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Nodal Type Bound States for Nonlinear Schrödinger Equations with Decaying Potentials

LUO Tingjian and WANG Zhengping*

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, China.

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Abstract. In this paper, we are concerned with the existence of nodal type bound state for the following stationary nonlinear Schrödinger equation

$$-\Delta u(x)+V(x)u(x)=|u|^{p-1}u, \qquad x\in\mathbb{R}^N,\ N\geq 3,$$

where $1<p<(N+2)/(N-2)$ and the potential $V(x)$ is a positive radial function and may decay to zero at infinity. Under appropriate assumptions on the decay rate of $V(x)$, Souplet and Zhang [1] proved the above equation has a positive bound state. In this paper, we construct a nodal solution with precisely two nodal domains and prove that the above equation has a nodal type bound state under the same conditions on $V(x)$ as in [1].

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1 Introduction

In this paper, we consider the time-independent nonlinear Schrödinger equation

$$-\Delta u(x)+V(x)u(x)=|u|^{p-1}u, \qquad x\in\mathbb{R}^N,\ N\geq 3, \tag{1.1}$$

where $1<p<(N+2)/(N-2)$. We assume that $V(x)$ satisfies the following conditions:

(V1) $V(x)$ is a radially symmetric and locally Hölder continuous function.

*Corresponding author. *Email addresses:* luotj2008@126.com (T. J. Luo), wangzp@wipm.ac.cn (Z. P. Wang)

(V2) There exist a, $A>0$ and $\alpha\in[0,2(N-1)(p-1)/(p+3))$ such that

$$\frac{a}{1+|x|^{\alpha}}\le V(x)\le A. \tag{1.2}$$

We call a function u a *bound state* of (1.1) if $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ and satisfies

$$\int_{\mathbb{R}^N}\nabla u\nabla v+V(x)uv\mathrm{d}x=\int_{\mathbb{R}^N}|u|^{p-1}uv\mathrm{d}x \qquad \text{for all } v\in C_0^\infty(\mathbb{R}^N). \tag{1.3}$$

Furthermore, a function u_0 is called a *nodal type bound state* of (1.1) if u_0 is a bound state of (1.1) and $u_0^\pm\not\equiv 0$, where $u_0^+(x)=\max\{u_0(x),0\}$ and $u_0^-(x)=\min\{u_0(x),0\}$.

In the past two decays, much attention has been paid to the existence of bound states for problem (1.1) under the assumption that $\lim_{|x|\to+\infty}V(x)>0$. For example, if $V(x)$ satisfies

(V3) there exists $V_0>0$ such that $V(x)\ge V_0$ for all $x\in\mathbb{R}^N$, and

(V4) $\lim\limits_{|x|\to+\infty}V(x)=+\infty$, Rabinowitz [3] proved that (1.1) has a bound state by a variant version of mountain pass theorem. If $V(x)$ satisfies (V3) and

(V5) $\lim\limits_{|x|\to+\infty}V(x)=\sup\limits_{x\in\mathbb{R}^N}V(x)<+\infty$.

Li, et al. [4] proved that there is a ground state, that is the least energy solution among all bound states, for problem (1.1). When $V(x)$ may change sign in $\mathbb{R}^N$ and satisfies $\lim_{|x|\to+\infty}V(x)>0$, Ding and Szulkin [5] showed the existence of bound states for problem (1.1). Under the conditions (V3) and (V4), Bartsch, et al. [2] proved the existence of nodal type bound states for problem (1.1).

