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Infinitely Many Solutions for an Elliptic Problem with Critical Exponent in Exterior Domain

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Abstract. We consider the following nonlinear problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, \quad u>0 & \text{in } \mathbf{R}^N\backslash\Omega, \\ u(x)\to 0 & \text{as } |x|\to+\infty, \\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbf{R}^N, N\geq 4$ is a smooth and bounded domain and n denotes inward normal vector of $\partial\Omega$. We prove that the above problem has infinitely many solutions whose energy can be made arbitrarily large when Ω is convex seen from inside (with some symmetries).

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1 Introduction and main result

In this paper we consider the nonlinear Neumann elliptic problem

$$\begin{cases} -\Delta u - u^{\frac{N+2}{N-2}} = 0, \quad u>0 & \text{in } \mathbf{R}^N\backslash\Omega, \\ u(x)\to 0 & \text{as } |x|\to+\infty, \\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where n denotes interior unit normal vector and Ω is a smooth bounded domain in $\mathbf{R}^N, N\geq 4$.

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Semilinear elliptic equations involving critical Sobolev exponents have been considered by various authors, e.g., [1–6]. Such kind of equations arise in various branches of mathematics as well as physics (see, e.g., [2,7] and the reference therein). The most notorious example is *Yamabe's* problem: let (M,g) be a Riemannian manifold of dimension $N, N\geq 3$, and R' be a given function on M. Can one find a new metric g_1 on M such that R' is the scalar curvature of g_1 and g_1 is conformal to g (i.e., $g_1 = u^{\frac{4}{N-2}} g$ for some function $u>0$ on M)? This is equivalent to the problem of finding positive solution of the equation

$$-4\frac{N-1}{N-2}\Delta_g u = R' u^{\frac{N+2}{N-2}} - R(x)u \quad \text{on } M, \tag{1.2}$$

where Δ_g is Laplace-Beltrami operator on M in the g metric and $R(x)$ is the scalar curvature of (M,g). In case M is compact, Eq. (1.2) has been considered by many authors, see [7] for a survey on its development and a brief history. In the special case where $M=\mathbf{R}^N$ and g is the usual metric we have $R\equiv 0$ and the equation is reduced to

$$\Delta u + R' u^{\frac{N+2}{N-2}} = 0. \tag{1.3}$$

From now on we are concerned with the case $R'\equiv$constant. Without loss of generality we may assume $R'\equiv 1$. According to [8] the functions

$$U_{\lambda,a}(x) = \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{N-2}{2}}}, \quad \lambda>0, \quad a\in\mathbf{R}^N,$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u>0 \ \text{ in } \mathbf{R}^N,$$

where $\alpha_N = N(N-2)$.

On the other hand, by Divergence Theorem there is no positive solution of the following problem

$$-\Delta u = u^{\frac{N+2}{N-2}} \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$. Hence it has been a matter of high interest to study the problem in exterior domain, which is Eq. (1.1). In [9], Pan and Wang proved that if the mean curvature of $\partial\Omega$ seen from inside is negative somewhere, then Eq. (1.1) has a least energy solution while Ω is a ball Eq. (1.1) has no least energy solution. A natural question is: how about higher energy solutions?

The purpose of this paper is to prove that Eq. (1.1) has infinitely many higher energy solutions while Ω is convex seen from inside. More precisely, we assume that Ω is a smooth and bounded domain satisfying the following properties:

Let $y=(y',y'')\in\mathbf{R}^2\times\mathbf{R}^{N-2}$, $r=|y'|$. *Then*

(H_1) $y\in\Omega$ *if and only if* $(y_1,y_2,y_3,\cdots,-y_i,\cdots,y_N)\in\Omega,\ \forall\, i=3,\cdots,N;$

(H_2) $(r\cos\theta,r\sin\theta,y'')\in\Omega$ *if* $(r,0,y'')\in\Omega,\ \forall\theta\in(0,2\pi);$

(H_3) *Let* $T:=\partial\Omega\cap\{y_3=\cdots=y_N=0\}$. *There exists a connected component* Γ *of* T, *such that* $H(x)\equiv\gamma>0,\ \forall x\in\Gamma$, *where* $H(x)$ *is the mean curvature of* $\partial\Omega$ *at* $x\in\partial\Omega$.

Note that by the assumption (H_2), Γ is a circle in the plane $y_3=\cdots=y_N=0$. Thus, we may assume that

$$\Gamma=\{y_1^2+y_2^2=r_0^2,\ y_3=\cdots=y_N=0\},$$

where $r_0>0$ is a constant. Note also that for $x\in\Gamma$,

$$H(x)=\frac{\sum\limits_{j=1}^{N-1}k_j(x)}{N-1},$$

where $k_j(x)$ are the principal curvatures and $k_1(x)=r_0^{-1}$. Thus

$$H(x)\equiv\gamma=r_0^{-1}.$$

Such domain is very common, e.g., ball, ellipsoid.

For normalization reason, we consider throughout the paper the equation

$$\begin{cases} -\Delta u-\alpha_N u^{\frac{N+2}{N-2}}=0, \quad u>0 & \text{in } \mathbf{R}^N\backslash\Omega,\\ u(x)\to 0 & \text{as } |x|\to+\infty,\\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\alpha_N=N(N-2)$. The solutions are identical up to the multiplicative constant $(\alpha_N)^{-(N-2)/4}$.

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that* $N\geq 4$ *and* Ω *is a smooth and bounded domain satisfying* (H_1), (H_2) *and* (H_3). *Then problem* (1.4) *has infinitely many* **non-radial** *positive solutions, whose energy can be made arbitrarily large.*

It is interesting to compare the results in this paper and [10] with recent work of Brendle on the non-compactness of Yamabe problem (1.2). When $M=\mathbf{R}^N$, i.e., R' is constant, Schoen conjectured all solutions to (1.2) are compact. This conjecture is proved to be true in dimensions less than 24. See [11–15]. In [16], Brendle constructed a metric g in dimension $N\geq 52$, with the following properties: (i) $g_{ij}=\delta_{ij}$ for $|x|\geq\frac{1}{2}$, (ii) g is not conformally flat. Then, for this metric, there exists a sequence of positive smooth solutions u_n to (1.2) such that

$$\sup_{|x|\leq 1} u_n(x) \to +\infty,$$

and u_n develops exactly one singularity. This disapproves Schoen's conjecture in dimensions $N \geq 52$. On one hand, both problems (1.2) and (1.4) have no parameters but possess infinitely many positive solutions. The proofs are similar: a kind of variational reduction method (we call it *localized energy method*) is used. On the other hand, the solutions constructed by Brendle has a single bubble near the origin, and the energy of the solutions remains uniformly bounded. Here we obtain solutions with *arbitrarily many bubbles*, and the energy of the solutions can be *arbitrarily large*.

