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NEW EXPRESSIONS OF PERIODIC WAVES AND A NOVEL PHENOMENON IN A COMPRESSIBLE HYPERELASTIC ROD*

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Dedicated to Prof. Boling Guo for his 70th Birthday

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Abstract A In this paper, we employ both bifurcation method of dynamical systems and numerical exploration of differential equations to investigate the periodic waves of a general compressible hyperelastic rod equation

$$u_t + 3uu_x - u_{xxt} - \gamma(2u_xu_{xx} + uu_{xxx}) = 0,$$

with parameter $\gamma < 0$. New expressions including explicit expressions and implicit expressions are obtained. Some previous results are extended. Specially, a new phenomenon is found: when the initial value tends to a certain number, the periodic shock wave suddenly changes into a smooth periodic wave. In dynamical systems, this represents that one of orbits can pass through the singular line. The coherency of numerical simulation and theoretical derivation implies the correctness of our results.

Key Words Hyperelastic rod; bifurcation method; numerical exploration; periodic waves.

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1. Introduction

Many authors have studied nonlinear waves in elastic rods. For instance, Wright [1] considered traveling waves in a rod composed of an incompressible hyperelastic material. Samsonov [2] obtained the so-called double dispersive equation and showed that it has a solitary wave solution. Coleman and Newman [3] derived the one-dimensional rod equation for a general incompressible hyperelastic material. Dai [4] studied disturbances in an initially stretched or compressed rod which is composed of a compressible Mooney-Rivlin material and derived a new type of nonlinear dispersive equation as the model equation which takes the following form:

$$u_t + 3uu_x - u_{xxt} - \gamma(2u_xu_{xx} + uu_{xxx}) = 0, \tag{1.1}$$

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where γ is a physical parameter. Dai and Huo [5, 6] used phase portraits of traveling wave system to show that Eq.(1.1) has a variety of travelling waves, including solitary shock waves, solitary waves, periodic shock waves, etc. Also a new phenomenon was found in [5]: a solitary wave can suddenly change into a periodic wave (with finite period). In dynamical systems, this represents that a homoclinic orbit suddenly changes into a closed orbit. Constantin and Strauss [7] proved that the solitary waves of Eq.(1.1) are orbitally stable, which establish that the shape of the waves are stable. Yin [8] studied the Cauchy problem of Eq.(1.1). Liu and Chen [9] got some implicit expressions of the compactons for Eq.(1.1), which include Jacobian elliptic functions. Dai et al [10] provided some theoretical results to deal with singular solutions and obtained an explicit expression of the compactons for Eq.(1.1).

With $\gamma = 1$ in Eq.(1.1), we find Camassa-Holm equation which has been studied successively by many authors; see for instance Camassa & Holm [11], Constantin et al [12-15], Lenells [16- 18], Wazwaz [19, 20], Liu and Wang [21]. When $\gamma = 0$, Eq.(1.1) becomes the BBM equation [22], a well-known model for surface waves in a channel.

Recently, bifurcation method of dynamical systems has been used to investigate the nonlinear waves of some partial differential equations; see for instance Li and Liu [23, 24], Liu et al [25- 27], Tang et al [28, 29].

In this paper, we employ the bifurcation method of dynamical systems and numerical exploration to study the periodic waves of Eq.(1.1) with parameter $\gamma < 0$. Firstly, we derive travelling wave equation and system. Then we draw bifurcation curves and bifurcation phase portraits of the travelling wave system. From the bifurcation phase portraits one can see all the closed orbits. By using these closed orbits, the implicit expressions or explicit expressions of periodic waves are obtained. Our work extends previous results.

Specially we find another new phenomenon in Eq.(1.1): When the initial value tends to a certain number, the periodic shock wave suddenly changes into a smooth periodic wave. In dynamical systems, this represents that one of orbits can pass through the singular line (see orbit $\overline{\Gamma}$ in Fig.1 (I)). We give not only theoretical derivation, but also numerical simulation. The coherency of the consequences deduced from these two methods implies the correctness of our conclusions.

This paper is organized as follows. In Section 2, we give a preliminary. Our main results and two sets of graphs of implicit functions and explicit functions are given in Section 3. We arrange the theoretic derivation of our main results in Section 4. In Section 5, two sets of numerical simulations are displayed to test the correctness of our theoretic derivation. A short conclusion is also given in this section.

2. Preliminary

In this section, firstly we derive travelling wave equation and system. Then we draw the bifurcation curves and bifurcation phase portraits of the travelling wave system. These phase portraits will be used to obtain our main results.

Now we substitute $u = \varphi(\xi)$ into Eq.(1.1), where $\xi = x - ct$ and c is an arbitrary constant. We get

$$-c\varphi' + 3\varphi\varphi' + c\varphi''' - \gamma(2\varphi'\varphi'' + \varphi\varphi''') = 0. \tag{2.1}$$

Integrating (2.1) once, we have travelling wave equation

$$\varphi''(\gamma\varphi - c) = 3\varphi^2/2 - c\varphi + g - \gamma(\varphi')^2/2, \tag{2.2}$$

where g is an arbitrary constant.

Letting $y = \varphi'$, we obtain travelling wave system

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\frac{3}{2}\varphi^2 - c\varphi + g - \frac{\gamma}{2}y^2}{\gamma\varphi - c}, \tag{2.3}$$

with first integral

$$H(\varphi, y) = (\gamma\varphi - c)y^2 - \varphi^3 + c\varphi^2 - 2g\varphi = h. \tag{2.4}$$

In the coming sections we will use the phase portraits of the system (2.3) to construct our results.

Note that the system (2.3) has a singular line $\varphi = c/\gamma$ which is inconvenient to our study. Thus we make transformation $d\tau = d\xi/(\gamma\varphi - c)$. Under this transformation the system (2.3) becomes the following Hamiltonian system

$$\frac{d\varphi}{d\tau} = (\gamma\varphi - c)y, \quad \frac{dy}{d\tau} = \frac{3}{2}\varphi^2 - c\varphi + g - \frac{\gamma}{2}y^2, \tag{2.5}$$

which has the same first integral as the system (2.3). Therefore (2.3) and (2.5) have the same topological phase portraits except the singular line $\varphi = c/\gamma$. This implies that one can get the topological phase portraits of the system (2.3) from that of the system (2.5).

Now we consider the system (2.5). In the following lemma the properties of the singular points and some inequalities will be pointed out.