In order to find the bound states of (1.1), ones usually use variational method to look for the nonzero critical points of the energy functional given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\mathrm{d}x, \qquad u\in H_V, \tag{1.4}$$

where

$$H_V=\left\{u\in D^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x<\infty\right\}$$

with the norm

$$\|u\|_{H_V}=\left(\int_{\mathbb{R}^N}[|\nabla u|^2+V(x)u^2]\mathrm{d}x\right)^{\frac{1}{2}} \quad \text{if } V(x)\ge 0 \quad \text{on } \mathbb{R}^N,$$

and otherwise, if $V(x)$ changes sign in $\mathbb{R}^N$, H_V is substituted by H_{V^+}. We note that H_V or H_{V^+} is a subspace of $H^1(\mathbb{R}^N)$ if $V(x)$ satisfies $\lim_{|x|\to+\infty}V(x)>0$. So, in this case the nonzero critical point of (1.4) must be the bound state of (1.1). Then it is natural to ask whether there is a bound state for problem (1.1) if $V(x)$ satisfies $\lim_{|x|\to+\infty}V(x)=0$.

By using a dynamical approach, Souplet and Zhang [1] proved (1.1) has a positive bound state under the conditions (V1) and (V2). In [6], Ambrosetti, et al. studied the existence and concentration behavior of ground states for problem (1.1) with vanishing potential in more general case. After the work of [6], there are many papers on the existence of bound states, ground states as well as semi-classical states (where $-\Delta$ is replaced by $-\epsilon^2\Delta$ for $\epsilon>0$ small) for problem (1.1) with decaying potentials. See [7–14]. The condition that the decaying rate of V is not more than 2 plays an important role in getting bound states of (1.1) among these papers. So, Ambrosetti and Malchiodi [15] posed an open question concerning the existence of bound states for (1.1) with fast decaying potentials. Recently, Cao and Peng [16] and Ba, Deng and Peng [17] made much progress in this open question and proved the existence of multi-peak bound states for (1.1) even if V has a compact support. In all the above mentioned papers, most of them are devoted to studying the positive solution of (1.1). To the authors' knowledge, there are few results on the existence of nodal solution for problem (1.1) with decaying potentials.

In this paper, inspired by [2], we construct a nodal solution with precisely two nodal domains, and prove further that this nodal solution has the same exponential decay as in [11]. Thus, we finally get a nodal type bound state under the same conditions on $V(x)$ as in [1]. Our result in this paper, for one thing, obtains a nodal solution of problem (1.1) under the same assumptions on $V(x)$ as those in [1]. For another, it is different from the result of [2], since the potential $V(x)$ under our conditions may not have the positive lower bound.

The main result of this paper is the following

Theorem 1.1. *Suppose that* (V1) *and* (V2) *hold. Then problem* (1.1) *has a nodal type bound state.*

2 Preliminary results

Let

$$D_r^{1,2}(\mathbb{R}^N)=\Big\{u\in D^{1,2}(\mathbb{R}^N):\ u(x)=u(|x|)\Big\},$$
$$H=\{u\in D_r^{1,2}(\mathbb{R}^N):\ \int_{\mathbb{R}^N}V(|x|)u^2\mathrm{d}x<\infty\}.$$

Then $(H,\langle,\rangle)$ is a real Hilbert space with the scalar product given by

$$\langle u,v\rangle=\int_{\mathbb{R}^N}\nabla u\nabla v+V(|x|)uv\mathrm{d}x,\qquad \forall\ u,\ v\in H.$$

The norm of H is defined by

$$\|u\|=(\int_{\mathbb{R}^N}|\nabla u|^2+V(|x|)u^2\mathrm{d}x)^{\frac{1}{2}},\qquad \forall\ u\in H.$$

By [14, Theorem1], we have the following

Proposition 2.1. *Suppose that* (V1) *and* (V2) *hold. Then for* $p\in(1,(N+2)/(N-2))$, *the embedding* $H\hookrightarrow L^{p+1}(\mathbb{R}^N)$ *is compact.*

We define the energy functional $I:H\to\mathbb{R}$ by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+V(|x|)u^2)\mathrm{d}x-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\mathrm{d}x. \tag{2.1}$$

Then any nonzero critical point of I is a nontrivial weak solution of problem (1.1). Moreover, by elliptic regularity theory we know that if $u\in H$ satisfies (1.3), then u must be a classical solution of problem (1.1).