We believe that the symmetric condition in Theorem 1.1 is technical. A more general result, as follows, should be true.

Conjecture 1.1. *Assume that* $\max_{x\in\partial\Omega} H(x) > 0$ *and the set* $\{x \in \partial\Omega | H(x) = \max_{x\in\partial\Omega} H(x)\}$ *is a smooth l-dimensional sub-manifold on* $\partial\Omega$*, with* $1 \leq l \leq N-1$*. Then there are infinitely many positive solutions to problem* (1.4).

In the following we will see that the idea of the proof depends on the critical exponent. That is using this idea we can't obtain similar results for sub-super critical case.

2 Outline of proof

In [17] the following is considered

$$\begin{cases} -\Delta u + \mu u - \alpha_N u^{\frac{N+2}{N-2}} = 0, \quad u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where μ is a fixed positive number. When Ω is a smooth and bounded domain satisfying (H_1), (H_2) and $(H_3^{'})$, then for $N \geq 3$, there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (2.1) has infinitely many solutions whose energy can be made arbitrary large, where

$H_3^{'}$: *Let* $T := \partial\Omega \cap \{y_3 = \cdots = y_N = 0\}$*. There exists a connected component* Γ *of* T*, such that* $H(x) \equiv \gamma < 0$*,* $\forall x \in \Gamma$.

From the above we see that the problem (1.4) is similar to (2.1) expect the domain and linear term μu. A natural idea is that we may try the same method to prove Theorem 1.1. We will use the techniques in the singularly perturbed elliptic problems to prove the following Theorem 2.1 which is equivalent to Theorem 1.1. In all the singularly perturbed problems, some small parameters are present either in the operator or in the nonlinearity or in the boundary condition. Here there is *no parameter*. Instead, we use *k*, *the number of* *the bubbles* of the solutions, as the parameter in the construction of bubble solutions for (1.4). This idea is also used in the recent paper [10]. We believe that the results will also hold in the case of $N=3$.

It is well-known that the functions

$$U_{\lambda,a}(y)=\Big(\frac{\lambda}{1+\lambda^2|y-a|^2}\Big)^{\frac{N-2}{2}},\quad \lambda>0,\ a\in\mathbf{R}^N$$

are the only solutions to the problem

$$-\Delta u=\alpha_N u^{\frac{N+2}{N-2}},\quad u>0,\quad \text{in }\ \mathbf{R}^N.$$

Let us fix a positive integer

$$k\geq k_0,$$

where k_0 is large, to be determined later.

Integral estimates (see Appendix A in [17]) suggest to make the additional a priori assumption that λ behaves as the following

$$\lambda=\frac{1}{\Lambda}k^{\frac{N-2}{N-3}},\quad N\geq 4,$$

where $\delta\leq\Lambda\leq\frac{1}{\delta}$ and δ is a small positive constant which is to be determined later.

Let $2^*=\frac{2N}{N-2}$. Using the transformation

$$u(y)\mapsto\varepsilon^{-\frac{N-2}{2}}u\Big(\frac{y}{\varepsilon}\Big),$$

we find that (1.4) becomes

$$\begin{cases}-\Delta u=\alpha_N u^{2^*-1},\quad u>0, & \text{in } \mathbf{R}^N\backslash\Omega_\varepsilon,\\ u(x)\to 0, & \text{as } |x|\to+\infty,\\ \dfrac{\partial u}{\partial n}=0, & \text{on } \partial\Omega_\varepsilon,\end{cases}\tag{2.2}$$

where

$$\varepsilon=k^{-\frac{N-2}{N-3}},\quad N\geq 4\tag{2.3}$$

and $\Omega_\varepsilon=\{y|\varepsilon y\in\Omega\}$. Define

$$\begin{aligned}H_s=\Big\{&u: u\in H^1(\mathbf{R}^N\backslash\Omega_\varepsilon),\ u \text{ is even in } y_h,\ h=2,\cdots,N,\\ &u(r\cos\theta,r\sin\theta,y'')=u\Big(r\cos(\theta+\frac{2\pi j}{k}),r\sin(\theta+\frac{2\pi j}{k}),y''\Big),\ j=1,\cdots,k-1\Big\},\end{aligned}$$

and

$$x_j=\Big(\frac{r_0}{\varepsilon}\cos\frac{2(j-1)\pi}{k},\frac{r_0}{\varepsilon}\sin\frac{2(j-1)\pi}{k},0\Big),\quad j=1,\cdots,k,$$

where 0 is the zero vector in $\mathbf{R}^{N-2}$. We define W_{Λ,x_j} to be the unique solution of

$$\begin{cases} -\Delta u = \alpha_N U^{2^*-1}_{\frac{1}{\Lambda},x_j} & \text{in } \mathbf{R}^N \backslash \Omega_\varepsilon, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.4}$$

Let

$$W(y) = \sum_{j=1}^{k} W_{\Lambda,x_j}.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 2.1. *Suppose that $N \geq 4$ and Ω is a smooth and bounded domain satisfying (H_1), (H_2) and (H_3). Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, problem* (2.2) *has a solution u_k of the form*

$$u_k = W(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^\infty} \to 0$.

Let

$$\varphi_{\Lambda,x_j}(y) = U_{\frac{1}{\Lambda},x_j}(y) - W_{\Lambda,x_j}(y), \tag{2.5}$$

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N \backslash \Omega_\varepsilon} |Du|^2 - \frac{\alpha_N}{2^*}\int_{\mathbf{R}^N \backslash \Omega_\varepsilon} |u|^{2^*}, \tag{2.6}$$

then by Appendix A in [17] we have the following lemma.

Lemma 2.1. *There is a constant $C > 0$, such that*

$$|\varphi_{\Lambda,x_j}| + |\partial_\Lambda \varphi_{\Lambda,x_j}| \leq \frac{C\varepsilon}{(1+|y-x_j|)^{N-3}}, \quad N \geq 4, \tag{2.7}$$

and

$$|W_{\lambda,x_j}| \leq C\Big(U_{\frac{1}{\Lambda},x_j} + \frac{\varepsilon}{(1+|y-x_j|)^{N-3}}\Big),$$
$$|\partial_\Lambda W_{\lambda,x_j}| \leq C\Big(U_{\frac{1}{\Lambda},x_j} + \frac{\varepsilon}{(1+|y-x_j|)^{N-3}}\Big).$$

For $N \geq 4$, we have

$$I(W) = k\Big(A_0 + A_1\Lambda\gamma\varepsilon - A_2\Lambda^{N-2}\varepsilon + o(\varepsilon)\Big), \tag{2.8}$$

where A_i, $i = 0,1,2$, is some positive constant.