Lemma 1 *For given parameter $\gamma < 0$ and arbitrary constant c, let*

$$q = \frac{c}{\gamma}, \tag{2.6}$$

$$g_1(c) = \frac{2\gamma - 3}{2\gamma^2}c^2, \tag{2.7}$$

$$g_2(c) = \frac{(\gamma - 1)(\gamma + 3)}{8\gamma^2}c^2, \tag{2.8}$$

$$g_3(c) = \frac{c^2}{6}, \tag{2.9}$$

$$\varphi_\pm^0 = \left(c \pm \sqrt{c^2 - 6g}\right)/3 \quad \text{for} \quad g \le g_3(c), \tag{2.10}$$

$$\varphi^*_\pm = \left(c \pm 2\sqrt{c^2 - 6g}\right)/3 \quad \text{for} \quad g \le g_3(c), \tag{2.11}$$

$$\overline{\varphi}_\pm = \frac{\left(c(\gamma-1) \pm \sqrt{\gamma^2(c^2-8g)+c^2(2\gamma-3)}\right)}{2\gamma}. \quad \text{for} \quad g \le g_2(c), \tag{2.12}$$

$$y^0_\pm = \pm\sqrt{\frac{1}{\gamma^3}(3c^2 - 2\gamma c^2 + 2\gamma^2 g)} \quad \text{for} \quad g \le g_1(c), \tag{2.13}$$

$$f_1(\varphi) = 3\varphi^2/2 - c\varphi + g, \tag{2.14}$$

$$f_2(\varphi) = \varphi^3 - 3\varphi^2 + 2g\varphi. \tag{2.15}$$

Then we have the following inequalities and properties:

(1) *$g_1(c)$, $g_2(c)$ and $g_3(c)$ have a unique intersection point $(0,0)$ and*

$$g_1(c) < g_2(c) < g_3(c) \quad \text{for} \quad c \ne 0. \tag{2.16}$$

(2) *φ^0_+ and φ^0_- are the roots of the equation $f_1(\varphi) = 0$. φ^*_+ is a root of the equation $f_2(\varphi) + H(\varphi^0_-, 0) = 0$. φ^*_- is a root of the equation $f_2(\varphi) + H(\varphi^0_+, 0) = 0$. $\overline{\varphi}_+$ is a root of the equation $f_2(\varphi)+H(\overline{\varphi}_-, 0) = 0$, $\overline{\varphi}_-$ is a root of the equation $f_2(\varphi)+H(\overline{\varphi}_+, 0) = 0$, y^0_+ and y^0_- are the roots of the equation $\frac{\gamma}{2}y^2 - f_1(q) = 0$.*

(3) *For any c, if $g < g_1(c)$, then (2.5) has four singular points $(\varphi^0_-, 0)$, $(\varphi^0_+, 0)$, (q, y^0_-) and (q, y^0_+). $(\varphi^0_-, 0)$ and $(\varphi^0_+, 0)$ are two center points surrounded by a family of closed orbits respectively. (q, y^0_-) and (q, y^0_+) are two saddle points connected by two heteroclinic orbits which pass $(\overline{\varphi}_+, 0)$ and $(\overline{\varphi}_-, 0)$ respectively. $\varphi^0_\pm$, $\varphi^*_\pm$ and $\overline{\varphi}_\pm$ satisfy that*

$$\varphi^*_- < \overline{\varphi}_+ < \varphi^0_- < q < \varphi^0_+ < \overline{\varphi}_- < \varphi^*_+. \tag{2.17}$$

Let $\overline{\Gamma}$ denote the closed curve composed by two heteroclinic orbits, Γ^* denote the closed orbits surrounding $(\varphi^0_-, 0)$, Γ^0 denote the closed orbits surrounding $(\varphi^0_+, 0)$ (see Fig.1 (I)).

(4) *If $c \ne 0$ and $g = g_1(c)$, then (2.5) has two singular points $(\varphi^0_-, 0)$, $(\varphi^0_+, 0)$.*

When $c < 0$, $(\varphi^0_-, 0)$ is a center point surrounded by a family of closed orbits, and $(\varphi^0_+, 0)$ is connected by a singular closed orbit $\overline{\Gamma}$ which passes $(\overline{\varphi}_+, 0)$. $\varphi^0_\pm$, $\varphi^*_\pm$ and $\overline{\varphi}_\pm$ satisfy that

$$\varphi^*_- = \overline{\varphi}_+ < \varphi^0_- < \varphi^0_+ = \overline{\varphi}_- = q < \varphi^*_+. \tag{2.18}$$

Let Γ^* denote the closed orbits surrounding $(\varphi^0_-, 0)$, (see Fig.1 (D)).

When $c > 0$, $(\varphi^0_+, 0)$ is a center point surrounded by a family of closed orbits, and $(\varphi^0_-, 0)$ is connected by a singular closed orbit $\overline{\Gamma}$ which passes $(\overline{\varphi}_-, 0)$. $\varphi^0_\pm$, $\varphi^*_\pm$ and $\overline{\varphi}_\pm$ satisfy that

$$\varphi^*_- < q = \varphi^0_- = \overline{\varphi}_+ < \varphi^0_+ < \overline{\varphi}_- = \varphi^*_+. \tag{2.19}$$

Let Γ^0 denote the closed orbits surrounding $(\varphi^0_+, 0)$ (see Fig.1 (H)).

(5) *If $c < 0$ and $g_1(c) < g < g_3(c)$, then (2.5) has two singular points $(\varphi^0_-, 0)$ and $(\varphi^0_+, 0)$. $(\varphi^0_-, 0)$ is a center point surrounded by a family of closed orbits. $(\varphi^0_+, 0)$ is* *a saddle point connected by a homoclinic orbit which passes* $(\varphi_-^*, 0)$. $\varphi_\pm^0$, $\varphi_\pm^*$ *and* $\overline{\varphi}_\pm$ *satisfy that*

$$\varphi_-^* < \overline{\varphi}_+ < \varphi_-^0 < \overline{\varphi}_- < \varphi_+^0 < q < \varphi_+^* \quad \text{for} \quad g_1(c) < g < g_2(c). \tag{2.20}$$

$$\varphi_-^* < \overline{\varphi}_+ = \varphi_-^0 = \overline{\varphi}_- < \varphi_+^0 < \varphi_+^* = q \quad \text{for} \quad g = g_2(c), \tag{2.21}$$

and

$$\varphi_-^* < \varphi_-^0 < \varphi_+^0 < \varphi_+^* < q \ \text{ for } \ g_2(c) < g < g_3(c) \ \ (\overline{\varphi}_- \text{ and } \overline{\varphi}_+ \text{ vanish}). \tag{2.22}$$

Let $\overline{\Gamma}$ *denote the closed orbit passing* $(\overline{\varphi}_+, 0)$ *(we will show that* $\overline{\Gamma}$ *passes* $(\overline{\varphi}_-, 0)$ *too),* Γ^{**} *denote the closed orbits locating outside* $\overline{\Gamma}$, Γ^* *denote the closed orbits locating inside* $\overline{\Gamma}$. *When* $g_1(c) < g < g_2(c)$, $\overline{\Gamma}$, Γ^* *and* Γ^{**} *exist* (see Fig.1 (C)). *When* $g_2(c) \le g < g_3(c)$, $\overline{\Gamma}$ *and* Γ^* *vanish,* Γ^{**} *exist* (see Fig.1 (A), (B)).