Now, let $u\in H$ be a nontrivial weak solution of problem (1.1). The set Ω is said to be a *nodal domain* of u if Ω is an open and connected subset of $\mathbb{R}^N$ such that u has fixed sign in Ω and $u(y)=0$ when $y\in\partial\Omega$.

We denote the characteristic function of Ω by χ_Ω. By [18, Lemma2], we have the following

Proposition 2.2. *Let* $u\in H$ *be a nontrivial weak solution of problem* (1.1) *and* Ω_1, Ω_2, $\cdots$, Ω_n *are nodal domains of* u. *Then* $u\chi_{\Omega_1\cup,\cdots,\cup\Omega_n}\in H$.

3 Proof of the main theorem

Let

$$\mathcal{M}=\{u\in H:\ u^{\pm}\not\equiv 0,\ \langle I'(u),u^+\rangle=\langle I'(u),u^-\rangle=0\},$$

where $u^+(x)=\max\{u(x),0\}$ and $u^-(x)=\min\{u(x),0\}$.

Remark 3.1. (i) Obviously, $\mathcal{M}$ contains all nodal solutions of problem (1.1). (ii) $\mathcal{M}\neq\varnothing$. Indeed, we can choose two functions φ_1, $\varphi_2\in C_0^\infty(\mathbb{R}^N)$, such that $\varphi_1\geq(\not\equiv)0$, $\varphi_2\leq(\not\equiv)0$, and $\mathrm{supp}\varphi_1\bigcap\mathrm{supp}\varphi_2=\varnothing$.

Let $u=t_1\varphi_1+t_2\varphi_2$, where t_1, $t_2>0$ will be decided later. By the construction of φ_1, φ_2, we have $u^+=t_1\varphi_1$, $u^-=t_2\varphi_2$ and

$$\langle I'(u),u^+\rangle=\langle I'(t_1\varphi_1+t_2\varphi_2),t_1\varphi_1\rangle=\langle I'(t_1\varphi_1),t_1\varphi_1\rangle.$$

Similarly, $\langle I'(u),u^-\rangle=\langle I'(t_2\varphi_2),t_2\varphi_2\rangle$.

Notice that

$$\begin{aligned}\langle I'(t_1\varphi_1),t_1\varphi_1\rangle&=\int_{\mathbb{R}^N}|\nabla(t_1\varphi_1)|^2+V(|x|)|(t_1\varphi_1)|^2\mathrm{d}x-\int_{\mathbb{R}^N}|(t_1\varphi_1)|^{p+1}\mathrm{d}x\\&=t_1^2\left(\int_{\mathbb{R}^N}|\nabla\varphi_1|^2+V(|x|)|\varphi_1|^2\mathrm{d}x-t_1^{p-1}\int_{\mathbb{R}^N}|\varphi_1|^{p+1}\mathrm{d}x\right).\end{aligned}$$

Then $\langle I'(t_1\varphi_1),t_1\varphi_1\rangle=0$ if we choose

$$t_1=\left(\frac{\int_{\mathbb{R}^N}|\nabla\varphi_1|^2+V(|x|)|\varphi_1|^2\mathrm{d}x}{\int_{\mathbb{R}^N}|\varphi_1|^{p+1}\mathrm{d}x}\right)^{\frac{1}{p-1}}>0.$$

Similarly, there exists $t_2>0$ such that $\langle I'(t_2\varphi_2),t_2\varphi_2\rangle=0$. By the definition of $\mathcal{M}$, we have $u\in\mathcal{M}$. Hence $\mathcal{M}\neq\varnothing$.