3 Finite-dimensional reduction

In this section, we perform a finite-dimensional reduction. Let

$$\|u\|_* = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1} |u(y)|, \tag{3.1}$$

$$\|f\|_{**} = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1} |f(y)|, \tag{3.2}$$

where we choose

$$\tau = \begin{cases} \dfrac{N-3}{N-2}, & N\geq 6,\ N=4; \\ \dfrac{5}{6}, & N=5. \end{cases} \tag{3.3}$$

For this choice of τ and the definition of ε, we have

$$\sum_{j=2}^k \frac{1}{|x_j-x_1|^\tau} \leq C, \quad \text{if } N\geq 4. \tag{3.4}$$

Let

$$Y_i = \frac{\partial W_{\Lambda,x_i}}{\partial \Lambda}, \quad Z_i = -\Delta Y_i = (2^*-1) U_{\frac{1}{\Lambda},x_i}^{2^*-2} \frac{\partial U_{\frac{1}{\Lambda},x_i}}{\partial \Lambda}.$$

For any $\|h\|_{L^\infty} < \infty$, we consider the following linear problem

$$\begin{cases} -\Delta\phi - N(N+2)W^{2^*-2}\phi = h + c_1 \sum\limits_{i=1}^k Z_i, & \text{in } \mathbf{R}^N\backslash\Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi\in H_s, \quad <\sum\limits_{i=1}^k Z_i,\phi> = 0 \end{cases} \tag{3.5}$$

for some number c_1, where

$$<u,v> = \int_{\mathbf{R}^N\backslash\Omega_\varepsilon} uv.$$

Let us remark that in general we should also include the translational derivatives of W in the right hand side of (3.5). However due to the symmetry assumption $\phi\in H_s$, this part of kernel automatically disappears. This is the main reason for imposing the symmetries.

First we state a lemma whose proof is in Appendix A.

Lemma 3.1. *Let f satisfy $\|f\|_{**}<\infty$ and u be the solution of*

$$\begin{aligned}&-\Delta u=f \quad \text{in } \mathbf{R}^N\backslash\Omega_\varepsilon,\\ &|u(x)|\to 0 \quad \text{as } |x|\to+\infty, \quad \frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega_\varepsilon.\end{aligned}$$

Then we have

$$|u(x)|\le C\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}\frac{|f(y)|}{|x-y|^{N-2}}\mathrm{d}y. \tag{3.6}$$

Next, we need the following lemma to carry out the reduction.

Lemma 3.2. *Assume that ϕ_k solves (3.5) for $h=h_k$. If $\|h_k\|_{**}\to 0$ as $k\to\infty$, so does $\|\phi_k\|_*$.*

Proof. We argue by contradiction. Suppose that there are $k\to+\infty$, $h=h_k$, $\Lambda_k\in[\delta,\delta^{-1}]$, and ϕ_k solving (3.5) for $h=h_k$, $\Lambda=\Lambda_k$, with $\|h_k\|_{**}\to 0$, and $\|\phi_k\|_*\ge c'>0$. We may assume that $\|\phi_k\|_*=1$. For simplicity, we drop the subscript k.

According to Lemma 3.1, we have

$$\begin{aligned}|\phi(y)|\le& C\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}W^{2^*-2}|\phi(z)|\mathrm{d}z\\ &+C\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}\big(|h(z)|+|c_1\sum_{i=1}^{k}Z_i(z)|\big)\mathrm{d}z.\end{aligned} \tag{3.7}$$

Using Lemma A.4, there is a strictly positive small number θ such that

$$\begin{aligned}&\Big|\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}W^{2^*-2}\phi(z)\mathrm{d}z\Big|\\ \le& C\|\phi\|_*\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}+o(1)\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big).\end{aligned} \tag{3.8}$$

It follows from Lemma A.3 that

$$\begin{aligned}&\Big|\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}h(z)\mathrm{d}z\Big|\\ \le& C\|h\|_{**}\int_{\mathbf{R}^N}\frac{1}{|z-y|^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\mathrm{d}z\\ \le& C\|h\|_{**}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\Big|\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^k Z_i(z)\,\mathrm{d}z\Big|\\
\le& C\sum_{i=1}^k\int_{\mathbf{R}^N}\frac{1}{|z-y|^{N-2}}\frac{1}{(1+|z-x_i|)^{N+2}}\mathrm{d}z\\
\le& C\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}\tag{3.10}$$

Next, we estimate c_1. Multiplying (3.5) by Y_1 and integrating, we see that c_1 satisfies

$$\Big\langle\sum_{i=1}^k Z_i,Y_1\Big\rangle c_1=\big\langle-\Delta\phi-N(N+2)W^{2^*-2}\phi,Y_1\big\rangle-\big\langle h,Y_1\big\rangle.\tag{3.11}$$

It follows from Lemma A.2 that

$$\begin{aligned}
\big|\langle h,Y_1\rangle\big|\le& C\|h\|_{**}\int_{\mathbf{R}^N}\Big(\frac{1}{(1+|z-x_1|)^{N-2}}\\
&+\frac{\varepsilon}{(1+|z-x_1|)^{N-3}}\Big)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\mathrm{d}z\\
\le& C\|h\|_{**}.
\end{aligned}$$

On one hand,

$$\begin{aligned}
&\big\langle-\Delta\phi-N(N+2)W^{2^*-2}\phi,Y_1\big\rangle\\
=&\big\langle-\Delta Y_1-N(N+2)W^{2^*-2}Y_1,\phi\big\rangle\\
=&N(N+2)\big\langle U^{2^*-2}_{\frac{1}{\Lambda},x_1}\partial_\Lambda U_{\frac{1}{\Lambda},x_1}-W^{2^*-2}Y_1,\phi\big\rangle.
\end{aligned}\tag{3.12}$$

Obviously,

$$|\phi(y)|\le C\|\phi\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}.$$

On the other hand, it follows from (2.7) that

$$\begin{aligned}
&|\varphi_{\Lambda,x_i}(y)|\le\frac{C\varepsilon}{(1+|y-x_i|)^{N-3}},\\
&|Y_1|\le\frac{C}{(1+|y-x_1|)^{N-2}}+\frac{C\varepsilon}{(1+|y-x_1|)^{N-3}}.
\end{aligned}$$

