56), we observe that monotone character of a and (5.52) give that for any $k\geq 0$ and any $T'<T$

$$[a(T_k(u_n),DT_k(u_n))-a(T_k(u),DT_k(u))][DT_k(u_n)-DT_k(u)]\to 0 \tag{5.58}$$

strongly in $L^1((0,T')\times\Omega)$ *as* $n\to+\infty$.

Moreover, weak convergence of $(T_k(u_n))$ and $a(DT_k(u_n))$, (5.58), (5.54) and (5.55) imply that

$$a(T_k(u_n),DT_k(u_n))DT_k(u)\rightharpoonup a(T_k(u),DT_k(u))DT_k(u), \text{ weakly in } L^1(Q),$$

and

$$a(T_k(u_n),DT_k(u))DT_k(u)\to a(T_k(u_n),DT_k(u))DT_k(u), \text{ strongly in } L^1(Q)$$

as $n\to+\infty$.

Using the above convergence results in (5.58) shows that for any $k \geq 0$ and any $T' < T$

$$a(T_k(u_n),DT_k(u_n))DT_k(u_n) \rightharpoonup a(T_k(u),DT_k(u))DT_k(u) \text{ weakly in } L^1((0,T')\times\Omega), \tag{5.59}$$

as $n \to +\infty$.

At the possible expense of extending the functions $a(x,t,s,d)$, f on a time interval $(0,\bar{T})$ with $\bar{T} > T$ in such a way that assumptions with a and f hold true with $\bar{T}$ in place of T, we can show that the convergence result (5.59) is still valid in $L^1(Q)$-weak, namely that (5.56) holds true. □

Step 4: In this step we prove that u satisfies (5.2).

Lemma 5.6. *The limit u of the approximate solution u_n of* (5.11) *satisfies*

$$\lim_{m\to+\infty}\int_{\{m\leq|u|\leq m+1\}} a(u,Du)Du\,\mathrm{d}x\mathrm{d}t = 0.$$

Proof. To this end, observe that for any fixed $m \geq 0$, one has

$$\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t = \int_Q a(u_n,Du_n)(DT_{m+1}(u_n) - DT_m(u_n))\mathrm{d}x\mathrm{d}t$$
$$= \int_Q a(T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n)\mathrm{d}x\mathrm{d}t - \int_Q a(T_m(u_n),DT_m(u_n))DT_m(u_n)\mathrm{d}x\mathrm{d}t.$$

According to (5.56), one is at liberty to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$ and to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\
&= \int_Q a(T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)\mathrm{d}x\mathrm{d}t - \int_Q a(T_m(u),DT_m(u))DT_m(u)\mathrm{d}x\mathrm{d}t\\
&= \int_{\{m\leq|u_n|\leq m+1\}} a(u,Du)Du\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{5.60}$$

Taking the limit as $m \to +\infty$ in (5.60) and using the estimate (5.30) show that u satisfies (5.2) and the proof of the lemma is complete. □

Step 5: In this step, u is shown to satisfy (5.3) and (5.4). Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let M be a positive real number such that $\mathrm{supp}(S') \subset [-M,M]$. Pointwise multiplication of the approximate equation (5.11) by $S'(u_n)$ leads to

$$\begin{aligned}
&\frac{\partial B_S^n(u_n)}{\partial t} - div[S'(u_n)a(u_n,Du_n)] + S''(u_n)a(u_n,Du_n)Du_n + div(S'(u_n)\phi_n(u_n))\\
&\qquad - S''(u_n)\phi_n(u_n)Du_n = fS'(u_n), \text{ in } D'(Q).
\end{aligned}\tag{5.61}$$

It was follows we pass to the limit as in (5.61) n tends to $+\infty$.

• Limit of $\partial B_S^n(u_n)/\partial t$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B_S^n(u_n)$ converges to $B_S(u)$ a.e. in Q and L^∞ weak*. Then $\partial B_S^n(u_n)/\partial t$ converges to $\partial B_S(u)/\partial t$ in $D'(Q)$ as n tends to $+\infty$.