Define the metric d on $\mathcal{M}$ by

$$\mathrm{d}(u,v)=\|u-v\|,\qquad \forall\ u,\ v\in H.$$

Then we have the following

Proposition 3.1. *If* (V1)*, and* (V2) *hold, then ($\mathcal{M}$, d) is a complete metric space.*

Proof. First we claim that there exists $\mu>0$ such that

$$\int_{\mathbb{R}^N}|u^{\pm}|^{p+1}\mathrm{d}x\geq\mu,\qquad \forall\ u\in\mathcal{M}. \tag{3.1}$$

Indeed, for $u\in\mathcal{M}$, we have

$$0=\langle I'(u),u^{\pm}\rangle=\int_{\mathbb{R}^N}|\nabla u^{\pm}|^2+V(|x|)|u^{\pm}|^2\mathrm{d}x-\int_{\mathbb{R}^N}|u^{\pm}|^{p+1}\mathrm{d}x.$$

Then by Proposition 2.1, we get (3.1). Let $\{u_n\}\subset\mathcal{M}$ satisfy

$$d(u_n,u_m)\to 0,\qquad \text{as } n,\ m\to+\infty.$$

Then $\{u_n\}$ is a Cauchy sequence in H. By the completeness of H, there exists $u\in H$ such that

$$\|u_n-u\|\to 0,\qquad \text{as } n\to+\infty.$$

Then by Proposition 2.1, we get

$$\|u_n-u\|_{L^{p+1}}\to 0,\qquad \text{as } n\to+\infty.$$

This implies

$$\|u_n^+-u^+\|_{L^{p+1}}\xrightarrow{n}0,\qquad \|u_n^--u^-\|_{L^{p+1}}\xrightarrow{n}0. \tag{3.2}$$

From (3.1) we see that $u^{\pm}\not\equiv 0$.

To complete the proof of this proposition, we need only to show that

$$\langle I'(u),u^+\rangle=\langle I'(u),u^-\rangle=0.$$

Since $I\in C^1(H,\mathbb{R})$ and $u_n\xrightarrow{n}u$ strongly in H, we have

$$\|I'(u_n)-I'(u)\|_{H^{-1}}\xrightarrow{n}0,$$

where H^{-1} denotes the dual space of H. Thus, we deduce that

$$\begin{aligned}|\langle I'(u),u_n^+\rangle| &= |\langle I'(u)-I'(u_n),u_n^+\rangle| \le \|I'(u_n)-I'(u)\|_{H^{-1}}\|u_n^+\| \\ &\le C\|I'(u_n)-I'(u)\|_{H^{-1}} \xrightarrow{n} 0. \end{aligned} \tag{3.3}$$

On the other hand, by $u_n^+ \overset{n}{\rightharpoonup} u^+$ weakly in H, we have

$$\langle I'(u),u_n^+\rangle \xrightarrow{n} \langle I'(u),u^+\rangle. \tag{3.4}$$

Combining (3.3) and (3.4), for the unique of the limit, we get $\langle I'(u),u^+\rangle = 0$. Similarly we can prove that $\langle I'(u),u^-\rangle = 0$. Hence, we see that $u \in \mathcal{M}$ and complete the proof of this proposition. ☐

Define

$$c_1 = \inf\{I(u) : u \in \mathcal{M}\}.$$

Now, we follow the argument of [19] to show that c_1 is achieved by some $u \in \mathcal{M}$ and u is a nodal solution of problem (1.1).

Lemma 3.1. *If* (V1), *and* (V2) *hold, then c_1 is achieved by some $u \in \mathcal{M}$. Moreover, u is a nodal solution with precisely two nodal domains of problem* (1.1).

Proof. By Proposition 3.1 we know that ($\mathcal{M}$, d) is a complete metric space. It is easy to check that I is bounded from below on $\mathcal{M}$. Then by Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \mathcal{M}$ such that

$$c_1 \le I(u_n) \le c_1 + \frac{1}{n}, \tag{3.5}$$

$$I(v) \ge I(u_n) - \frac{1}{n}\|v-u_n\|, \qquad \forall\, v \in \mathcal{M}. \tag{3.6}$$