We consider the cases $N\geq 6$ first. Note that now $4/(N-2)\leq 1$ for $N\geq 6$. Using Lemmas 2.1, we obtain

$$\begin{aligned}
&\Big|\langle U^{2^*-2}_{\frac{1}{\Lambda},x_1}\partial_\Lambda U_{\frac{1}{\Lambda},x_1}-W^{2^*-2}Y_1,\phi\rangle\Big|\\
\leq& C\|\phi\|_*\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=2}^{k}\Big(\frac{1}{(1+|z-x_j|)^4}+|\phi_{\Lambda,x_j}|\Big)\sum_{i=1}^{k}\frac{1}{(1+|z-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}z\\
&+C\|\phi\|_*\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\Big(U^{2^*-2}_{\frac{1}{\Lambda},x_1}|\partial_\Lambda\varphi_{\Lambda,x_1}|+|Y_1|\sum_{j=2}^{k}|\varphi_{\Lambda,x_j}|^{2^*-2}\Big)\sum_{i=1}^{k}\frac{1}{(1+|z-x_i|)^{\frac{N-2}{2}+\tau}}\\
\leq& C\|\phi\|_*\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{1+\sigma}}+o(1)\|\phi\|_*\\
&+C\|\phi\|_*\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{\varepsilon^{\frac{N+2}{N-2}}}{(1+|y-x_1|)^{N-3}}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y. \qquad (3.13)
\end{aligned}$$

Note that the following calculation is also valid for $N=4,5$. Let

$$\Omega_j=\Big\{y=(y',y'')\in\mathbf{R}^N\setminus\Omega_\varepsilon:\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\geq\cos\frac{\pi}{k}\Big\}.$$

If $y\in\Omega_1$, then

$$\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
\leq&\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\Big(\sum_{i=2}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\tau}}\Big)\\
\leq&\frac{C\varepsilon^{\beta}k^2}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{4(N-3)}{N-2}+\tau-\beta}}+\frac{C\varepsilon^{\frac{\beta}{2}}k}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{4(N-3)}{N-2}+\tau}},
\end{aligned}$$

where $\beta=\frac{N-7}{N-2}$ if $N\geq 8$ and $\beta=0$ if $N=6,7$ since

$$\begin{aligned}
\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}&\leq\frac{C\varepsilon^{\frac{\beta}{2}}k}{(1+|y-x_1|)^{\frac{4(N-3)}{N-2}-\frac{\beta}{2}}},\\
\sum_{i=2}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}&\leq\frac{C\varepsilon^{\frac{\beta}{2}}k}{(1+|y-x_1|)^{\frac{N-2}{2}+\tau-\frac{\beta}{2}}}.
\end{aligned}$$

So, we obtain

$$\int_{\Omega_1}\frac{\varepsilon^{\frac{N+2}{N-2}}}{(1+|y-x_1|)^{N-3}}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y=o(1).$$

If $y\in\Omega_l$, $l\geq 2$, then

$$\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\leq\frac{C\varepsilon^{\frac{\beta}{2}}k}{(1+|y-x_l|)^{\frac{4(N-3)}{N-2}-\frac{\beta}{2}}},$$
$$\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\leq\frac{C}{(1+|y-x_l|)^{\frac{N-2}{2}+\tau}}+\frac{C\varepsilon^{\frac{\beta}{2}}k}{(1+|y-x_l|)^{\frac{N-2}{2}+\tau-\frac{\beta}{2}}}.$$

As a result,

$$\begin{aligned}&\int_{\Omega_l}\frac{\varepsilon^{\frac{N+2}{N-2}}}{(1+|y-x_1|)^{N-3}}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y\\ \leq&C\int_{\Omega_l}\frac{1}{(1+|y-x_1|)^{N-3}}\frac{1}{(1+|y-x_l|)^{\frac{4(N-3)}{N-2}+\frac{N-2}{2}+\tau-\beta}}\mathrm{d}y\\ \leq&\frac{C}{|x_l-x_1|^{\frac{N-1}{2}}}.\end{aligned}$$

Hence

$$\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{\varepsilon^{\frac{N+2}{N-2}}}{(1+|y-x_1|)^{N-3}}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{4(N-3)}{N-2}}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y=o(1),$$

where we have used the fact

$$\sum_{l=2}^{k}\frac{1}{|x_1-x_l|^{\frac{N-1}{2}}}=o(1).$$

Similarly, for $N=4,5$ we have

$$\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{C\varepsilon}{(1+|z-x_1|)^3}\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{N-2}}\sum_{i=1}^{k}\frac{1}{(1+|z-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}z=o(1)\tag{3.14}$$

which will be used later.

For $N=4,5$, we have $\frac{4}{N-2}>1$. By Lemmas 2.1,

$$\begin{aligned}&\Big|\langle U_{\frac{1}{\Lambda},x_1}^{2^*-2}\partial_\Lambda U_{\frac{1}{\Lambda},x_1}-W^{2^*-2}Y_1,\phi\rangle\Big|\\ \leq&C\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}U_{\frac{1}{\Lambda},x_1}^{2^*-3}\sum_{j=2}^{k}(U_{\frac{1}{\Lambda},x_j}+\phi_{\Lambda,x_j})|Y_1\phi|+C\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\big(\sum_{j=2}^{k}U_{\frac{1}{\Lambda},x_j}\big)^{\frac{4}{N-2}}|Y_1\phi|\\ &+\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\Big(U_{\frac{1}{\Lambda},x_1}^{2^*-2}|\partial_\Lambda\varphi_{\Lambda,x_1}|+U_{\frac{1}{\Lambda},x_1}^{2^*-3}|\varphi_{\Lambda,x_1}||Y_1|+\sum_{j=2}^{k}|\varphi_{\Lambda,x_j}|^{2^*-2}|Y_1|\Big)|\phi|\end{aligned}$$

$$
\begin{aligned}
&\le C\|\phi\|_*\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\Big(\frac{1}{(1+|z-x_1|)^4}\sum_{j=2}^k\frac{1}{(1+|z-x_j|)^{N-2}}\\
&\quad+\frac{C\varepsilon}{(1+|z-x_1|)^3}\sum_{j=2}^k\frac{1}{(1+|z-x_j|)^{N-2}}\sum_{i=1}^k\frac{1}{(1+|z-x_i|)^{\frac{N-2}{2}+\tau}}\Big)\\
&\quad+C\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j})^{\frac{4}{N-2}}|Y_1\phi|+o(1)\|\phi\|_*\\
&\le C\|\phi\|_*\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\Big(\frac{1}{(1+|y-x_1|)^{N-2}}(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j})^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
&\quad+\frac{\varepsilon}{(1+|y-x_1|)^{N-3}}(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j})^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\Big)\mathrm{d}y+o(1)\|\phi\|_*. \qquad (3.15)
\end{aligned}
$$

If $y\in\Omega_1$, then

$$
\begin{aligned}
\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}&\le\frac{1}{(1+|y-x_1|)^{N-2-\tau-\theta}}\sum_{j=2}^k\frac{1}{|x_j-x_1|^{\tau+\theta}}\\
&=o(1)\frac{1}{(1+|y-x_1|)^{N-2-\tau-\theta}},
\end{aligned}
$$

and

$$
\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\le\frac{C}{(1+|y-x_1|)^{\frac{N-2}{2}}}.
$$

