JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **24**, No. 1, pp. 70-82

doi: 10.4208/jpde.v24.n1.5
February 2011

Remarks on the Regularity Criteria of Three-Dimensional Navier-Stokes Equations in Margin Case

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Received 23 July 2010; Accepted 7 December 2010

Abstract. In the study of the regularity criteria for Leray weak solutions to three-dimensional Navier-Stokes equations, two sufficient conditions such that the horizontal velocity $\tilde{u}$ satisfies $\tilde{u}\in L^2(0,T;BMO(\mathbf{R}^3))$ or $\tilde{u}\in L^{2/1+r}(0,T;\dot{B}^r_{\infty,\infty}(\mathbf{R}^3))$ for $0<r<1$ are considered.

AMS Subject Classifications: 35Q35

Chinese Library Classifications: O175.24

Key Words: Regularity criteria; Navier-Stokes equations; BMO; Besov space.

1 Introduction and main results

The incompressible fluid motion in the whole space $\mathbf{R}^3$ is governed by the Navier-Stokes equations with unit viscosity

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla\pi=\Delta u, \\ \nabla\cdot u=0, \\ u(x,0)=u_0. \end{cases} \tag{1.1}$$

Here $u=(u_1,u_2,u_3)$ and π present the unknown velocity field and the unknown scalar pressure field, u_0 is a given initial velocity.

Since the pioneer study of Leray [1] in 1930s, there is a large literature on the well-posedness of weak solutions to the incompressible Navier-Stokes equations. Many contributions have been made in an effort to understand the regularity of the weak solutions. However, the problem on the regularity or finite time singularity for the weak solution

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still remains unsolved. Regularity can only been derived when certain growth conditions are satisfied. This is known as a regularity criterion problem. The investigation of the regularity criterion on the weak solution stems from the celebrated work of Serrin [2]. With the extended examinations given by Struwe [3], Serrin's regularity criterion can be described as follows:

A weak solution u is regular if the growth condition

$$u\in L^p(0,T;L^q(\boldsymbol{R}^3))\equiv L^pL^q,\qquad \text{for}\quad \frac{2}{p}+\frac{3}{q}=1,\quad 3<q\leq\infty, \tag{1.2}$$

holds true.

The condition described by (1.2) which involves all components of the velocity vector field $u=(u_1,u_2,u_3)$ is known as degree -1 growth condition (see Chen and Xin [4]), since

$$\|u(\lambda\cdot,\lambda^2\cdot)\|_{L^pL^q}=\|u\|_{L^p(0,\lambda^2T;L^q(\mathbf{R}^3))}\lambda^{-\frac{2}{p}-\frac{3}{q}}=\|u\|_{L^p(0,\lambda^2T;L^q(\mathbf{R}^3))}\lambda^{-1}.$$

The degree -1 growth condition is critical due to the scaling invariance property. That is, $u(x,t)$ solves (1.1) if and only if $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$ is a solution of (1.1).

Moreover, this result has been extended by many authors in terms of velocity $u(x,t)$, the gradient of velocity $\nabla u(x,t)$ or vorticity $w(x,t)=\nabla\times u$ in Lebesgue spaces, BMO space or Besov spaces, respectively (refer to [5–10] and reference therein).

Actually, the weak solution remains regular when a part of the velocity components is involved in some growth conditions. For example, regularity of the weak solution was recently obtained by Beirão da Veiga [11] (see also Dong and Chen [12]) when the horizontal velocity denoted by

$$\tilde{u}=(u_1,u_2,0)$$

satisfies the critical growth condition

$$\tilde{u}\in L^pL^q,\qquad \text{for}\quad \frac{2}{p}+\frac{3}{q}=1,\quad 3<q\leq\infty. \tag{1.3}$$

And some critical growth conditions on the two vorticity components were obtained by Kozono and Yatsu [13], Zhang and Chen [14]. One may also mentioned that the weak solution remains regular if the single velocity component satisfies the higher (subcritical) growth conditions (see Zhou [15, 16], Penel and Pokorý [17], Kukavica and Ziane [18], Cao and Titi [19]).