From $u_n \in \mathcal{M}$, we have $\langle I'(u_n),u_n\rangle = 0$. Thus, for n large enough, we get

$$c_1 + 1 \ge I(u_n) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2.$$

So, $\{u_n\}$ is bounded in H. Now we will prove that

$$I'(u_n) \xrightarrow{n} 0 \qquad \text{in } H^{-1}. \tag{3.7}$$

For each n and $\varphi \in H$, we define the functions $h_n^\pm : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} h_n^\pm(t,s,l) = &\int_{\mathbb{R}^N} |\nabla(u_n + t\varphi + su_n^+ + lu_n^-)^\pm|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(|x|)|(u_n + t\varphi + su_n^+ + lu_n^-)^\pm|^2 \mathrm{d}x \\ &- \int_{\mathbb{R}^N} |(u_n + t\varphi + su_n^+ + lu_n^-)^\pm|^{p+1} \mathrm{d}x. \end{aligned} \tag{3.8}$$

Then we see that $h_n^{\pm}$ are of class C^1 and $h_n^{\pm}(0,0,0)=0$. Moreover, let

$$v=u_n+t\varphi+su_n^+ +lu_n^-,$$

we have that

$$\frac{\partial h_n^+}{\partial s}(0,0,0)=2\int_{\mathbb{R}^N}(\nabla v^+\nabla u_n^+ +V(|x|)v^+u_n^+)\mathrm{d}x-(p+1)\int_{\mathbb{R}^N}|v^+|^p u_n^+\mathrm{d}x. \tag{3.9}$$

This implies

$$\frac{\partial h_n^+}{\partial s}(0,0,0)=(1-p)\int_{\mathbb{R}^N}|u_n^+|^{p+1}\mathrm{d}x. \tag{3.10}$$

Similarly, we can deduce that

$$\frac{\partial h_n^-}{\partial l}(0,0,0)=(1-p)\int_{\mathbb{R}^N}|u_n^-|^{p+1}\mathrm{d}x. \tag{3.11}$$

$$\frac{\partial h_n^+}{\partial l}(0,0,0)=0, \qquad \frac{\partial h_n^-}{\partial s}(0,0,0)=0. \tag{3.12}$$

Combining (3.1), (3.10) and (3.11), we get

$$\frac{\partial h_n^+}{\partial s}(0,0,0)<0, \qquad \frac{\partial h_n^-}{\partial l}(0,0,0)<0. \tag{3.13}$$

Thus, by the implicit function theorem, there exits $\delta_n>0$ and functions $s_n(t)$, $l_n(t)\in C^1((-\delta_n,\delta_n),\mathbb{R})$ such that $s_n(0)=l_n(0)=0$ and

$$h_n^{\pm}(t,s_n(t),l_n(t))=0, \qquad \forall\, t\in(-\delta_n,\,\delta_n). \tag{3.14}$$

This implies that

$$v_n=u_n+t\varphi+s_n(t)u_n^+ +l_n(t)u_n^-\in\mathcal{M}, \qquad \forall\, t\in(-\delta_n,\,\delta_n). \tag{3.15}$$

We claim that the sequence $\{s_n'(0)\}$ is bounded.

In fact, from (3.14) we have

$$\frac{\partial h_n^+}{\partial t}(0,0,0)+\frac{\partial h_n^+}{\partial s}(0,0,0)s_n'(0)+\frac{\partial h_n^+}{\partial l}(0,0,0)l_n'(0)=0.$$

Then (3.12) implies

$$s_n'(0)=\frac{-\frac{\partial h_n^+}{\partial t}(0,0,0)}{\frac{\partial h_n^+}{\partial s}(0,0,0)}=\frac{-\left[2\int_{\mathbb{R}^N}(\nabla u_n^+\nabla\varphi+V(|x|)u_n^+\varphi)\mathrm{d}x-(p+1)\int_{\mathbb{R}^N}|u_n^+|^p\varphi\mathrm{d}x\right]}{(1-p)\int_{\mathbb{R}^N}|u_n^+|^{p+1}\mathrm{d}x}.$$

Thus it follows from (3.1) and $\|u_n\|\le C$ that $|s_n'(0)|\le C$. Similarly, we can also deduce that the sequence $\{l_n'(0)\}$ is bounded.