(6) *If* $c > 0$ *and* $g_1(c) < g < g_3(c)$, *then* (2.5) *has two singular points* $(\varphi_-^0, 0)$ *and* $(\varphi_+^0, 0)$. $(\varphi_-^0, 0)$ *is a saddle point connected with a homoclinic orbit which passes* $(\varphi_+^*, 0)$. $(\varphi_+^0, 0)$ *is a center point surrounded by a family of closed orbits.* $\varphi_\pm^0$, $\varphi_\pm^*$ *and* $\overline{\varphi}_\pm$ *satisfy that*

$$\varphi_-^* < q < \varphi_-^0 < \overline{\varphi}_+ < \varphi_+^0 < \overline{\varphi}_- < \varphi_+^* \quad \text{for} \quad g_1(c) < g < g_2(c). \tag{2.23}$$

$$\varphi_-^* = q < \varphi_-^0 < \varphi_+^0 = \overline{\varphi}_- = \overline{\varphi}_+ < \varphi_+^* \quad \text{for} \quad g = g_2(c), \tag{2.24}$$

and

$$q < \varphi_-^* < \varphi_-^0 < \varphi_+^0 < \varphi_+^* \ \text{ and } \ g_2(c) < g < g_3(c) \ \ (\overline{\varphi}_- \text{ and } \overline{\varphi}_+ \text{ vanish}). \tag{2.25}$$

Let $\overline{\Gamma}$ *denote the closed orbit passing* $(\overline{\varphi}_+, 0)$ *(we will show that* $\overline{\Gamma}$ *passes* $(\overline{\varphi}_-, 0)$ *too),* Γ^{00} *denote the closed orbits locating outside* $\overline{\Gamma}$, Γ^0 *denote the closed orbits locating inside* $\overline{\Gamma}$. *When* $g_1(c) < g < g_2(c)$, $\overline{\Gamma}$, Γ^0 *and* Γ^{00} *exist* (see Fig.1 (G)). *When* $g_2(c) \le g < g_3(c)$, $\overline{\Gamma}$ *and* Γ^0 *vanish,* Γ^{00} *exist* (see Fig.1 (E), (F)).

(7) *For any* c, *if* $g = g_3(c)$, *then* (2.5) *has a unique singular point* $(0, c/3)$ *which is a degenerate saddle point.* φ_+^0 *and* φ_-^0 *satisfy that*

$$\varphi_+^0 = \varphi_-^0 = c/3 < q \quad \text{for} \quad c < 0, \tag{2.26}$$

and

$$\varphi_+^0 = \varphi_-^0 = c/3 > q \quad \text{for} \quad c > 0. \tag{2.27}$$

(8) *For any* c, *if* $g > g_3(c)$, *then* (2.5) *has no singular point.*

On the basis of Lemma 1 we draw the bifurcation curves and phase portraits of systems (2.3) and (2.5) as Fig.1.

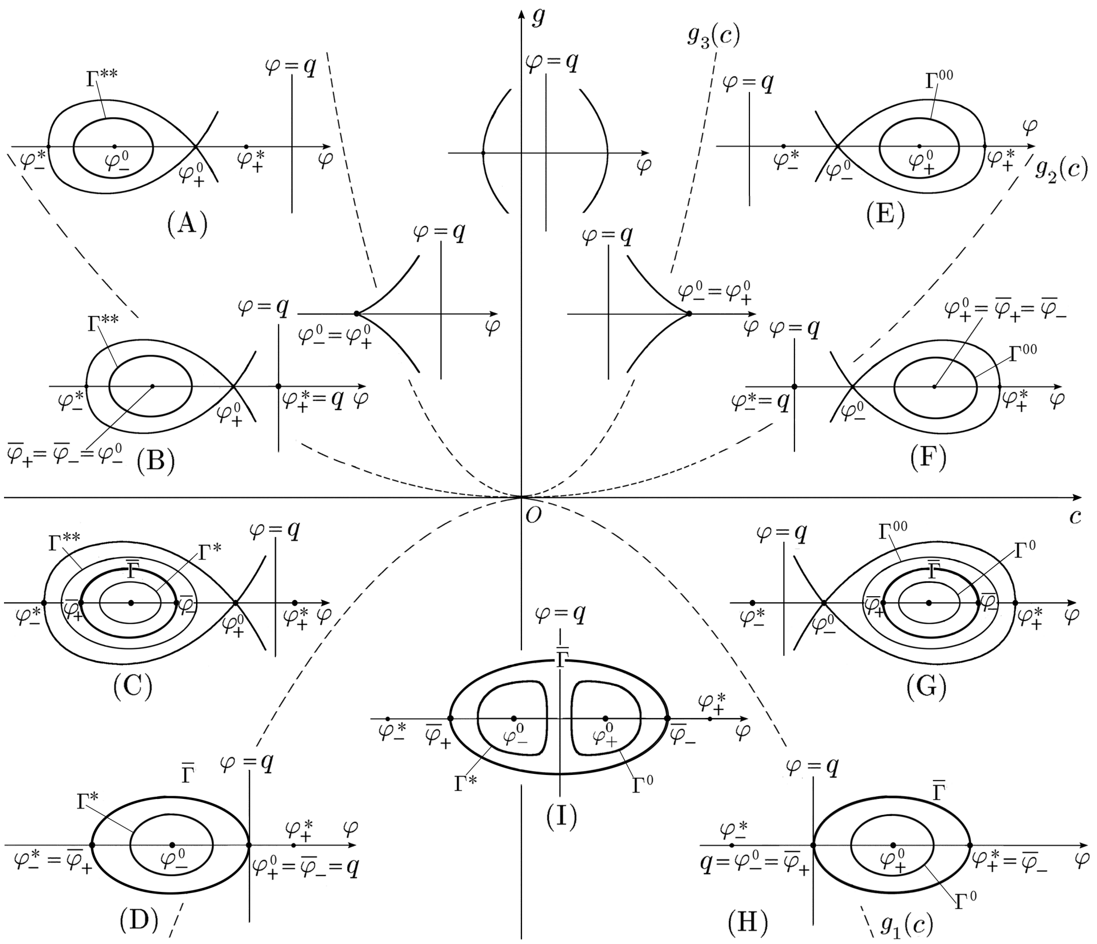


Fig.1. Bifurcation curves and bifurcation phase portraits of systems (2.3) and (2.5) for $\gamma < 0$. (The topological phase portraits of systems (2.3) and (2.5) are the same except the line $\varphi = q$)

Proof From (2.7)-(2.9) we have

$$g_3(c) - g_2(c) = \frac{(\gamma-3)^2c^2}{24\gamma^2}, \tag{2.28}$$

and

$$g_2(c) - g_1(c) = \frac{(\gamma-3)^2c^2}{8\gamma^2}. \tag{2.29}$$

Note that $\gamma < 0$. This implies that the inequality (2.16) holds and three curves $g = g_i(c)(i = 1, 2, 3)$ have a unique intersection point $(0, 0)$.

From the solutions of equations

$$\begin{cases} (\gamma\varphi - c)y = 0, \\ f_1(\varphi) - \frac{\gamma}{2}y^2 = 0, \end{cases} \tag{2.30}$$

we see that the statements about the number and distribution of the singular points are true.

If $\lambda_\pm(\varphi_\pm^0, 0)$ and $\lambda_\pm(q, y_\pm^0)$ respectively denote the eigenvalues of the linearized system of (2.5) at the singular points $(\varphi_\pm^0, 0)$ and $(q, y_\pm^0)$, then we have

$$\lambda_\pm(\varphi_-^0, 0) = \pm\sqrt{\gamma(\varphi_-^0 - q) f_1'(\varphi_-^0)}, \tag{2.31}$$

$$\lambda_\pm(\varphi_+^0, 0) = \pm\sqrt{\gamma(\varphi_+^0 - q) f_1'(\varphi_+^0)}, \tag{2.32}$$

and

$$\lambda_\pm(q, y_-^0) = \lambda_\pm(q, y_+^0) = \pm|\gamma y_\pm^0|, \tag{2.33}$$

Note that φ_-^0 and φ_+^0 are the two roots of $f_1(\varphi)$ and $\varphi_-^0 < \varphi_+^0$. Thus we have

$$f_1'(\varphi_-^0) < 0 \quad \text{and} \quad f_1'(\varphi_+^0) > 0. \tag{2.34}$$

On the other hand, from the expressions (2.10)-(2.12) we obtain equalities and inequalities (2.17)-(2.27). Apply (2.17)-(2.27) and (2.34) to (2.31)-(2.32), (2.13) to (2.33), the signs of $\lambda_\pm(\varphi_-^0, 0)$, $\lambda_\pm(\varphi_+^0, 0)$ and $\lambda_\pm(q, y_\pm^0)$ can be determined. According to the qualitative theory and bifurcation method of dynamical systems (e.g., Guckenheimer & Holmes [30]), it is seen that the statements about the properties of singular points are true.