The margin case $q=\infty$ in (1.3) appears to be more challenging. The aim of the present paper is to improve the regularity criterion (1.3) from Lebesgue space L^∞ to BMO space and Besov space (see the definitions in Section 2), respectively.

Before statement the main results, we firstly recall the definition of Leray weak solution of Navier-Stokes equations (see, for example, [20]).

Definition 1.1. *Let $u_0\in L^2(\boldsymbol{R}^3)$ and $\nabla\cdot u_0=0$. A vector field $u(x,t)$ is termed as a Leray weak solution of* (1.1) *if u satisfies the following properties:*

(i) $u\in L^\infty(0,T;L^2(\boldsymbol{R}^3))\cap L^2(0,T;H^1(\boldsymbol{R}^3))$ *for* $\forall\ T>0$;

(ii) $\partial_t u+(u\cdot\nabla)u+\nabla\pi=\Delta u$ *in the distribution space* $\mathcal{D}'((0,T)\times\boldsymbol{R}^3)$;

(iii) $\nabla\cdot u=0$ *in the distribution space* $\mathcal{D}'((0,T)\times\boldsymbol{R}^3)$;

(iv) *u satisfies the energy inequality*

$$\|u(t)\|^2_{L^2}+2\int_0^t\int_{\boldsymbol{R}^3}|\nabla u(x,s)|^2\mathrm{d}x\mathrm{d}s\le\|u_0\|^2_{L^2},\qquad for\quad 0\le t\le T. \tag{1.4}$$

By a strong solution we mean a weak solution $u(x,t)$ of Navier-Stokes equations (1.1) with the initial velocity $u_0\in H^1(\mathbf{R}^3)$ satisfies

$$u\in L^\infty(0,T;H^1(\mathbf{R}^3))\cap L^2(0,T;H^2(\mathbf{R}^3)). \tag{1.5}$$

It is well known that strong solution is regular and unique. In this case one also has the energy equality in (1.4) instead of the inequality.

The main results now read:

Theorem 1.1. *Suppose* $\forall\ T>0$, $u_0\in H^1(\boldsymbol{R}^3)$ *and* $\nabla\cdot u_0=0$ *in the sense of distributions. Assume that u is a Leray weak solution of* (1.1) *in* $(0,T)$*. If the horizontal velocity denoted by* $\tilde{u}=(u_1,u_2,0)$ *satisfies the following growth condition*

$$\int_0^T\|\tilde{u}(t)\|^2_{BMO}\mathrm{d}t<\infty. \tag{1.6}$$

Then u is a regular solution on $(0,T]$.

Theorem 1.2. *On substitution of the condition* (1.6) *by the following growth condition*

$$\int_0^T\|\tilde{u}(t)\|^{\frac{2}{1+r}}_{B^r_{\infty,\infty}}\mathrm{d}t<\infty,\qquad 0<r<1, \tag{1.7}$$

the conclusion of Theorem 1.1 *holds true.*

Remark 1.1. Theorems 1.1 and 1.2 improve the earlier results [6, 10–12] and it is easy to verify that the spaces (1.6) and (1.7) satisfy the degree -1 growth conditions. The results are in the spirit of the Beale-Kato-Majda [21] criterion for 3D Euler equations. It should be mentioned that for the case $r=1$ in Theorem 1.2, Dong and Zhang [22] have recently refined the regularity of weak solution if the horizontal derivatives of the horizontal velocity satisfies

$$\int_0^T\|\nabla_h\tilde{u}(t)\|_{\dot{B}^0_{\infty,\infty}}\mathrm{d}t<\infty,\qquad \nabla_h\tilde{u}=(\partial_1\tilde{u},\partial_2\tilde{u},\,0).$$

The case $r=0$, however, still remains unsolved.