By (3.6) and (3.15) we get

$$I(u_n+t\varphi+s_n(t)u_n^++l_n(t)u_n^-)-I(u_n)\geq\frac{-1}{n}\|t\varphi+s_n(t)u_n^++l_n(t)u_n^-\|.$$

Letting $t\to 0^+$ and $t\to 0^-$, we have

$$\langle I'(u_n),\varphi\rangle\geq\frac{-1}{n}\|\varphi+s_n'(0)u_n^++l_n'(0)u_n^-\|, \tag{3.16}$$

$$\langle I'(u_n),\varphi\rangle\leq\frac{1}{n}\|\varphi+s_n'(0)u_n^++l_n'(0)u_n^-\|. \tag{3.17}$$

Since the sequences $\{s_n'(0)\}$ and $\{l_n'(0)\}$ are bounded, combining (3.16) and (3.17) we have

$$\langle I'(u_n),\varphi\rangle\xrightarrow{n}0,\qquad\forall\varphi\in H.$$

Thus, (3.7) holds. Then by Proposition 2.1, there exists $u\in H$ such that

$$u_n\to u\text{ strongly in }H,\qquad I(u)=c_1\qquad\text{and}\qquad I'(u)=0.$$

From (3.1) we know that $u^{\pm}\not\equiv 0$. Hence, u is a nodal solution of problem (1.1).

Next, motivated by [2] we prove that u has precisely two nodal domains. Setting

$$\bar{c}=\inf\{I(u):\ u\in\mathcal{N}\},\ \ \mathcal{N}=\{u\in H\backslash\{0\}:\ \langle I'(u),u\rangle=0\}.$$

Similar to the proof of (3.1) we have $\bar{c}>0$. If Ω is a nodal domain of u, by Proposition 2.2 we get $u\chi_\Omega\in H$ and

$$\langle I'(u\chi_\Omega),u\chi_\Omega\rangle=\langle I'(u),u\chi_\Omega\rangle=0.$$

So, $I(u\chi_\Omega)\geq\bar{c}>0$.

Suppose by contradiction that u has three nodal domains Ω_1, Ω_2, Ω_3 such that

$$u>0\qquad\text{on }\Omega_1\qquad\text{and}\qquad u<0\qquad\text{on }\Omega_2.$$

Then by Proposition 2.2 we have $u\chi_{\Omega_1\cup\Omega_2}\in H$. Noting that

$$(u\chi_{\Omega_1\cup\Omega_2})^+=u\chi_{\Omega_1}\qquad\text{and}\qquad(u\chi_{\Omega_1\cup\Omega_2})^-=u\chi_{\Omega_2},$$

we see that $u\chi_{\Omega_1\cup\Omega_2}\in\mathcal{M}$. Hence, we get a contradiction as

$$c_1<c_1+\bar{c}\leq I(u\chi_{\Omega_1\cup\Omega_2})+I(u\chi_{\Omega_3})\leq I(u)=c_1.$$

This completes the proof of this lemma. □

Lemma 3.2. *Suppose that* (V1), (V2) *hold and let u be the nodal solution of problem* (1.1) *given by Lemma* 3.1. *Then there exist* C, ϵ, $R>0$ *such that*

$$|u(x)|\leq C\exp\{-\epsilon|x|^{\frac{2-\alpha}{2}}\},\qquad\forall\ |x|\geq R, \tag{3.18}$$

where $\alpha\in[0,\ 2(N-1)(p-1)/(p+3))$ *is given by* (V2).

Proof. By Lemma 3.1, we know that $u(|x|)$ has precisely two nodal domains. That is, letting $r=|x|$, then $\{r\in[0,+\infty):u(r)\neq 0\}$ has two connected components. Thus, there exists $R_0>0$ such that $u(|x|)$ on the interval $[R_0,+\infty)$ has fixed sign. We may as well assume that $u(|x|)\geq 0$ for $|x|\geq R_0$.