Now let us turn to the implicit expressions of the closed orbits on $\varphi - y$ plane. Notice that the center points are on φ axis. Thus any closed orbit has two intersection points with φ axis. From (2.4) we see that the closed orbit passing $(\varphi_0, 0)$ has expression

$$(\gamma\varphi - c)y^2 + (\varphi_0 - \varphi)(\varphi - m)(\varphi - n) = 0, \tag{2.35}$$

where

$$m = m(\varphi_0) = \frac{1}{2}\left(c - \varphi_0 + \sqrt{c^2 - 8g + 2c\varphi_0 - 3\varphi_0^2}\right), \tag{2.36}$$

and

$$n = n(\varphi_0) = \frac{1}{2}\left(c - \varphi_0 - \sqrt{c^2 - 8g + 2c\varphi_0 - 3\varphi_0^2}\right). \tag{2.37}$$

About m and n we have the following two special cases:

(i) For $c < 0$ and $g_1(c) \le g < g_2(c)$, if $\varphi_0 = \overline{\varphi}_+$, then $m(\varphi_0) = q$ and $n(\varphi_0) = \overline{\varphi}_-$; if $\varphi_0 = \overline{\varphi}_-$, then $m(\varphi_0) = q$ and $n(\varphi_0) = \overline{\varphi}_+$.

(ii) For $c > 0$ and $g_1(c) \le g < g_2(c)$, or for any c and $g < g_1(c)$, if $\varphi_0 = \overline{\varphi}_+$, then $m(\varphi_0) = \overline{\varphi}_-$ and $n(\varphi_0) = q$; if $\varphi_0 = \overline{\varphi}_-$, then $m(\varphi_0) = \overline{\varphi}_+$ and $n(\varphi_0) = q$.

These imply that when $\varphi_0 = \overline{\varphi}_+$ or $\varphi_0 = \overline{\varphi}_-$, (2.35) becomes

$$y^2 - \left|\frac{1}{\gamma}(\varphi - \overline{\varphi}_+)(\varphi - \overline{\varphi}_-)\right| = 0. \tag{2.38}$$

Note that $\overline{\varphi}_+$ and $\overline{\varphi}_-$ exist for $g < g_1(c)$ too. Therefore (2.38) holds for $g < g_1(c)$. From (2.38) we find that there is a special closed orbit $\overline{\Gamma}$ passing points $(\overline{\varphi}_+, 0)$ and $(\overline{\varphi}_-, 0)$. Hereto, the proof is completed.

3. Main Results and Graphs of Implicit Functions

In this section we will give the expressions of periodic wave solutions and draw their planar graphs. The expressions contain the sets. One of them is implicit and the other is explicit. We state them in the following theorem 1 and corollaries.

Theorem 1 *For given parameter $\gamma < 0$, let c, g, φ_0 be constants and $\xi = x - ct$, $\sin^{-1}(\cdot)$ be the inverse function of sine function $\sin(\cdot)$, $\mathrm{sn}^{-1}(\cdot,\cdot)$ be the inverse function of Jacobin elliptic function $\mathrm{sn}(\cdot,\cdot)$, $\prod(\cdot,\cdot,\cdot)$ be Legendre's incomplete elliptic integral of the third kind, $K(\cdot)$ be the complete elliptic integral of the first kind. Then we have the following results:*

Result 1 *Under one of the two conditions:* (i) $c < 0$, $g_2(c) \le g < g_3(c)$ *and* $\varphi_-^* < \varphi_0 < \varphi_-^0$ (see Fig.1 (A), (B)), (ii) $c < 0$, $g_1(c) < g < g_2(c)$ *and* $\varphi_-^* < \varphi_0 < \overline{\varphi}_+$ (see Fig.1 (C)), *Eq.*(1.1) *has a periodic wave solution $u = \varphi(\xi)$ corresponding to φ_0 and its period is $2T_1$. On $(-T_1, T_1)$ φ possesses implicit expression*

$$k_1^2\,\mathrm{sn}^{-1}\left(\sqrt{\frac{n-\varphi}{\alpha_1^2(m-\varphi)}},k_1\right)+(\alpha_1^2-k_1^2)\prod\left(\sin^{-1}\sqrt{\frac{n-\varphi}{\alpha_1^2(m-\varphi)}},\alpha_1^2,k_1\right)=\beta_1(T_1-|\xi|),$$

$$\text{for } |\xi| < T_1, \tag{3.1}$$

where

$$k_1^2 = \frac{(m-q)(n-\varphi_0)}{(n-q)(m-\varphi_0)}, \tag{3.2}$$

$$\alpha_1^2 = \frac{n-\varphi_0}{m-\varphi_0}, \tag{3.3}$$

$$\beta_1 = \frac{n-\varphi_0}{2\sqrt{\gamma(m-\varphi_0)(n-q)}}, \tag{3.4}$$

$$T_1 = \left[k_1^2 K(k_1) + (\alpha_1^2-k_1^2)\prod\left(\frac{\pi}{2},\alpha_1^2,k_1\right)\right]/\beta_1, \tag{3.5}$$

and

$$\varphi_0 \le \varphi < n < m < q. \tag{3.6}$$

Result 2 *Under one of the two conditions:* (i) $c > 0$, $g_2(c) \le g < g_3(c)$ *and* $\varphi_+^0 < \varphi_0 < \varphi_+^*$ (see Fig.1 (E), (F)), (ii) $c > 0$, $g_1(c) < g < g_2(c)$ *and* $\overline{\varphi}_- < \varphi_0 < \varphi_+^*$ (see Fig.1 (G)), *Eq.*(1.1) *has a periodic wave solution $u = \varphi(\xi)$ corresponding to φ_0 and its period is $2T_2$. On $(-T_2, T_2)$, φ possesses implicit expression*

$$k_2^2\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\varphi-m}{\alpha_2^2(\varphi-n)}},k_2\right)+(\alpha_2^2-k_2^2)\prod\left(\sin^{-1}\sqrt{\frac{\varphi-m}{\alpha_2^2(\varphi-n)}},\alpha_2^2,k_2\right)=\beta_2(T_2-|\xi|),$$