2 Preliminaries

Throughout this paper, c stands for a generic positive constant which may vary from line to line. $L^p(\mathbf{R}^3)$ with $1\le p\le\infty$ denotes the usual Lebesgue space of all L^p integral functions associated with the norm

$$\|f\|_{L^p}=\begin{cases}\left(\int_{\mathbf{R}^3}|f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \operatorname*{ess\,sup}_{x\in\mathbf{R}^3}|f(x)|, & p=\infty.\end{cases}$$

In order to define Besov space and Triebel-Lizorkin space, let us first recall the Littlewood-Paley decomposition theory (see Chemin [23]). Let $\mathcal{S}(\mathbf{R}^3)$ be the Schwartz class of rapidly decreasing function, given $f\in\mathcal{S}(\mathbf{R}^3)$, its Fourier transformation $\mathcal{F}$ or $\hat{f}$ is defined by

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=(2\pi)^{-\frac{3}{2}}\int_{\mathbf{R}^3}e^{-ix\cdot\xi}f(x)\mathrm{d}x.$$

Choose two nonnegative radial functions $\chi,\phi\in\mathcal{S}(\mathbf{R}^3)$ supported in $\mathcal{B}=\{\xi\in\mathbf{R}^3:|\xi|\le 4/3\}$ and $\mathcal{C}=\{\xi\in\mathbf{R}^3:3/4\le|\xi|\le 8/3\}$, respectively, such that

$$\sum_{j\in Z}\varphi(2^{-j}\xi)=1,\qquad \xi\in\mathbf{R}^3\setminus\{0\}.$$

Let $h=\mathcal{F}^{-1}\varphi$ and $\tilde{h}=\mathcal{F}^{-1}\chi$, and then we define the dyadic blocks as follows:

$$\begin{aligned}\Delta_j f&=\varphi(2^{-j}D)f=2^{3j}\int_{\mathbf{R}^3}h(2^jy)f(x-y)\,\mathrm{d}y,\\ S_j f&=\chi(2^{-j}D)f=\sum_{k\le j-1}\Delta_k f=2^{3j}\int_{\mathbf{R}^3}\bar{h}(2^jy)f(x-y)\,\mathrm{d}y.\end{aligned}$$

By telescoping the series, we thus have the following Littlewood-Paley decomposition

$$f=\sum_{j=-\infty}^{\infty}\Delta_j f. \tag{2.1}$$

Moreover, from the Young inequality, the following classic Bernstein inequality reads:

Lemma 2.1. (Chemin [23]) *Assume $1\le p\le q\le\infty$. Then*

$$\sup_{|\alpha|=k}\|\partial^\alpha\Delta_j f\|_{L^q}\le c\,2^{jk+3j(1/p-1/q)}\,\|\Delta_j f\|_{L^p}, \tag{2.2}$$

with c being a positive constant independent of f,j.

With the introduction of Δ_j, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R}^3)$ for $s\in\mathbf{R}, p,q\in[1,\infty]$ is defined by the full-dyadic decomposition such as

$$\dot{B}^s_{p,q}(\mathbf{R}^3)=\Big\{f\in\mathcal{S}'(\mathbf{R}^3)/\mathcal{P}(\mathbf{R}^3):\|f\|_{\dot{B}^s_{p,q}}<\infty\Big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}}=\begin{cases}\left(\displaystyle\sum_{j=-\infty}^{\infty}2^{jsq}\|\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & 1\le q<\infty,\\ \displaystyle\sup_{j\in Z}2^{js}\|\Delta_j f\|_{L^p}, & q=\infty,\end{cases}$$

and $\mathcal{S}'(\mathbf{R}^3),\mathcal{P}(\mathbf{R}^3)$ are the spaces of all tempered distributions on $\mathbf{R}^3$ and the set of all scalar polynomials defined on $\mathbf{R}^3$, respectively. For $p=q=2, \dot{B}^s_{2,2}(\mathbf{R}^3)\cong\dot{H}^s(\mathbf{R}^3)$, where $\dot{H}^s(R^2)$ is the homogeneous Sobolev space.