So, we obtain

$$
\int_{\Omega_1}\frac{1}{(1+|y-x_1|)^{N-2}}\Big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y=o(1).
$$

If $y\in\Omega_l$, $l\ge 2$, then

$$
\begin{aligned}
&\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\le\frac{C}{(1+|y-x_l|)^{N-2-\tau}},\\
&\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\le\frac{C}{(1+|y-x_l|)^{\frac{N-2}{2}}}.
\end{aligned}
$$

As a result,

$$\begin{aligned}&\int_{\Omega_l}\frac{1}{(1+|y-x_1|)^{N-2}}\Big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y\\ \le& C\int_{\Omega_l}\frac{1}{(1+|y-x_1|)^{N-2}}\frac{1}{(1+|y-x_l|)^{4-\frac{4\tau}{N-2}+\frac{N-2}{2}}}\mathrm{d}y\\ \le&\frac{C}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}}}.\end{aligned}$$

Note that $\frac{N+2}{2}-\frac{4\tau}{N-2}>\tau$. Thus

$$\begin{aligned}&\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{1}{(1+|y-x_1|)^{N-2}}\Big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y\\ \le& o(1)+C\sum_{l=2}^k\frac{1}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}}}=o(1).\end{aligned}$$

By the same calculations, we have

$$\int_{\mathbf{R}^N\setminus\Omega_\varepsilon}\frac{\varepsilon}{(1+|y-x_1|)^{N-3}}\Big(\sum_{j=2}^k U_{\frac{1}{\Lambda},x_j}\Big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\mathrm{d}y=o(1)$$

since now $\frac{N}{2}-\frac{4}{N-2}\tau>\tau$ is also valid. So, we have proved

$$\Big|\big\langle U_{\frac{1}{\Lambda},x_1}^{2^*-2}\partial_\Lambda U_{\frac{1}{\Lambda},x_1}-W^{2^*-2}Y_1,\phi\big\rangle\Big|=o(1)\|\phi\|_*.$$

But there is a constant $\bar{c}>0$,

$$\big\langle\sum_{i=1}^k Z_i,Y_1\big\rangle=\bar{c}+o(1).$$

Thus we obtain that

$$c_1=o(\|\phi\|_*)+\mathcal{O}(\|h\|_{**}).$$

Consequently,

$$\|\phi\|_*\le\left(o(1)+\|h_k\|_{**}+\frac{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\right).\tag{3.16}$$

Since $\|\phi\|_*=1$, we obtain from (3.16) that there is $R>0$, such that

$$\|\phi(y)\|_{B_R(x_i)}\ge c_0>0,\tag{3.17}$$

for some i. But $\bar{\phi}(y)=\phi(y-x_i)$ converges uniformly in any compact set of $\mathbf{R}^N_+$ to a solution u of

$$\Delta u+N(N+2)U^{2^*-2}_{\frac{1}{\Lambda},0}u=0 \tag{3.18}$$

for some $\Lambda\in[\delta,\delta^{-1}]$, and u is perpendicular to the kernel of (3.18). So, $u=0$. This is a contradiction to (3.17). □

From Lemma 3.2, using the same argument as in the proof of Proposition 4.1 in [18], Proposition 3.1 in [19], we can prove the following result:

Proposition 3.1. *There exists $k_0>0$ and a constant $C>0$, independent of k, such that for all $k\geq k_0$ and all $h\in L^\infty(\mathbf{R}^N\backslash\Omega_\varepsilon)$, problem* (3.5) *has a unique solution $\phi\equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_*\leq C\|h\|_{**},\qquad |c_1|\leq C\|h\|_{**}. \tag{3.19}$$

Moreover, the map $L_k(h)$ is C^1 with respect to Λ.

Now, we consider the existence of solution ϕ for the problem

$$\begin{cases}-\Delta\big(W+\phi\big)=\alpha_N\big(W+\phi\big)^{2^*-1}+c_1\sum\limits_{i=1}^{k}Z_i, & \text{in } \mathbf{R}^N\backslash\Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial n}=0, & \text{on } \partial\Omega_\varepsilon,\\ \phi\in H_s,\quad <\sum\limits_{i=1}^{k}Z_i,\phi>=0. \end{cases} \tag{3.20}$$

We have the following result.

Proposition 3.2. *There is an integer $k_0>0$, such that for each $k\geq k_0$, $\delta\leq\Lambda\leq\delta^{-1}$, where δ is a fixed small constant,* (3.20) *has a unique solution ϕ, satisfying*

$$\|\phi\|_*\leq C\varepsilon^{\frac{1}{2}+\sigma},$$

where $\sigma>0$ is a fixed small constant. Moreover, $\Lambda\to\phi(\Lambda)$ is C^1.

Rewrite (3.20) as

$$\begin{cases}-\Delta\phi-N(N+2)W^{2^*-2}\phi=N(\phi)+l_k+c_1\sum\limits_{i=1}^{k}Z_i, & \text{in } \mathbf{R}^N\backslash\Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial n}=0, & \text{on } \partial\Omega_\varepsilon,\\ \phi\in H_s,\quad <\sum\limits_{i=1}^{k}Z_i,\phi>=0, \end{cases} \tag{3.21}$$

where

$$N(\phi)=\alpha_N\Big(\big(W+\phi\big)^{2^*-1}-W^{2^*-1}-(2^*-1)W^{2^*-2}\phi\Big),$$
$$l_k=\alpha_N\Big(W^{2^*-1}-\sum_{j=1}^{k}U^{2^*-1}_{\frac{1}{\Lambda},x_j}\Big).$$

In order to use the contraction mapping theorem to prove that (3.21) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

In the following, we always assume that $\|\phi\|_*\leq\varepsilon|\ln\varepsilon|$.