$$\text{for } |\xi| < T_2, \tag{3.7}$$

where

$$k_2^2 = \frac{(n-q)(m-\varphi_0)}{(m-q)(n-\varphi_0)}, \tag{3.8}$$

$$\alpha_2^2 = \frac{m-\varphi_0}{n-\varphi_0}, \tag{3.9}$$

$$\beta_2 = \frac{\varphi_0 - m}{2\sqrt{\gamma(m-q)(n-\varphi_0)}}, \tag{3.10}$$

$$T_2 = \left[k_2^2 K(k_2) + (\alpha_2^2 - k_2^2)\prod\left(\frac{\pi}{2}, \alpha_2^2, k_2\right)\right]/\beta_2, \tag{3.11}$$

and

$$q < n < m < \varphi \leq \varphi_0. \tag{3.12}$$

Result 3 *Under one of the two conditions:* (i) $c < 0$, $g_1(c) \leq g < g_2(c)$ *and* $\overline{\varphi}_+ < \varphi_0 < \varphi_-^0$ (see Fig.1 (C), (D)), (ii) *any* c, $g < g_1(c)$ *and* $\overline{\varphi}_+ < \varphi_0 < \varphi_-^0$ (see Fig.1 (I)), *Eq.*(1.1) *has a periodic wave solution* $u = \varphi(\xi)$ *corresponding to* φ_0 *and its period is* $2T_3$. *On* $(-T_3, T_3)$, φ *possesses implicit form*

$$\prod\left(\sin^{-1}\sqrt{\frac{n-\varphi}{\alpha_3^2(q-\varphi)}}, \alpha_3^2, k_3\right) = \beta_3(T_3 - |\xi|), \quad \text{for} \quad |\xi| < T_3, \tag{3.13}$$

where

$$k_3^2 = \frac{(m-q)(n-\varphi_0)}{(m-n)(q-\varphi_0)}, \tag{3.14}$$

$$\alpha_3^2 = \frac{n-\varphi_0}{q-\varphi_0}, \tag{3.15}$$

$$\beta_3 = \frac{\sqrt{(m-n)(q-\varphi_0)}}{2\sqrt{-\gamma}(q-n)}, \tag{3.16}$$

$$T_3 = \prod\left(\frac{\pi}{2}, \alpha_3^2, k_3\right)/\beta_3, \tag{3.17}$$

and

$$\varphi_0 \leq \varphi < n < q < m. \tag{3.18}$$

Result 4 *Under one of the two conditions:* (i) $c > 0$, $g_1(c) \leq g < g_2(c)$ *and* $\varphi_+^0 < \varphi_0 < \overline{\varphi}_-$ (see Fig.1 (G), (H)), (ii) *any* c, $g < g_1(c)$ *and* $\varphi_+^0 < \varphi_0 < \overline{\varphi}_-$ (see Fig.1 (I)), *Eq.*(1.1) *has a periodic wave solution* $u = \varphi(\xi)$ *corresponding to* φ_0 *and its period is* $2T_4$. *On* $(-T_4, T_4)$, φ *possesses implicit form*

$$\prod\left(\sin^{-1}\sqrt{\frac{\varphi-m}{\alpha_4^2(\varphi-q)}}, \alpha_4^2, k_4\right) = \beta_4(T_4 - |\xi|), \quad \text{for} \quad |\xi| < T_4, \tag{3.19}$$

where

$$k_4^2 = \frac{(q-n)(m-\varphi_0)}{(m-n)(q-\varphi_0)}, \tag{3.20}$$

$$\alpha_4^2 = \frac{m-\varphi_0}{q-\varphi_0}, \tag{3.21}$$

$$\beta_4 = \frac{\sqrt{(m-n)(\varphi_0-q)}}{2\sqrt{-\gamma}(m-q)}, \tag{3.22}$$

$$T_4 = \prod\left(\frac{\pi}{2}, \alpha_4^2, k_4\right)/\beta_4, \tag{3.23}$$

and

$$n < q < m < \varphi \le \varphi_0. \tag{3.24}$$

Result 5 *For any c and $g < g_2(c)$, there are two cases as follows:*

(i) *If $\varphi_0 = \overline{\varphi}_-$, then corresponding to φ_0 Eq.*(1.1) *has an explicit periodic wave solution*

$$\begin{aligned} u_+(\xi) &= \frac{\overline{\varphi}_+ + \overline{\varphi}_-}{2} + \frac{\overline{\varphi}_- - \overline{\varphi}_+}{2}\cos\frac{\xi}{\sqrt{-\gamma}} \\ &= \frac{(\gamma-1)c}{2\gamma} + \frac{\sqrt{\gamma^2(c^2-8g)+c^2(2\gamma-3)}}{2\gamma}\cos\frac{\xi}{\sqrt{-\gamma}}. \end{aligned} \tag{3.25}$$

(ii) *If $\varphi_0 = \overline{\varphi}_+$, then corresponding to φ_0 Eq.*(1.1) *has an explicit periodic wave solution*

$$\begin{aligned} u_-(\xi) &= \frac{\overline{\varphi}_+ + \overline{\varphi}_-}{2} - \frac{\overline{\varphi}_- - \overline{\varphi}_+}{2}\cos\frac{\xi}{\sqrt{-\gamma}} \\ &= \frac{(\gamma-1)c}{2\gamma} - \frac{\sqrt{\gamma^2(c^2-8g)+c^2(2\gamma-3)}}{2\gamma}\cos\frac{\xi}{\sqrt{-\gamma}}. \end{aligned} \tag{3.26}$$

Corollary 1 *If parameter $\gamma < 0$, then Eq.*(1.1) *has two explicit periodic wave solutions*

$$u_1(\xi) = \frac{(\gamma-1)c}{2\gamma} + A\cos\frac{\xi}{\sqrt{-\gamma}}, \tag{3.27}$$

and

$$u_2(\xi) = \frac{(\gamma-1)c}{2\gamma} + A\sin\frac{\xi}{\sqrt{-\gamma}}, \tag{3.28}$$

where $A \neq 0$, $c \neq 0$ are arbitrary constants and $\xi = x - ct$.

Corollary 2 *If parameter $\gamma < 0$, then Eq.*(1.1) *has an explicit complex solution*

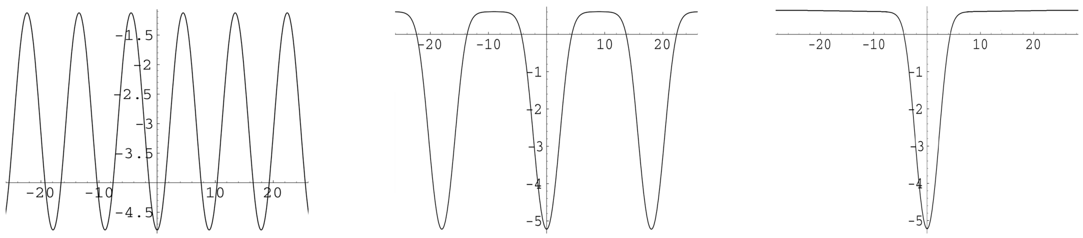
$$u(\xi) = \frac{(\gamma-1)c}{2} + A\,\mathrm{e}^{i\xi/\sqrt{-\gamma}}, \tag{3.29}$$

where A, c are arbitrary constants and $\xi = x - ct$.

Remark 1 Results 3, 4 had been obtained by Dai and Huo [8]. When $g = g_1(c)$, Result 5 also had been got by Dai and Huo [8]. In other words, Results 1, 2 are new. Result 5 and corollaries extend previous results.