In a similar way, the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbf{R}^3)$ can also be defined by

$$\dot{F}^s_{p,q}(\mathbf{R}^3)=\Big\{f\in\mathcal{S}'(\mathbf{R}^3)/\mathcal{P}(\mathbf{R}^3):\|f\|_{\dot{F}^s_{p,q}}<\infty\Big\},$$

where

$$\|f\|_{\dot{F}^s_{p,q}}=\left\|\left(\sum_{j=-\infty}^{\infty}2^{jsq}|\Delta_j f|^q\right)^{\frac{1}{q}}\right\|_{L^p},$$

for $1\le p<\infty$, $1\le q\le\infty$ and $p=\infty$, $1\le q<\infty$.

It is readily seen that space definitions imply the following continuous embeddings

$$L^\infty(\mathbf{R}^3)\subset\dot{F}^0_{\infty,2}(\mathbf{R}^3)\subset\dot{B}^0_{\infty,\infty}(\mathbf{R}^3). \tag{2.3}$$

Especially, the following interesting relation between the Lizorkin-Triebel spaces and the BMO is due to Triebel [24, Section 2.3.5].

Lemma 2.2. $\dot{F}^0_{\infty,2}\cong BMO$. *Namely, there exist two positive constants c_1,c_2 such that*

$$c_1\|f\|_{\dot{F}^0_{\infty,2}}\le\|f\|_{BMO}\le c_2\|f\|_{\dot{F}^0_{\infty,2}}, \tag{2.4}$$

where BMO is the space of the bounded mean oscillations defined by

$$BMO=\left\{f\in L^1_{loc}(\boldsymbol{R}^3);\sup_{x,r}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)-\bar{f}_{B_r(x)}|\mathrm{d}y<\infty\right\},$$

with

$$\bar{f}_{B_r(x)}=\frac{1}{|B_r(x)|}\int_{B_r(x)}f(y)\mathrm{d}y.$$

For more properties of these function spaces, one may refer to [24].

3 *A priori* estimates

In order to prove Theorems 1.1 and 1.2, it is sufficient to examine *a priori* estimates for smooth solutions of (1.1) described in the following.

Theorem 3.1. *Let $T>0$, $u_0\in H^1(\boldsymbol{R}^3)$ with $\nabla\cdot u_0=0$. Assume that $u(x,t)$ is a smooth solution of* (1.1) *on $\boldsymbol{R}^3\times(0,T)$ and satisfies the growth conditions* (1.6). *Then*

$$\sup_{0\le t<T}\|\nabla u(t)\|_{L^2}\le c\big(\|\nabla u_0\|_{L^2}+e\big)^{\exp\left\{c\int_0^T\|\tilde{u}(s)\|^2_{BMO}\mathrm{d}s\right\}}$$

holds true.

Theorem 3.2. *Under the same conditions on Theorem* 3.1 *with the smooth solution $u(x,t)$ satisfies* (1.7). *Then*

$$\sup_{0<t<T}\|\nabla u(t)\|_{L^2}\le\|\nabla u_0\|_{L^2}\exp\left\{c\int_0^T(e+\|\tilde{u}(t)\|_{\dot{B}^r_{\infty,\infty}})^{\frac{2}{1+r}}\mathrm{d}t\right\}$$

holds true.

3.1 Proof of Theorem 3.1

Taking inner product of the momentum equations of (1.1) with Δu and integrating by parts, one shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2_{L^2}+\|\Delta u(t)\|^2_{L^2}\le-\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}u_i\partial_i u_j\partial_{kk}u_j\ \mathrm{d}x. \tag{3.1}$$

We will show that the right hand side of (3.1) is bounded by

$$-\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}u_i\partial_i u_j\partial_{kk}u_j\ \mathrm{d}x\le c\int_{\mathbf{R}^3}|\tilde{u}|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x. \tag{3.2}$$

It should mentioned that the assertion (3.2) is more or less obtained by Beirão da Veiga [11], for the readers' convenience, we present a simple proof.