• Limit of $-\mathrm{div}[S'(u_n)a_n(u_n,Du_n)]$.

Since $\mathrm{supp}(S') \subset [-M,M]$, we have for $n \geq M$

$$S'(u_n)a_n(u_n,Du_n) = S'(u_n)a(T_M(u_n),DT_M(u_n)), \text{ a.e. in } Q.$$

The pointwise convergence of u_n to u and (5.55) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(u_n,Du_n) \rightharpoonup S'(u)a(T_M(u),DT_M(u)), \text{ in } \prod_{i=1}^N L^{p'}(Q,w_i^*), \tag{5.62}$$

as n tends to $+\infty$. $S'(u)a(T_M(u),DT_M(u))$ has been denoted by $S'(u)a(u,Du)$ in (5.3).

• Limit of $S''(u_n)a(u_n,Du_n)Du_n$.

As far as the 'energy' term

$$S''(u_n)a(u_n,Du_n)Du_n = S''(u_n)a(T_M(u_n),DT_M(u_n))DT_M(u_n), \text{ a.e. in } Q.$$

The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (5.56) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(u_n,Du_n)Du_n \rightharpoonup S''(u)a(T_M(u),DT_M(u))DT_M(u), \text{ weakly in } L^1(Q). \tag{5.63}$$

Recall that

$$S''(u)a(T_M(u),DT_M(u))DT_M(u) = S''(u)a(u,Du)Du, \text{ a.e. in } Q.$$

• Limit of $S'(u_n)\phi_n(u_n)$.

Since $\mathrm{supp}(S') \subset [-M,M]$, we have

$$S'(u_n)\phi_n(u_n) = S'(u)\phi_n(T_M(u)), \text{ a.e. in } Q.$$

As a consequence of (5.8) and $u_n \to u$, a.e. in Q, it follows that

$$S'(u_n)\phi_n(u_n) \to S'(u)\phi(T_M(u)), \text{ strongly in } \prod_{i=1}^N L^{p'}(Q,w_i^*),$$

as n tends to $+\infty$. The term $S'(u)\phi(T_M(u))$ is denoted by $S'(u)\phi(u)$.

• Limit of $S''(u_n)\phi_n(u_n)Du_n$.

Since $S'\in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp}(S')\subset[-M,M]$, we have

$$S''(u_n)\phi_n(u_n)Du_n=\phi_n(T_M(u_n))DS'(u_n),\ \text{ a.e. in } Q.$$

Moreover, $DS'(u_n)$ converges to $DS'(u)$ weakly in $L^p(Q,w)$ as n tends to $+\infty$, while $\phi_n(T_M(u_n))$ is uniformly bounded with respect to n and converges a.e. in Q to $\phi(T_M(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\phi_n(u_n)Du_n\rightharpoonup\phi(T_M(u))DS'(u),\ \text{ weakly in } L^p(Q,w).$$

The term $\phi(T_M(u))DS'(u)=S''(u_n)\phi(u)Du$.

• Limit of $S'(u_n)f_n$.

Due to (5.9) and $u_n\to u$ a.e. in Q, we have

$$S'(u_n)f_n\to S'(u)f,\ \text{ strongly in } L^1(Q) \text{ as } n\to+\infty.$$

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in Eq. (5.61) and to conclude that u satisfies (5.3).

It remains to show that $B_S(u)$ satisfies the initial condition (5.4). To this end, firstly remark that, S being bounded, $B_S^n(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.61) and the above considerations on the behavior of the terms of this equation show that $\partial B_S^n(u_n)/\partial t$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. As a consequence, an Aubin's type lemma (see, e.g., [14]) implies that $B_S^n(u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_S^n(u_n)(t=0)=B_S^n(u_0^n)$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that

$$B_S(u)(t=0)=B_S(u_0),\quad \text{in } \Omega.$$

As a conclusion of step 1 to step 5, the proof of Theorem 5.1 is complete. □

6 Example

Let us consider the following special case:

$$b(r)=\exp(\beta r)-1,\qquad \phi\colon\ r\in\mathbb{R}\to(\phi_i)_{i=1,\cdots,N}\in\mathbb{R}^N,$$

where

$$\phi_i(r)=\exp(\alpha_i r),\qquad i=1,\cdots,N,\ \ \alpha_i\in\mathbb{R},$$

ϕ is a continuous function. And,

$$a_i(x,t,d)=w_i(x)\,|d_i|^{p-1}\operatorname{sgn}(d_i),\qquad i=1,\cdots,N,$$

with $w_i(x)$ a weight function ($i=1,\cdots,N$). For simplicity, we suppose that

$$w_i(x)=w(x),\ \text{ for } i=1,\cdots,N-1,\quad w_N(x)\equiv 0.$$

It is easy to show that the $a_i(t,x,d)$ are Caratheodory functions satisfying the growth condition (3.8) and the coercivity (3.10). On the order hand the monotonicity condition is verified. In fact,

$$\begin{aligned}&\sum_{i=1}^{N}\big(a_i(x,t,d)-a(x,t,d')\big)(d_i-d_i')\\=&w(x)\sum_{i=1}^{N-1}\Big(|d_i|^{p-1}\mathrm{sgn}(d_i)-|d_i'|^{p-1}\mathrm{sgn}(d_i')\Big)(d_i-d_i')\geq 0,\end{aligned}$$

for almost all $x\in\Omega$ and for all $d,d'\in\mathbb{R}^N$. This last inequality can not be strict, since for $d\neq d'$ with $d_N\neq d_N'$ and $d_i=d_i'$, $i=1,\cdots,N-1$, the corresponding expression is zero.

In particular, let us use special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x)=\mathrm{dist}(x,\partial\Omega)$ and set $w(x)=d^{\lambda}(x)$, such that

$$\lambda<\min\Big(\frac{p}{N},p-1\Big). \tag{6.1}$$

Remark 6.1. The condition (6.1) is sufficient for (3.4).

Finally, the hypotheses of Theorem 5.1 are satisfied. Therefore, for all $f\in L^1(Q)$, the following problem:

$$\left\{\begin{aligned}&u\in L^{\infty}([0,T];L^1(\Omega));\\&T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega,w)),\\&\lim_{m\to+\infty}\int_{\{m\leq|u|\leq m+1\}}a(u,Du)Du\mathrm{d}x\mathrm{d}t=0;\\&B_S(r)=\int_0^r\beta(\exp\beta\sigma)S(\sigma)\mathrm{d}\sigma,\\&\qquad-\int_Q B_S(u)\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t+\int_Q S(u)\sum_{i=1}^N w_i\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\mathrm{sgn}\Big(\frac{\partial u}{\partial x_i}\Big)\frac{\partial\varphi}{\partial x_i}\mathrm{d}x\mathrm{d}t\\&\qquad+\int_Q S'(u)\sum_{i=1}^N w_i\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\mathrm{sgn}\Big(\frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}\varphi\mathrm{d}x\mathrm{d}t\\&\qquad+\int_Q\sum_{i=1}^N S(u)\exp(\alpha_i u)\frac{\partial\varphi}{\partial x_i}\mathrm{d}x\mathrm{d}t-\int_Q\sum_{i=1}^N S'(u)\exp(\alpha_i u)\frac{\partial u}{\partial x_i}\varphi\mathrm{d}x\mathrm{d}t\\&\quad=\int_Q fS'(u)\varphi\mathrm{d}x\mathrm{d}t,\\&B_S(u)(t=0)=B_S(u_0),\ \text{ in }\ \Omega,\\&\forall\ \ \varphi\in C_0^{\infty}(Q)\ \text{ and }\ S\in W^{1,\infty}(\mathbb{R})\ \text{ with }\ S'\in C_0^{\infty}(\mathbb{R}),\end{aligned}\right. \tag{6.2}$$

has at least one renormalised solution.

Remark 6.2. For uniqueness of a renormalized solution of (1.1) we are currently working with doubling variable technique.

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