We will give proof for Theorem 1 and the corollaries in the next section. Now we take six sets of data to display the graphs of the implicit functions and the explicit functions above. These graphs will be compared with the numerical simulations appearing in Section 5.

Example 1 Taking $\gamma = -2$, $c = -4$, we have $q = 2$, $g_1(c) = -14$, $g_2(c) = -1.5$. Letting $g = -3$, we get $\varphi_-^* = -5.22063$, $\overline{\varphi}_+ = -4.73205$. Take three numbers $-4.8, -5.2, -5.22062$ as the value of φ_0 respectively. Clearly, c, g and φ_0 satisfy the condition (ii) in Result 1, that is, $c < 0$, $g_1(c) < g < g_2(c)$ and $\varphi_-^* < \varphi_0 < \overline{\varphi}_+$ (see Fig.1 (C)). Therefore corresponding to φ_0, Eq.(1.1) possesses periodic wave solution $u = \varphi(\xi)$ which is of implicit form (3.1). Substituting $\gamma = -2$, $c = -4$, $q = 2$, $g = -3$ and $\varphi_0 = -4.8$, -5.2, -5.22062 into (3.1) respectively and using Maple we obtain the graph of the implicit function $u = \varphi(\xi)$ as Fig.2 (a), (b), (c).



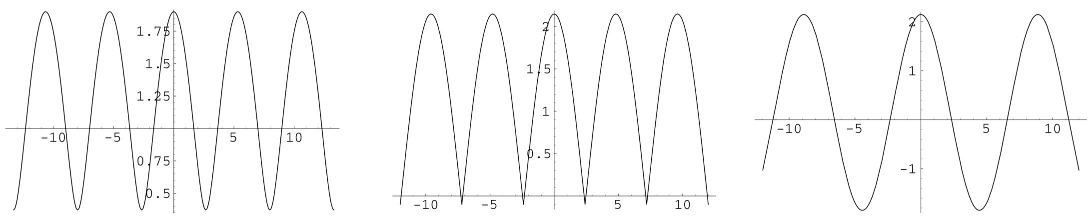
(a) (b) (c)

Fig.2. Graph of implicit function $\varphi(\xi)$ in (3.1) when $\gamma = -2$, $c = -4$, $g = -3$. (a) $\varphi_0 = -4.8$, (b) $\varphi_0 = -5.2$, (c) $\varphi_0 = -5.22062$.

Remark 2 From Fig.2 we see that when $c < 0$, $g_1(c) < g < g_2(c)$, φ_0 satisfies $\varphi_-^* < \varphi_0 < \overline{\varphi}_+$ and φ_0 tends to φ_-^*, the periodic wave becomes a solitary wave.

Example 2 Take $\gamma = -2$, $c = 0.2$, it follows that $q = -0.1$, $g_1(c) = -0.035$. Letting $g = -2$, we have $\overline{\varphi}_+ = -1.84812412$, $\varphi_+^0 = 1.22329$, and $\overline{\varphi}_- = 2.14812412$. Let φ_0 equal 1.9 and 2.14812 respectively. Clearly, g and φ_0 satisfy the condition (ii) in Result 4, that is, $g < g_1(c)$ and $\varphi_+^0 < \varphi_0 < \overline{\varphi}_-$ (see Fig.1 (I)). Thus corresponding to φ_0, Eq.(1.1) possesses periodic wave solution $u = \varphi(\xi)$ which is of implicit form (3.19). Substituting $\gamma = -2$, $c = 0.2$, $q = -0.1$, $g = -2$ and $\varphi_0 = 1.9$, 2.14812, into (3.19) respectively and using Maple we obtain the graphs of the implicit function $u = \varphi(\xi)$ as Fig.3 (a), (b).

Meantime, since $g = -2$ satisfies the condition in Result 5 (namely, $g < g_2(c)$), Eq.(1.1) has a periodic wave solution of explicit form (3.25). Substituting $\gamma = -2$, $\overline{\varphi}_- = 2.14812412$, and $\overline{\varphi}_+ = -1.84812412$ into (3.25) and using Maple we get its graph as Fig.3 (c).



(a) (b) (c)

Fig.3. For $\gamma = -2$, $c = 0.2$, $g = -2$, the graphs of implicit function $\varphi(\xi)$ in (3.19) and explicit function $\varphi(\xi)$ in (3.25). (a) the graph of $\varphi(\xi)$ in (3.19) when $\varphi_0 = 1.9$, (b) the graph of $\varphi(\xi)$ in (3.19) when $\varphi_0 = 2.14812$, (c) the graph of $\varphi(\xi)$ in (3.25) when $\varphi_0 = \overline{\varphi}_- = 2.14812412$ and $\overline{\varphi}_+ = -1.84812412$.

Remark 3 From Fig.2 we see that when $g < g_1(c)$, φ_0 satisfies $\varphi_+^0 < \varphi_0 < \overline{\varphi}_-$ and φ_0 tends to $\overline{\varphi}_-$, the periodic wave loses the smoothness and becomes a periodic shock wave. When $\varphi_0 = \overline{\varphi}_-$, the periodic shock wave becomes a smooth periodic wave suddenly.

4. The Proof of Theorem 1 and Corollary 1, 2

Proof From Lemma 1 it is seen that when φ_0 satisfies one of the conditions of Theorem 1, the orbit passing $(\varphi_0, 0)$ is a closed orbit. Assume that its parameter expression is $u = \varphi(\xi)$ and $y = y(\xi)$. From the derivation in Section 2, $u = \varphi(\xi)$ is a periodic wave solution of Eq.(1.1). Corresponding to those five results of Theorem 1 and Corollary 1, 2, we give the proof as follows respectively.

(1°) Proof of Result 1 Under one of the two conditions in Result 1, let $\Gamma_{\varphi_0}^{**}$ denote the closed orbit passing $(\varphi_0, 0)$ (see Fig.4). Thus on $\varphi - y$ plane it has expression

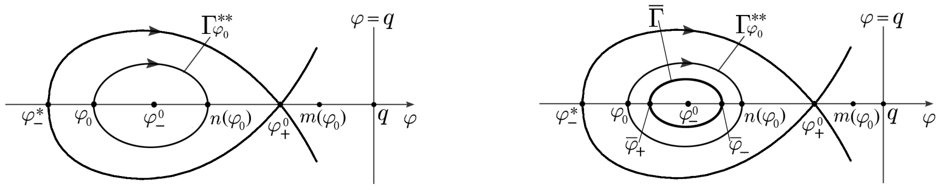
$$y = \pm\frac{1}{\sqrt{-\gamma}}\sqrt{\frac{(m-\varphi)(n-\varphi)(\varphi-\varphi_0)}{q-\varphi}}, \quad \text{for} \quad \varphi_0 \le \varphi \le n < m < q. \tag{4.1}$$

Let its period be $2T_1$ and $\varphi(0) = \varphi_0$, it follows that $\varphi(T_1) = \varphi(-T_1) = n$ (see Fig.4).