Firstly, with the aid of the divergence free condition $\sum_{i=1}^3\partial_i u_i=0$ and integration by parts, observe that,

$$\begin{aligned}&-\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}u_i\partial_i u_j\partial_{kk}u_j\ \mathrm{d}x=\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}\partial_k(u_i\partial_i u_j)\ \partial_k u_j\ \mathrm{d}x\\&=\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\ \mathrm{d}x+\frac{1}{2}\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}u_i\partial_i(\partial_k u_j\partial_k u_j)\ \mathrm{d}x\\&=\sum_{i,j,k=1}^3\int_{\mathbf{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\ \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j \mathrm{d}x+\sum_{j=1}^{2}\sum_{k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_3\partial_3 u_j\partial_k u_j \mathrm{d}x+\sum_{k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_3\partial_3 u_3\partial_k u_3 \mathrm{d}x\\
&=\sum_{m=1}^{3} I_m. \qquad (3.3)
\end{aligned}
$$

The estimation of the terms I_m is demonstrated one by one in the following.

In order to estimate I_1 and I_2. we apply integration by parts to have

$$
\begin{aligned}
I_1&=\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbf{R}^3} u_i\partial_k(\partial_i u_j\partial_k u_j)\mathrm{d}x\le c\int_{\mathbf{R}^3}|\tilde{u}|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x,\\
I_2&=\sum_{j=1}^{2}\sum_{k=1}^{3}\int_{\mathbf{R}^3} u_j\partial_3(\partial_k u_3\partial_k u_j)\mathrm{d}x\le c\int_{\mathbf{R}^3}|\tilde{u}|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x.
\end{aligned}
$$

For I_3, the divergence free condition $\partial_3 u_3=-\partial_1 u_1-\partial_2 u_2$ and integration by parts imply

$$
\begin{aligned}
I_3&=\sum_{k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_3(\partial_1 u_1+\partial_2 u_2)\partial_k u_3 \mathrm{d}x\\
&\le-\sum_{k=1}^{3}\int_{\mathbf{R}^3}\big(u_1\partial_1(\partial_k u_3\partial_k u_3)+u_2\partial_2(\partial_k u_3\partial_k u_3)\big)\ \mathrm{d}x\\
&\le c\int_{\mathbf{R}^3}|\tilde{u}|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x.
\end{aligned}
$$

Thus plugging the above inequalities into (3.3) to derive (3.2) and then (3.1)implies

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+2\|\Delta u(t)\|_{L^2}^2\le c\int_{\mathbf{R}^3}|\tilde{u}|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x. \qquad (3.4)
$$

Making use of the Littlewood-Paley decomposition (2.1) for $\tilde{u}$ reads firstly,

$$
\tilde{u}=\sum_{j<-N}\Delta_j\tilde{u}+\sum_{j=-N}^{N}\Delta_j\tilde{u}+\sum_{j>N}\Delta_j\tilde{u},
$$

and then applying that to the right hand side of (3.4) gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2\\
\le&\int_{\mathbf{R}^3}\Big|\sum_{j<-N}\Delta_j\tilde{u}\Big|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x+c\int_{\mathbf{R}^3}\Big|\sum_{j=-N}^{N}\Delta_j\tilde{u}\Big|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x\\
&\quad+c\int_{\mathbf{R}^3}\Big|\sum_{j>N}\Delta_j\tilde{u}\Big|\ |\nabla u|\ |\nabla^2 u|\ \mathrm{d}x\\
=&J_1+J_2+J_3, \qquad (3.5)
\end{aligned}
$$

where the positive integer N will be chosen later.

We now estimate J_l (l=1,2,3) one by one. For J_1, applying Hölder inequality, Minkowski inequality and Bernstein inequality (2.2), one shows that

$$\begin{aligned}J_1=&c\int_{\mathbf{R}^3}\Big|\sum_{j<-N}\Delta_j\tilde{u}\Big|\,|\nabla u|\,|\nabla^2 u|\,\mathrm{d}x\\ \le& c\sum_{j<-N}\|\Delta_j\tilde{u}\|_{L^3}\|\nabla u\|_{L^6}\|\Delta u\|_{L^2}\le c\sum_{j<-N}2^{\frac{j}{2}}\|\Delta_j\tilde{u}\|_{L^2}\|\Delta u\|_{L^2}^2\\ \le& c\Big(\sum_{j<-N}2^j\Big)^{\frac12}\Big(\sum_{j<-N}\|\Delta_j\tilde{u}\|_{L^2}^2\Big)^{\frac12}\|\Delta u\|_{L^2}^2\\ \le& c2^{-\frac{N}{2}}\|u\|_{\dot{B}^0_{2,2}}\|\Delta u\|_{L^2}^2\cong c2^{-\frac{N}{2}}\|u\|_{L^2}\|\Delta u\|_{L^2}^2\le c2^{-\frac{N}{2}}\|\Delta u\|_{L^2}^2,\end{aligned}\tag{3.6}$$