Lemma 3.3. *We have*

$$\|N(\phi)\|_{**}\leq C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)|\leq\begin{cases}C|\phi|^{2^*-1}, & N\geq 6;\\ C\big(W^{\frac{6-N}{N-2}}\phi^2+|\phi|^{2^*-1}\big), & N=4,5.\end{cases}$$

Firstly, we consider $N\geq 6$. We have

$$\begin{aligned}|N(\phi)|&\leq C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\ &\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}},\end{aligned}\tag{3.22}$$

where we use the inequality

$$\sum_{j=1}^{k}a_jb_j\leq\Big(\sum_{j=1}^{k}a_j^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{k}b_j^q\Big)^{\frac{1}{q}},\quad \frac{1}{p}+\frac{1}{q}=1,\quad a_j,b_j\geq 0,\ \ j=1,\cdots,k.\tag{3.23}$$

By Lemma A.1 and (3.3), we find,

$$\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\tau}}\leq C+\sum_{j=2}^{k}\frac{C}{|x_1-x_j|^{\tau}}\leq C.$$

Thus,

$$|N(\phi)|\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

For $N=5$, $\frac{6-N}{N-2}=\frac{1}{3}$, we have

$$
\begin{aligned}
W^{\frac{1}{3}}\phi^2 \le & C\|\phi\|_*^2\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^3}+\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\\
\le & C\|\phi\|_*^2\Big(\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^3}\Big)^{\frac{1}{3}}+\Big(\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\Big)\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\\
\le & C\|\phi\|_*^2\Big\{\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^{\frac{7}{3}}+\Big(\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\Big\}\\
\le & C\|\phi\|_*^2\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{7}{2}+\frac{5}{6}}}+C\|\phi\|_*^2\Big(\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2.
\end{aligned}
$$

For $y\in\Omega_l$, $l\ge 1$, using (3.23) we have that

$$
\begin{aligned}
&\Big(\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\le C\varepsilon^{\frac{1}{3}}\Big(\frac{1}{(1+|y-x_l|)^{\frac{2}{3}}}+(\varepsilon k)^{\frac{2}{3}}\Big),\\
&\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\le C\varepsilon^{\frac{1}{9}}k\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{3+\frac{5}{3}-\frac{1}{9}}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\\
\le & C\Big(\frac{1}{(1+|y-x_l|)^{3+\frac{5}{3}}}+\frac{1}{(1+|y-x_l|)^{3+\frac{5}{3}-1}}\big(\sum_{j\neq l}\frac{1}{|x_l-x_j|^{\frac{1}{2}}}\big)^2\Big)\\
\le & C\Big(\frac{1}{(1+|y-x_l|)^{3+\frac{5}{3}}}+\varepsilon k^2\frac{1}{(1+|y-x_l|)^{3+\frac{5}{3}-1}}\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\Big(\sum_{j=1}^k\frac{\varepsilon}{(1+|y-x_j|)^2}\Big)^{\frac{1}{3}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\frac{5}{6}}}\Big)^2\\
\le & \frac{C\varepsilon^{\frac{1}{3}}}{(1+|y-x_l|)^{3+\frac{7}{3}}}+\frac{C\varepsilon^{\frac{4}{3}}k^2}{(1+|y-x_l|)^{3+\frac{7}{3}-1}}+C\sum_{j=1}^k\frac{\varepsilon^{\frac{1}{3}}(\varepsilon k)^{\frac{2}{3}}\varepsilon^{\frac{1}{9}}k}{(1+|y-x_j|)^{3+\frac{5}{3}-\frac{1}{9}}}\\
\le & C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{7}{2}+\frac{5}{6}}},
\end{aligned}
$$

which leads to

$$W^{\frac{1}{3}}\phi^2\le C\|\phi\|_*^2\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{7}{2}+\frac{5}{6}}}.$$

Similarly, as the case of $N\ge 6$,

$$|\phi|^{2^*-1}\le C\|\phi\|_*^{2^*-1}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{7}{2}+\frac{5}{6}}}.$$

Thus

$$|N(\phi)|\le C\|\phi\|_*^2\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{7}{2}+\frac{5}{6}}}.$$

For $N=4$, $\frac{6-N}{N-2}=1$, we have

$$\begin{aligned}|N(\phi)|\le& C\|\phi\|_*^2\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^2}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\\&+C\|\phi\|_*^2\sum_{j=1}^k\frac{\varepsilon}{1+|y-x_j|}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\\&+C\|\phi\|_*^{2^*-1}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{3+\frac{1}{2}}}\\\le& C\|\phi\|_*^2\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^{2^*-1}+C\|\phi\|_*^{2^*-1}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{3+\frac{1}{2}}}\\&+C\|\phi\|_*^2\sum_{j=1}^k\frac{\varepsilon}{1+|y-x_j|}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\\\le& C\|\phi\|_*^2\Big\{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{3+\frac{1}{2}}}+\sum_{j=1}^k\frac{\varepsilon}{1+|y-x_j|}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\Big\}.\end{aligned}\tag{3.24}$$

If $y\in\Omega_l$, $l\ge 1$, then

$$\begin{aligned}&\sum_{j=1}^k\frac{\varepsilon}{1+|y-x_j|}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\\\le&\frac{C\varepsilon}{(1+|y-x_l|)^{\frac{3}{4}}}\sum_{j\ne l}\frac{1}{|x_l-x_j|^{\frac{1}{4}}}\Big(\frac{C}{(1+|y-x_l|)^{\frac{3}{2}-\frac{1}{8}}}\sum_{j\ne l}\frac{1}{(|x_l-x_j|)^{\frac{1}{8}}}\Big)^2\\\le& C\varepsilon^{1+\frac{1}{4}+\frac{1}{4}}k^3\frac{1}{(1+|y-x_l|)^{3+\frac{1}{2}}}\\\le& C\frac{1}{(1+|y-x_l|)^{3+\frac{1}{2}}},\end{aligned}$$

which gives us that for any $y\in\mathbf{R}^N\backslash\Omega_\varepsilon$

$$\sum_{j=1}^k\frac{\varepsilon}{1+|y-x_j|}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{1+\frac{1}{2}}}\Big)^2\le C\sum_{l=1}^k\frac{1}{(1+|y-x_l|)^{3+\frac{1}{2}}}.$$

As a result we have

$$\|N(\phi)\|_{**}\le C\|\phi\|_*^2,\quad N=4.$$

Thus

$$\|N(\phi)\|_{**}\le C\|\phi\|_*^{\min(2^*-1,2)}.$$

This completes the proof of the lemma. □

Next, we estimate l_k. By almost the same calculus of Lemma 3.6 in [17], we have

Lemma 3.4. *We have*

$$\|l_k\|_{**}\le C\varepsilon^{\frac{1}{2}+\sigma},$$

where $\sigma>0$ is a fixed small constant.

Using Lemmas 3.3 and 3.4 the remaining proof of Proposition 3.2 is just the same as that of Proposition 3.4 in [17]. Here we will not repeat it.

4 Proof of Theorem 2.1

Let

$$F(\Lambda)=I\big(W+\phi\big),$$

where ϕ is the function obtained in Proposition 3.2, and let

$$I(u)=\frac{1}{2}\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}|Du|^2-\frac{(N-2)^2}{2}\int_{\mathbf{R}^N\backslash\Omega_\varepsilon}|u|^{2^*}.$$

Using the symmetry, we can check that if Λ is a critical point of $F(\Lambda)$, then $W+\phi$ is a solution of (1.4). According to Proposition 4.1 in [17], we have that

Proposition 4.1. *For $N\ge4$, we have*

$$F(\Lambda)=k\big(A_0+A_1\gamma\Lambda\varepsilon-A_2\Lambda^{N-2}\varepsilon+o(\varepsilon)\big),$$

where the constant $A_i>0,i=0,1,2$ are positive constants, which are given in Proposition 2.8.