Substituting (4.1) into $\frac{d\varphi}{d\xi} = y$ and integrating it along $\Gamma_{\varphi_0}^{**}$, we have

$$\int_{\varphi}^{n}\sqrt{\frac{q-s}{(m-s)(n-s)(s-\varphi_0)}}ds = \frac{1}{\sqrt{-\gamma}}(T_1 - |\xi|), \quad \text{where} \quad \varphi_0 \le \varphi < n < m < q. \tag{4.2}$$

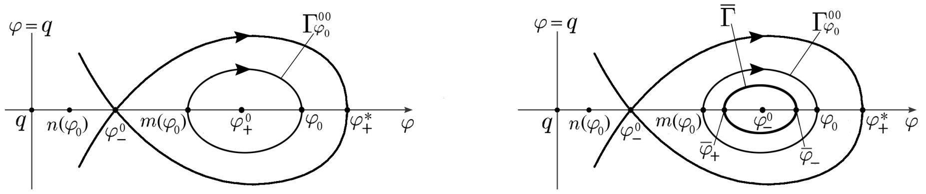
Completing the integral in (4.2) and noticing that $\varphi(0) = \varphi_0$, we get Result 1 as (3.1)-(3.6).



(a) (b)

Fig.4. The closed orbit $\Gamma_{\varphi_0}^{**}$ corresponding to φ_0 in Result 1. (a) $c<0$, $g_2(c)\le g<g_3(c)$ and $\varphi_-^*<\varphi_0<\varphi_-^0$, (b) $c<0$, $g_1(c)<g<g_2(c)$ and $\varphi_-^*<\varphi_0<\overline{\varphi}_+$.

(2°) Proof of Result 2 Under one of the two conditions in Result 2, let $\Gamma_{\varphi_0}^{00}$ denote the closed orbit passing $(\varphi_0,0)$ (see Fig.5). Thus on $\varphi-y$ plane it has expression



(a) (b)

Fig.5. The closed orbit $\Gamma_{\varphi_0}^{00}$ corresponding to φ_0 in Result 2. (a) $c>0$, $g_2(c)\le g<g_3(c)$ and $\varphi_+^0<\varphi_0<\varphi_+^*$, (b) $c>0$, $g_1(c)<g<g_2(c)$ and $\overline{\varphi}_-<\varphi_0<\varphi_+^*$.

$$y=\pm\frac{1}{\sqrt{-\gamma}}\sqrt{\frac{(\varphi_0-\varphi)(\varphi-m)(\varphi-n)}{\varphi-q}},\quad \text{for}\quad q<n<m\le\varphi\le\varphi_0. \tag{4.3}$$

Let its period be $2T_2$ and $\varphi(0)=\varphi_0$, it follows that $\varphi(T_2)=\varphi(-T_2)=m$ (see Fig.5).

Substituting (4.1) into $\frac{d\varphi}{d\xi}=y$ and integrating it along $\Gamma_{\varphi_0}^{00}$, we have

$$\int_m^{\varphi}\sqrt{\frac{s-q}{(\varphi_0-s)(s-m)(s-n)}}ds=\frac{1}{\sqrt{-\gamma}}(T_2-|\xi|),\quad \text{where}\quad q<n<m<\varphi\le\varphi_0. \tag{4.4}$$

Completing the integral in (4.4) and noticing that $\varphi(0)=\varphi_0$, we obtain Result 2 as (3.7)-(3.12).

(3°) Proof of Result 3 Under one of the two conditions in Result 3, let $\Gamma_{\varphi_0}^{*}$ denote the closed orbit passing $(\varphi_0,0)$ (see Fig.6). Thus on $\varphi-y$ plane it possesses expression

$$y=\pm\frac{1}{\sqrt{-\gamma}}\sqrt{\frac{(m-\varphi)(n-\varphi)(\varphi-\varphi_0)}{q-\varphi}},\quad \text{for}\quad \varphi_0\le\varphi\le n<q<m. \tag{4.5}$$

Let its period be $2T_3$ and $\varphi(0)=\varphi_0$, it follows that $\varphi(T_3)=\varphi(-T_3)=n$ (see Fig.6).

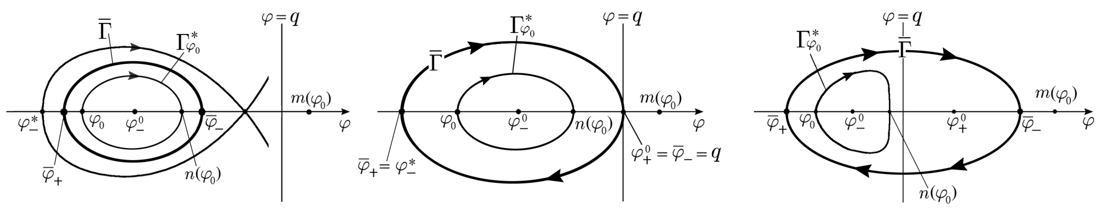


Fig.6. The closed orbit $\Gamma^*_{\varphi_0}$ corresponding to φ_0 in Result 3. (a) $c<0$ and $g_1(c)<g<g_2(c)$, (b) $c<0$ and $g=g_1(c)$, (c) for any c, $g<g_1(c)$.

Substituting (4.5) into $\frac{d\varphi}{d\xi}=y$ and integrating it along $\Gamma^*_{\varphi_0}$, we have

$$\int_\varphi^n\sqrt{\frac{q-s}{(m-s)(n-s)(s-\varphi_0)}}ds=\frac{1}{\sqrt{-\gamma}}(T_3-|\xi|),\quad \text{where}\quad \varphi_0\le\varphi<n<q<m. \tag{4.6}$$

Completing the integral in (4.6) and noticing that $\varphi(0)=\varphi_0$, we get Result 3 as (3.13)-(3.18).

(4°) Proof of Result 4 Under one of the two conditions in Result 4, let $\Gamma^0_{\varphi_0}$ denote the closed orbit passing $(\varphi_0,0)$ (see Fig.7). Thus on $\varphi-y$ plane it has expression

$$y=\pm\frac{1}{\sqrt{-\gamma}}\sqrt{\frac{(\varphi-m)(\varphi_0-\varphi)(\varphi-n)}{\varphi-q}},\quad \text{for}\quad n<q<m\le\varphi\le\varphi_0. \tag{4.7}$$

Let $2T_4$ be its period and $\varphi(0)=\varphi_0$, it follows that $\varphi(T_4)=\varphi(-T_4)=m$ (see Fig.7).

Substituting (4.7) into $\frac{d\varphi}{d\xi}=y$ and integrating it along $\Gamma^0_{\varphi_0}$, we have

$$\int_m^\varphi\sqrt{\frac{s-q}{(\varphi_0-s)(s-m)(s-n)}}ds=\frac{1}{\sqrt{-\gamma}}(T_4-|\xi|),\quad \text{where}\quad n<q<m<\varphi\le\varphi_0. \tag{4.8}$$

Completing the integral and noticing that $\varphi(0)=\varphi_0$, we obtain Result 4 as (3.19)-(3.24).