where we have used the inequality $\|u(t)\|_{L^2}\le\|u_0\|_{L^2}$ which is derived from the energy inequality (1.4).

For J_2, with the aid of the definition of Triebel-Lizorkin space $\dot{F}^0_{\infty,2}(\mathbf{R}^3)$ and Lemma 2.2, we have

$$\begin{aligned}J_2=&c\int_{\mathbf{R}^3}\Big|\sum_{j=-N}^{N}\Delta_j\tilde{u}\Big|\,|\nabla u|\,|\nabla^2 u|\,\mathrm{d}x\\ \le& c\Big\|\sum_{j=-N}^{N}\Delta_j\tilde{u}\Big\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\ \le& c\Big\|\Big(\sum_{j=-N}^{N}1\Big)^{\frac12}\Big(\sum_{j=-N}^{N}|\Delta_j\tilde{u}|^2\Big)^{\frac12}\Big\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\ \le& cN^{1/2}\|\tilde{u}\|_{\dot{F}^0_{\infty,2}}\,\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}\\ \le& cN^{1/2}\|\tilde{u}\|_{BMO}\,\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}\\ \le& cN\|\tilde{u}\|_{BMO}^2\,\|\nabla u\|_{L^2}^2+\frac14\|\Delta u\|_{L^2}^2.\end{aligned}\tag{3.7}$$

Similarly, by Hölder inequality and Bernstein inequality (2.2), J_3 yields

$$\begin{aligned}J_3=&c\int_{\mathbf{R}^3}\Big|\sum_{j>N}\Delta_j\tilde{u}\Big|\,|\nabla u|\,|\nabla^2 u|\,\mathrm{d}x\\ \le& c\sum_{j>N}\|\Delta_j\tilde{u}\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\ \le& c\sum_{j>N}2^{\frac{3j}{2}}\|\Delta_j\tilde{u}\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\end{aligned}$$

$$\begin{aligned}&\le c\Big(\sum_{j>N}2^{-j}\Big)^{\frac{1}{2}}\Big(\sum_{j>N}2^{4j}\|\Delta_j\tilde{u}\|_{L^2}^2\Big)^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\&\le c2^{-\frac{N}{2}}\|u\|_{\dot{B}^2_{2,2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2\le c2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2.\end{aligned}\tag{3.8}$$

Inserting (3.6-3.8) into the inequality (3.5) to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2\le cN\|\tilde{u}\|_{BMO}^2\;\|\nabla u\|_{L^2}^2,\tag{3.9}$$

here we have used the inequality

$$c\,2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\le\frac{1}{8}\quad\text{and}\quad c\,2^{-\frac{N}{2}}\le\frac{1}{8},$$

for suitable integer N. In fact, N may be chosen by

$$N\ge\max\left\{\frac{\ln(\|\nabla u\|_{L^2}^2+e)+\ln c}{\ln 2}+3,\frac{\ln c}{\ln 2}+3\right\},$$

hence (3.9) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2\le c\|\nabla u\|_{L^2}^2\;\|\tilde{u}\|_{BMO}^2\big(\ln(\|\nabla u\|_{L^2}^2+e)\big).\tag{3.10}$$

Integrating in time from 0 to t to produce

$$\|\nabla u(t)\|_{L^2}^2\le\|\nabla u_0\|_{L^2}^2\exp\left\{c\int_0^t\|\tilde{u}(s)\|_{BMO}^2\big(\ln(\|\nabla u(s)\|_{L^2}^2+e)\big)\,\mathrm{d}s\right\},$$

and so

$$\ln(\|\nabla u(t)\|_{L^2}^2+e)\le\ln(\|\nabla u_0\|_{L^2}^2+e)+c\int_0^t\|\tilde{u}(s)\|_{BMO}^2\big(\ln(\|\nabla u(s)\|_{L^2}^2+e)\big)\,\mathrm{d}s.$$