Proof of Theorem 2.1. We just need to prove that $F(\Lambda)$ has a critical point.

For $N\ge4$, the function

$$A_1\gamma\Lambda-A_2\Lambda^{N-2}$$

has a maximum point at

$$\Lambda_0=\Big(\frac{A_1\gamma}{A_2(N-2)}\Big)^{\frac{1}{N-3}}.$$

Thus, $F(\Lambda)$ attains its maximum in the interior of $[\delta,\delta^{-1}]$ if $\delta>0$ is small. As a result, $F(\Lambda)$ has a critical point in $[\delta,\delta^{-1}]$. □

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A Basic estimates

Firstly, we prove Lemma 3.1.

Proof. Through scaling we may assume $\varepsilon=1$. Let $G(x,y)$ be the Green's function satisfying

$$\begin{aligned} &-\Delta G(x,y)=\delta_y \quad \text{in } \mathbf{R}^N\backslash\Omega, \\ &|G(x,y)|\to 0 \quad \text{as } |x-y|\to+\infty, \quad \frac{\partial G(x,y)}{\partial n}=0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then we have for $x\in R^N\backslash\Omega$,

$$u(x)=\int_{\mathbf{R}^N\backslash\Omega} G(x,y)f(y)\mathrm{d}y.$$

So it is enough to show that there exists a constant C, independent of x and y such that

$$|G(x,y)|\leq\frac{C}{|x-y|^{N-2}}.$$

To this end we decompose G as in two parts

$$G(x,y)=H(|x-y|)+K(x,y)$$

where $K(|x-y|)$ is the singular part of G and $H(x,y)$ is the regular part of G. Certainly we have

$$|K(|x-y|)|\leq\frac{C}{|x-y|^{N-2}}.$$

It remains to show that

$$|H(x,y)|\leq\frac{C}{|x-y|^{N-2}}. \tag{A.1}$$

Note that, if for any $x\in\partial\Omega, d(y,x)>d_0>0$, then $\frac{\partial K(|x-y|)}{\partial n}$ is bounded. Take C_0 large enough, consider the inhomogeneous Neumann problem

$$\begin{cases} \Delta\psi=0 & \text{in } \mathbf{R}^N\backslash\Omega, \quad \psi(\infty)=0, \\ \dfrac{\partial\psi}{\partial n}=C_0 & \text{on } \partial\Omega. \end{cases} \tag{A.2}$$

By the classical potential theory (see [20]), (A.2) has a unique positive solution and is given by the single layer potential

$$\psi(x)=\int_{\partial\Omega}\sigma(z)K_0(|x-z|)\mathrm{d}S_z,$$

where σ is a continuous function on $\partial\Omega$ and K_0 is the fundamental solution of $-\Delta$ in $\mathbf{R}^N$. Therefore for some constant C,

$$0<\psi(x)<CK_0(|x-y|),\quad x\in\mathbf{R}^N\backslash\Omega.$$

Observe that ψ is an upper solution and $-\psi$ is a lower solution of

$$\begin{cases}\Delta H=0 \quad \text{in } \mathbf{R}^N\backslash\Omega, \quad |H(x,y)|\to 0 \quad \text{as } |x-y|\to+\infty,\\ \dfrac{\partial H}{\partial n}=-\dfrac{\partial K(|x-y|)}{\partial n} \quad \text{on } \partial\Omega.\end{cases}$$

Hence the comparison principle implies

$$|H(x,y)|\le CK_0(|x-y|)\le\frac{C}{|x-y|^{N-2}}.$$

Otherwise we first consider $d(x,\partial\Omega)$ and $d(y,\partial\Omega)$ small. Let $y\in\mathbf{R}^N\backslash\Omega$ be such that $d=d(y,\partial\Omega)$ is small. So there exists a unique point $\bar{y}\in\partial\Omega$ such that $d=|y-\bar{y}|$. Without loss of generality we may assume $\bar{y}=0$ and the outer normal at $\bar{y}$ is pointing toward $-x_N$-direction. Let y^* be the reflection point $y^*=(0,\cdots,0,-d)$ and consider the following auxiliary function

$$H^*(x,y)=K(|x-y^*|).$$

Then H^* satisfies $\Delta H^*=0$ in $\mathbf{R}^N\backslash\Omega$ and on $\partial\Omega$

$$\frac{\partial}{\partial n}(H^*(x,y))=-\frac{\partial}{\partial n}(K(|x-y|))+\mathcal{O}\Big(\frac{1}{d^{N-2}}\Big).$$

Hence we derive that

$$H(x,y)=H^*(x,y)+\mathcal{O}\Big(\frac{1}{d^{N-3}}\Big),$$

which proves (A.1) for $x,y\in\mathbf{R}^N\backslash\Omega$. This implies that for $x\in\mathbf{R}^N\backslash\Omega$

$$|u(x)|\le C\int_{\mathbf{R}^N\backslash\Omega}\frac{|f(y)|}{|x-y|^{N-2}}\mathrm{d}y. \tag{A.3}$$

If $x\in\partial\Omega$, we consider a sequence of points $x_i\in\mathbf{R}^N\backslash\Omega, x_i\to x\in\partial\Omega$ and take the limit in (A.3). Lebesgue's Dominated Convergence Theorem applies and (3.6) is proved. □

Now we start to prove that $W\le C$, where $C>0$ is a constant, independent of k. We have a more general result whose proof can be found in [17].

Lemma A.1. *For any $\alpha>0$,*

$$\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\alpha}}\le C\Big(1+\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{\alpha}}\Big),$$

where $C>0$ is a constant, independent of k.

For each fixed i and j, $i\neq j$, consider the following function

$$g_{ij}(y)=\frac{1}{(1+|y-x_j|)^{\alpha}}\frac{1}{(1+|y-x_i|)^{\beta}},\tag{A.4}$$

where $\alpha\ge 1$ and $\beta\ge 1$ are two constants. The following two lemmas can be found in Appendix B in [10].