(5°) Proof of Result 5 Under one of the two conditions in Result 5, let $\overline{\Gamma}$ denote the closed orbit passing $(\overline{\varphi}_+,0)$ and $(\overline{\varphi}_-,0)$ (see Fig.6, 7). Thus on $\varphi-y$ plane it has expression

$$y=\pm\frac{1}{\sqrt{-\gamma}}\sqrt{(\overline{\varphi}-\varphi)(\varphi-\overline{\varphi}_+)},\quad \text{for}\quad \overline{\varphi}_+\le\varphi\le\overline{\varphi}_-. \tag{4.9}$$

Substituting (4.9) into $\frac{d\varphi}{d\xi}=y$ and integrating it along $\Gamma_{\overline{\varphi}_\pm}$, we have

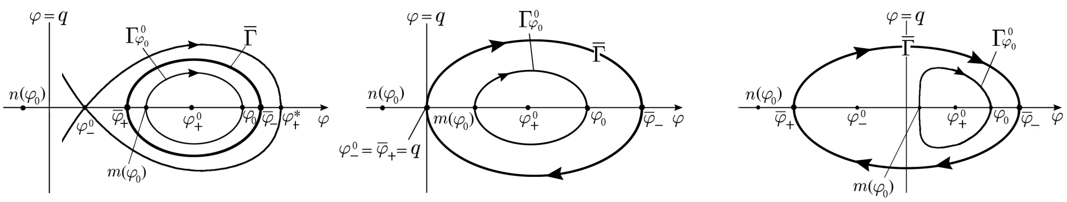
$$\int_\varphi^{\overline{\varphi}_-}\frac{ds}{\sqrt{(\overline{\varphi}_--s)(s-\overline{\varphi}_+)}}=\frac{|\xi|}{\sqrt{-\gamma}},\quad \text{where}\quad \overline{\varphi}_+\le\varphi\le\overline{\varphi}_-. \tag{4.10}$$

or

$$\int_{\overline{\varphi}_+}^{\varphi}\frac{ds}{\sqrt{(\overline{\varphi}_--s)(s-\overline{\varphi}_+)}}=\frac{|\xi|}{\sqrt{-\gamma}},\quad \text{where}\quad \overline{\varphi}_+\le\varphi\le\overline{\varphi}_-. \tag{4.11}$$

Completing two integrals above we obtain Result 5 as (3.25) and (3.26).

(6°) Proof of Corollary 1, 2 Note that g is an arbitrary constant and satisfies $g < g_2(c)$. Thus from (3.25) and (3.26) we get $u_1(\xi)$ as (3.27). Since $u_1(\frac{\pi}{2} - \xi)$ is also a solution of Eq.(1.1), we obtain $u_2(\xi)$ as (3.28). From (3.27) and (3.28) we guess that in (3.29) $u(\xi)$ is a solution of Eq.(1.1). Through test, our guess is right. Hereto we finish the proof.



(a) (b) (c)

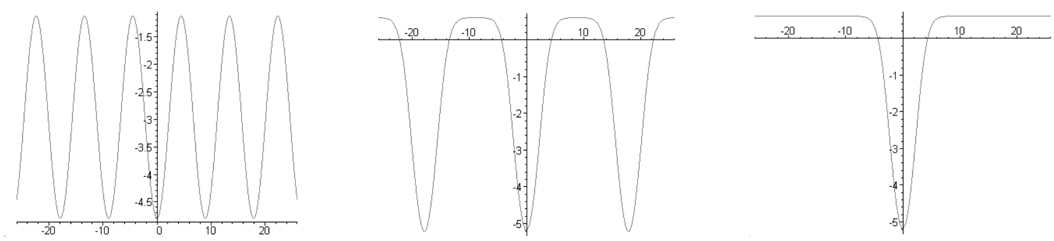
Fig.7. The closed orbit $\Gamma^0_{\varphi_0}$ corresponding to φ_0 in Result 4. (a) $c > 0$ and $g_1(c) < g < g_2(c)$, (b) $c > 0$ and $g = g_1(c)$, (c) for any c, $g < g_1(c)$.

5. Numerical Simulations and Conclusion

In this section, we give some numerical simulations and a short conclusion. The numerical simulations will be compared with the graphs drawn as Fig.2, 3.

From the derivation of travelling wave equation and system, one sees that the periodic integral curves of travelling wave equation (2.2) are the planar graphs of periodic wave solutions of Eq.(1.1). Thus through comparing the graphs of functions given in Theorem 1 with the simulations of the integral curves, we can test the correctness of the results in Theorem 1. Now we take six sets of data used in Examples 1, 2 to simulate the integral curves of Eq.(2.2) by using Maple.

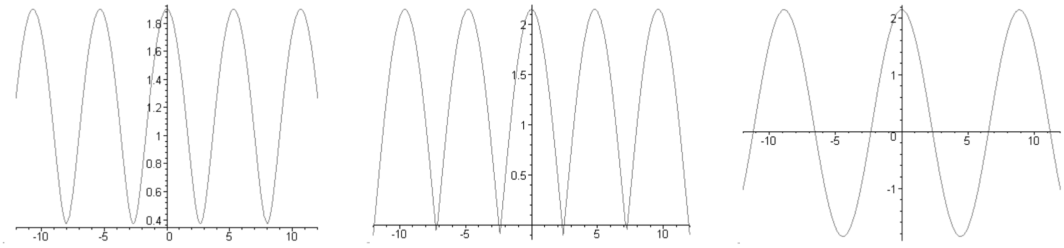
Example 3(Corresponding to Example 1) For those data given in Example 1, namely $\gamma = -2$, $c = -4$, $g = -3$, take respectively $\varphi(0) = -4.8, -5.2, -5.22062$ and $\varphi'(0) = 0$ as initial values of Eq.(2.2), the simulations of the integral curves are in Fig.8 (a), (b) and (c).



(a) (b) (c)

Fig.8. The simulations of integral curves of Eq.(2.2) when $\gamma = -2$, $c = -4$, $g = -3$. (a) initial values $\varphi(0) = -4.8$ and $\varphi'(0) = 0$, (b) initial values $\varphi(0) = -5.2$ and $\varphi'(0) = 0$, (c) initial values $\varphi(0) = -5.22062$ and $\varphi'(0) = 0$.

Example 4(Corresponding to Example 2) For those data given in Example 2, that is, $\gamma = -2$, $c = 0.2$, $g = -2$, take respectively $\varphi_0 = 1.9$, 2.14812, 2.14812412 and $\varphi'(0) = 0$ as initial values of Eq.(2.2), the simulations of the integral curves are as Fig.9 (a), (b) and (c).



(a) (b) (c)

Fig.9. The simulations of integral curves of Eq.(2.2) when $\gamma = -2$, $c = 0.2$, $g = -2$. (a) initial values $\varphi(0) = 1.9$ and $\varphi'(0) = 0$, (b) initial values $\varphi(0) = 2.14812$ and $\varphi'(0) = 0$, (c) initial values $\varphi(0) = 2.14812412$ and $\varphi'(0) = 0$.

Comparing the graphs of functions with the simulations of integral curves, one sees the following two facts: (i) Fig.2 and Fig.8 are identical. (ii) Fig.3 and Fig.9 are the same. These imply the correctness of our results.

Note that Fig.3(c) and Fig.9(c) correspond to the orbit passing $(\overline{\varphi}_+, 0)$ or $(\overline{\varphi}_-, 0)$ (see Fig.1 (I)). This implies the orbit $\overline{\Gamma}$ (see Fig.1 (I)) can pass through the singular line $\varphi = q$. This seems to be a new phenomenon which has not been found in any other system.

Just as we mentioned in Remark 1. We have obtained four new expressions of periodic wave solution as (3.1), (3.7), (3.27) and (3.28). Meantime, we have also derived previous expressions as (3.13) and (3.19) again which have been obtained by Dai and Huo [8]. Therefore our results have extended the periodic wave solutions of Eq.(1.1).

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