Taking the Gronwall inequality into consideration, one shows that

$$\sup_{0\le t<T}\|\nabla u(t)\|_{L^2}\le c(\|\nabla u_0\|_{L^2}+e)^{\exp\left\{c\int_0^T\|\tilde{u}(s)\|_{BMO}^2\mathrm{d}s\right\}}.$$

This completes the proof of Theorem 3.1. □

3.2 Proof of Theorem 3.2

By developing the idea of Chen, Miao and Zhang [25, Lemma 3.1], we first give a decomposition on the critical space (1.7).

Lemma 3.1. *Suppose a measurable function* $f\in L^{2/(1+r)}(0,T;\dot{B}^r_{\infty,\infty}(\boldsymbol{R}^3))$ *for* $0<r<1$, *then there exists a decomposition such that*

$$f=f^l+f^h,\quad \nabla f^l\in L^1(0,T;L^\infty(\boldsymbol{R}^3))\ \ \textit{and}\ \ f^h\in L^2(0,T;L^\infty(\boldsymbol{R}^3)). \tag{3.11}$$

Proof. According to the Littlewood-Paley decomposition

$$f=\sum_{j=-\infty}^{\infty}\Delta_j f=\sum_{j=-\infty}^{K}\Delta_j f+\sum_{j=K+1}^{\infty}\Delta_j f=f^l+f^h,$$

where K is an integer which will be chosen later. Employing Bernstein inequality (2.2), we have for f^l

$$\|\nabla f^l\|_{L^\infty}\le c\sum_{j=-\infty}^{K}\|\nabla\Delta_j f\|_{L^\infty}\le c\sum_{j=-\infty}^{K}2^j\|\Delta_j f\|_{L^\infty}\le c2^{(1-r)K}\|f\|_{\dot{B}^r_{\infty,\infty}},$$

and for f^h

$$\|f^h\|_{L^\infty}\le c\sum_{j>K}\|\Delta_j f\|_{L^\infty}\le c2^{-rK}\|f\|_{\dot{B}^r_{\infty,\infty}}.$$

By choosing $K=\frac{1}{1+r}\log(e+\|f\|_{\dot{B}^r_{\infty,\infty}})$ such that

$$\int_0^T\|\nabla f^l(t)\|_{L^\infty}\mathrm{d}t\le c\int_0^T(e+\|f(t)\|_{\dot{B}^r_{\infty,\infty}})^{\frac{2}{1+r}}\mathrm{d}t, \tag{3.12}$$

$$\int_0^T\|f^h(t)\|^2_{L^\infty}\mathrm{d}t\le c\int_0^T(e+\|f(t)\|_{\dot{B}^r_{\infty,\infty}})^{\frac{2}{1+r}}\mathrm{d}t. \tag{3.13}$$

This completes the proof of Lemma 3.2. □

Employing Lemma 3.1, we now carry out the estimation of (3.1) based on the assumption described by (1.7).

With the slight modification in the proof of (3.2), the right hand side of (3.1) yields

$$\begin{aligned}
&-\sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3}u_i\partial_i u_j\partial_{kk}u_j\ \mathrm{d}x=\sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\ \mathrm{d}x\\
=&\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\partial_k(u_i^h+u_i^l)\partial_i u_j\partial_k u_j\mathrm{d}x+\sum_{j=1}^{2}\sum_{k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_3\partial_3 u_j\partial_k(u_j^h+u_j^l)\mathrm{d}x\\
&\quad-\sum_{k=1}^{3}\int_{\mathbf{R}^3}\partial_k u_3\Big(\partial_1(u_1^h+u_1^l)+\partial_2(u_2^h+u_2^l)\Big)\partial_k u_3\mathrm{d}x\\
\le&\ c\int_{\mathbf{R}^3}|\nabla\tilde{u}^l||\nabla u|^2\ \mathrm{d}x+c\int_{\mathbf{R}^3}|\tilde{u}^h||\nabla u|\ |\nabla^2 u|\ \mathrm{d}x=:\tilde{J}_1+\tilde{J}_{L^2}.
\end{aligned} \tag{3.14}$$