Lemma A.2. *For any constant $0\le\sigma\le\min(\alpha,\beta)$, there is a constant $C>0$, such that*

$$g_{ij}(y)\le\frac{C}{|x_i-x_j|^{\sigma}}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).$$

Lemma A.3. *For any constant $0<\sigma<N-2$, there is a constant $C>0$, such that*

$$\int_{\mathbf{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z|)^{2+\sigma}}\mathrm{d}z\le\frac{C}{(1+|y|)^{\sigma}}.$$

Let us recall that

$$\varepsilon=k^{-\frac{N-2}{N-3}},\quad N\ge 4.$$

Lemma A.4. *Recall that $\tau=\frac{N-3}{N-2}$ if $N\ge 6$, $N=4$ and $\tau=\frac{5}{6}$ if $N=5$. Then there is a small $\theta>0$, such that*

$$\begin{aligned}&\int_{\mathbf{R}^N}\frac{1}{|y-z|^{N-2}}W^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\mathrm{d}z\\ \le& C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}+o(1)\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},\end{aligned}$$

where $o(1)\to 0$ as $k\to+\infty$.

Proof. Firstly, we consider $N\ge 6$. Then

$$\frac{4}{N-2}\le 1,\quad\frac{4(N-3)}{N-2}\ge 4,$$

which makes the proof the same as in [17]. Hence we only give out the proof of the case $N=4,5$.

Suppose now that $N=5$. Recall that $\epsilon=k^{-\frac{3}{2}}$ and

$$\Omega_j=\Big\{y=(y',y'')\in\mathbf{R}^N\backslash\Omega_\varepsilon:\ \langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\geq\cos\frac{\pi}{k}\Big\}.$$

For $z\in\Omega_1$, we have $|z-x_j|\geq|z-x_1|$. Using Lemma A.2, we obtain

$$\begin{aligned}\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^3}\leq&\frac{1}{(1+|z-x_1|)^{\frac{3}{2}}}\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}}}\\ \leq&\frac{C}{(1+|z-x_1|)^{\frac{7}{3}}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{2}{3}}}\leq\frac{C}{(1+|z-x_1|)^{\frac{7}{3}}}\end{aligned}$$

since

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{2}{3}}}\leq C(\varepsilon k)^{\frac{2}{3}}\sum_{j=2}^{k}\frac{1}{j^{\frac{2}{3}}}=\mathcal{O}(\varepsilon^{\frac{2}{3}}k)=\mathcal{O}(1).$$

Similarly,

$$\sum_{j=2}^{k}\frac{\varepsilon}{(1+|z-x_j|)^2}\leq\frac{C\varepsilon^{\frac{1}{3}}}{(1+|z-x_1|)^2}.$$

Thus,

$$\begin{aligned}W^{\frac{4}{3}}(z)\leq&\Big(\frac{C}{1+|z-x_1|)^3}+\frac{C}{(1+|z-x_1|)^{\frac{7}{3}}}+\frac{C\varepsilon^{\frac{1}{3}}}{(1+|z-x_1|)^2}\Big)^{\frac{4}{3}}\\ \leq&\frac{C}{(1+|z-x_1|)^{\frac{28}{9}}}+\frac{C\varepsilon^{\frac{4}{9}}}{(1+|z-x_1|)^{\frac{8}{3}}}.\end{aligned}$$

As a result, for $z\in\Omega_1$, using Lemma 2.1 again, we find that for $\theta>0$ small,

$$\begin{aligned}&W^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\\ \leq&\frac{C}{(1+|z-x_1|)^{\frac{28}{9}+\frac{3}{2}+\tau}}+\frac{C}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau+\theta}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{10}{9}-\theta}}\\ &+\frac{C\varepsilon^{\frac{4}{9}}}{(1+|z-x_1|)^{\frac{8}{3}+\frac{3}{2}+\tau}}+\frac{C\varepsilon^{\frac{4}{9}}}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{1}{3}}}\\ \leq&\frac{C}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau+\theta}}+o(1)\frac{1}{(1+|z-x_1|)^{2+\frac{3}{2}+\tau}}.\end{aligned}$$

So, we obtain

$$
\begin{aligned}
&\int_{\Omega_1}\frac{1}{|y-z|^3}W^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\mathrm{d}z\\
\le&\frac{C}{(1+|y-x_1|)^{\frac{3}{2}+\tau+\theta}}+o(1)\frac{1}{(1+|z-x_1|)^{\frac{3}{2}+\tau}},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\frac{1}{|y-z|^3}W^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\mathrm{d}z\\
=&\sum_{i=1}^{k}\int_{\Omega_i}\frac{1}{|y-z|^3}W^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\mathrm{d}z\\
\le&\sum_{i=1}^{k}\frac{C}{(1+|y-x_i|)^{\frac{3}{2}+\tau+\theta}}+o(1)\frac{1}{(1+|y-x_i|)^{\frac{3}{2}+\tau}}.
\end{aligned}
$$

Suppose that $N=4$. In this case, $\epsilon=k^{-2}$. We have that for $z\in\Omega_1$,

$$
\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^2}\le\frac{C}{(1+|z-x_1|)^{\frac{3}{2}}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{1}{2}}}\\
\le&\frac{C\epsilon^{\frac{1}{2}}k}{(1+|z-x_1|)^{\frac{3}{2}}}\le\frac{C}{(1+|z-x_1|)^{\frac{3}{2}}}
\end{aligned}
$$

and

$$
\sum_{j=2}^{k}\frac{\varepsilon}{(1+|z-x_j|)}\le\frac{C\varepsilon^{\frac{1}{2}}}{(1+|z-x_j|)}.
$$

Thus

$$
\begin{aligned}
&W^2(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{1+\tau}}\\
\le&\left(\frac{C}{(1+|z-x_1|)^3}+\frac{C\varepsilon}{(1+|z-x_j|)^2}\right)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{1+\tau}}\\
\le&\frac{C}{(1+|z-x_1|)^{4+\tau}}+\frac{C}{(1+|z-x_1|)^{2+1+\tau+\frac{1}{2}}}\sum_{j=1}^{k}\frac{1}{|x_1-x_j|^{\frac{1}{2}}}\\
&+\frac{C\varepsilon^{\frac{1}{2}}}{(1+|z-x_1|)^{2+1+\tau}}\\
\le&\frac{C}{(1+|z-x_1|)^{2+1+\tau+\frac{1}{2}}}+o(1)\frac{C}{(1+|z-x_1|)^{2+1+\tau}},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\frac{1}{|y-z|^2}W^2(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{1+\tau}}\mathrm{d}z\\
=&\sum_{i=1}^{k}\int_{\Omega_i}\frac{1}{|y-z|^2}W^2(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{1+\tau}}\mathrm{d}z\\
\leq&\sum_{i=1}^{k}\frac{C}{(1+|y-x_i|)^{\frac{1}{2}+1+\tau}}+o(1)\sum_{i=1}^{k}\frac{C}{(1+|z-x_1|)^{1+\tau}}.
\end{aligned}
$$

This completes the proof of the lemma. □

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