Since $u\in H$, by (V2) and similar to the proof of (3.3) in [1], there exist $C_1>0$ and $\gamma=N-1-\alpha/2$ such that

$$0\leq u(|x|)\leq C_1|x|^{-\frac{\gamma}{2}},\qquad \forall\ |x|\geq R_0. \tag{3.19}$$

By (V2), there exists $C_2>0$ such that

$$V(x)\geq C_2|x|^{-\alpha},\qquad \forall\ |x|\geq R_0. \tag{3.20}$$

Noting that $\gamma/2>\alpha/(p-1)$, then combining (1.1), (3.19) and (3.20), we can find $R_1>R_0$ such that

$$\begin{aligned}
&-\Delta u(x)+\frac{1}{2}V(x)u(x)\\
=&u(x)\left[(u(x))^{p-1}-\frac{1}{2}V(x)\right]\leq u(x)[C_1|x|^{-\frac{\gamma(p-1)}{2}}-C_2|x|^{-\alpha}]\\
\leq&0,\qquad \text{for } |x|\geq R_1.
\end{aligned} \tag{3.21}$$

Let $w(x)=\exp\{-\epsilon|x|^{\frac{2-\alpha}{2}}\}$, where $\epsilon>0$ will be determined later. Then we have

$$-\Delta\omega(x)=-\varepsilon^2\left(\frac{2-a}{2}\right)^2|x|^{-\alpha}\omega(x)+\varepsilon\left[N\cdot\frac{2-\alpha}{2}+\frac{\alpha^2-4}{4}\right]|x|^{\frac{2-\alpha}{2}-2}\omega(x).$$

This and (3.20) imply that, for $|x|\geq R_0$,

$$\begin{aligned}
&-\Delta w(x)+\frac{1}{2}V(x)w(x)\\
\geq&\left[\frac{C_2}{2}-\epsilon^2\left(\frac{2-\alpha}{2}\right)^2\right]|x|^{-\alpha}w(x)+\varepsilon\left[N\frac{2-\alpha}{2}+\frac{\alpha^2-4}{4}\right]|x|^{\frac{2-\alpha}{2}-2}\omega(x).
\end{aligned} \tag{3.22}$$

We choose $\epsilon>0$ small enough such that

$$C_3=\frac{C_2}{2}-\epsilon^2\left(\frac{2-\alpha}{2}\right)^2>0.$$

Note that $2-(2-\alpha)/2>\alpha$. Then there exists $R_2>R_0$ such that

$$C_3|x|^{-\alpha}+\varepsilon\left[N\cdot\frac{2-\alpha}{2}+\frac{\alpha^2-4}{4}\right]|x|^{\frac{2-\alpha}{2}-2}>0,\qquad \forall\ |x|\geq R_2.$$

This and (3.22) imply that

$$-\Delta w(x)+\frac{1}{2}V(x)w(x)\geq 0, \qquad \text{for } |x|\geq R_2. \tag{3.23}$$

Setting $R=\max\{R_1,R_2\}$, we have from (3.21) and (3.23) that

$$-\Delta u(x)+\frac{1}{2}V(x)u(x)\leq -\Delta w(x)+\frac{1}{2}V(x)w(x), \qquad \text{for } |x|\geq R. \tag{3.24}$$

We can choose $C>0$ such that $u(x)\leq Cw(x)$ for $|x|=R$. Then by the comparison theorem we know that (3.18) holds. □

Proof of Theorem 1.1: Noting that $2(N-1)(p-1)/(p+3)<2$ for $1<p<(N+2)/(N-2)$, then from (3.18) we have $u\in L^2(\mathbb{R}^N)$. Hence, by Lemmas 3.1 and 3.2 we see that problem (1.1) has a nodal type bound state. □

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