For $\tilde{J}_1$, applying Hölder inequality, Young inequality and (3.12) to produce

$$\tilde{J}_1=\int_{\mathbf{R}^3}|\nabla\tilde{u}^l|\;|\nabla u|^2\,\mathrm{d}x\le c\|\nabla\tilde{u}^l\|_{L^\infty}\;\|\nabla u\|_{L^2}^2, \tag{3.15}$$

and for $\tilde{J}_{L^2}$, similarly

$$\begin{aligned}\tilde{J}_{L^2}=&\int_{\mathbf{R}^3}|\tilde{u}^h|\;|\nabla u|\;|\nabla^2 u|\,\mathrm{d}x\le c\|\tilde{u}^h\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\ \le& c\|\tilde{u}^h\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\Delta u\|_{L^2}^2,\end{aligned} \tag{3.16}$$

Plugging (3.15-3.16) into (3.14) and then (3.1), one shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2\le c\left(\|\nabla\tilde{u}^l\|_{L^\infty}+\|\tilde{u}^h\|_{L^\infty}^2\right)\|\nabla u\|_{L^2}^2. \tag{3.17}$$

Hence, taking Gronwall inequality into account, it follows that

$$\sup_{0<t<T}\|\nabla u(t)\|_{L^2}^2\le\|\nabla u_0\|_{L^2}^2\exp\left\{\int_0^T c\left(\|\nabla\tilde{u}^l\|_{L^\infty}+\|\tilde{u}^h\|_{L^\infty}^2\right)\mathrm{d}s\right\}, \tag{3.18}$$

and applying (3.12) and (3.13) to the right hand side of (3.18) to give

$$\sup_{0<t<T}\|\nabla u(t)\|_{L^2}\le\|\nabla u_0\|_{L^2}\exp\left\{c\int_0^T(e+\|\tilde{u}(t)\|_{\dot{B}^r_{\infty,\infty}})^{\frac{2}{1+r}}\mathrm{d}t\right\}. \tag{3.19}$$

Hence the proof of Theorem 3.2 is complete.

4 Proof of Theorems 1.1 and 1.2

According to *a priori* estimates of smooth solutions described in Theorems 3.1 and 3.2, the proofs of Theorems 1.1 and 1.2 are standard.

Since $u_0\in H^1(\mathbf{R}^3)$ with $\nabla\cdot u_0=0$, by the local existence theorem of strong solutions to the Navier-Stokes equations (see, for example, Fujita and Kato [26]), there exist a $T^*>0$ and a smooth solution $\bar{u}$ of (1.1) satisfying $\bar{u}\in C([0,T^*);H^1)\cap C^1((0,T^*);H^1)\cap C([0,T^*);H^3)$, $\bar{u}(x,0)=u_0$. Note that the Leray weak solution satisfies the energy inequality (1.4). It follows from Serrin's weak-strong uniqueness criterion [27] that $\bar{u}\equiv u$ *on* $[0,T^*)$. Thus it is sufficient to show that $T^*=T$. Suppose that $T^*<T$. Without loss of generality, we may assume that T^* is the maximal existence time for $\bar{u}$. Since $\bar{u}\equiv u$ on $[0,T^*)$ and by the assumptions (1.6) or (1.7), it follows from Theorems 3.1 and 3.2 that the existence time of $\bar{u}$ can be extended after $t=T^*$ which contradicts with the maximality of $t=T^*$. This completes the proofs of Theorems 1.1 and 1.2.

Acknowledgments

This work is partially supported by the NSF of China (10801001),NSF of Anhui Province(11040606M02) and is also financed by the 211 Project of Anhui University (KJTD002B, KJJQ005).The authors also want to express their sincere thanks to the editor for his kind help.

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