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EXISTENCE OF PERIODIC SOLUTIONS FOR 3-D COMPLEX GINZBERG-LANDAU EQUATION

Li Donglong
(Guangxi University of Technology, Liuzou 545006, China)
(E-mail: lidl@21cn.com)
Guo Boling
(Institute of Applied Physics and Computational Mathematics,
P. O. Box 8009, Beijing 100088, China)
(E-mail: gbl@mail.iapem.ac.cn)
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Abstract In this paper, the authors consider complex Ginzburg-Landau equation (CGL) in three spatial dimensions

$$u_t = \rho u + (1 + i\gamma)\Delta u - (1 + i\mu)\left|u\right|^{2\sigma} u \ + f,$$

where u is an unknown complex-value function defined in 3+1 dimensional space-time R^{3+1}, Δ is a Laplacian in R^3, $\rho > 0$, γ, μ are real parameters, $\Omega \in R^3$ is a bounded domain. By using the method of Galërkin and Faedo-Schauder fix point theorem we prove the existence of approximate solution u_N of the problem. By establishing the uniform boundedness of the norm $\|u_N\|$ and the standard compactness arguments, the convergence of the approximate solutions is considered. Moreover, the existence of the periodic solution is obtained .

Key Words complex Ginzburg-Landau equation; Galërkin method; approximate solution; time periodic solution.
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1. Introduction

The generalized complex Ginzburg-Landau (CGL) equation describes the evolution of a complex-valued $u = u\,(x, t)$ by

$$u_t = \rho u + (1 + i\gamma)\Delta u - (1 + i\mu)\left|u\right|^{2\sigma} u \ .$$

It has a long history in physics as a generic amplitude equation near the onset of instabilities that lead to chaotic dynamics in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity. It is a particularity interesting model because it is a dissipative version of the nonlinear Schrödinger equation−A Hamiltonian equation which can possess solutions that form localized singularities in finite time.

Ghidaglia and Héorn [1], Doering et al [2], Promislow [3], etc. studied the finite dimensional Global attractor and related dynamic issues for the one or two spatial dimensional GLE with cubic nonlinearity ($\sigma = 1$) :

$$u_t - (1 + i\gamma)\Delta u + (1 + i\mu)\left|u\right|^2 u - \rho u = 0.$$

where $i = \sqrt{-1}$, $a > 0$. and γ, μ are given real numbers. Bartuccelli, Constantin, Doering, Gibbon and Gisselfalt [4] deal with the "soft" and "hard" turbulent behavior for this equation. In [5], Bu considered the global existence of the Cauchy problem of the following 2D GLE:

$$u_t - (\nu + i\alpha)\Delta u + (\mu + i\beta)\left|u\right|^{2q} u - \gamma u = 0$$

with $q = 1$ and $q = 2$, $\alpha\beta > 0$, or $|\beta| \le \dfrac{\sqrt{5}}{2}$. Doering, Gibbon and Levermore [6] investigated weak and strong solutions for this equation. Mielke [7] discussed the solution of this equation in weighted L^p space and derived some new bounds and investigated some properties of attractors. We consider the equation with non-homogeneous term in three spatial dimensions as follows

$$u_t = \rho u + (1 + i\gamma)\Delta u - (1 + i\mu)\left|u\right|^{2\sigma} u + f(x,t), \tag{1.1}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega \tag{1.2}$$

with periodic boundary condition

$$\Omega = (0, L) \times (0, L) \times (0, L), \quad u \text{ is } \Omega - \text{periodic}, \tag{1.3}$$

where u is an unknown complex-value function defined in 3+1dimensional space-time R^{3+1}, Δ is a Laplacian in $R^3, \rho > 0, \gamma, \mu$ are real parameters, the function $f(x,t)$ is $\omega-$periodic in time t.
Here, by using the Galerkin method and Leray-Schauder fixed point theorem, we will show the existence of approximate solution $u_N(t)$ of the problem $(1.1) - (1.3)$. We establish the uniform boundedness of the norm $\|u_N(t)\|$, by standard compactness arguments get convergence of the approximate solution, and obtain the existence of the time periodic solution for the problem $(1.1) - (1.3)$.
Our assumptions on σ, γ, μ are (**A**):

(i) By choosing suitable γ,

$$\sigma \le \min\left\{ \frac{\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2} - 1} - 1, \frac{1}{4}\frac{\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2} - 1} \right\};$$

(ii) There is a positive number $\delta > 0$, such that

$$0 < \sigma < \frac{1}{\sqrt{1+\left(\frac{\mu-\gamma\delta^2}{1+\delta^2}\right)^2}-1}.$$

The rest of this paper is arranged as follows. First, we introduce the work space and abstract problem in Section 2. We give the definition of approximate solution for our problem and construct approximate solution by the method of Faedo-Galërkin in Section 3. Next, in Sections 4 and 5 respectively we derive a uniform a priori estimate of approximate solution in L^2 and in H^1, H^2 space. Finally we prove the consequence of sequence of the approximate solutions and obtain the existence of the periodic solution for the problem (1.1) − (1.3) in three-dimension space.

2. Work Space and Abstract Problem

Let

$$L^2_{per}(\Omega) = \{u \in L^2(\Omega), u \text{ is a periodic function}\},$$

the norm in $L^2_{per}(\Omega)$ is the same as one in $L^2(\Omega)$ with the inner product

$$(u.v) = \int_\Omega u(x)\, v^*(x) dx.$$

And let

$$H^k_{per}(\Omega) = \{u \in H^k(\Omega), u \text{ is a periodic function}\},$$

the norm in $H^k_{per}(\Omega)$ is the same as one in $H^k(\Omega)$. Let X be a Banach space, we denote that $C^k(\omega, X) = \{f : (-\infty, +\infty) \to X,\ f^{(j)}$ is continuous, $j = 0, 1, \cdots k$, f is a periodic function$\}$
When $k = 0$, we replace $C^0(\omega, X)$ with $C(\omega, X)$. Let $A = -[(1+i\gamma)\Delta + d]$, where d is a real number, the domain of operator A is $D(M) = H^2_{per}(\Omega)$. Then by the result of [8], we know that $-A$ generates an analytic semigroups on $L^2_{per}(\Omega)$. The set of all linear independent eigenvectors of operator A is an orthogonal basis of $L^2_{per}(\Omega)$, and we can choose a real number $d < 0$ such that $0 \in \rho(A)$, where $\rho(A)$ denotes the resolve set of operator A.

Let

$$N(u) = (\rho - d)\, u - (1 + i\mu)\, |u|^{2\sigma} u,$$

N is a nonlinear operator from $H^2_{per}(\Omega)$ to $L^2_{per}(\Omega)$.
The problem (1.1)−(1.3) can be written into an abstract problem in the space $C\left(\omega, L^2_{per}(\Omega)\right)$ as follows

$$u_t + Au = N(u) + f, \tag{2.1}$$

$$u(\cdot, t) = u(\cdot, t+\omega), \tag{2.2}$$

where $f \in C\left(\omega, H^1_{per}(\Omega)\right)$.

3. Approximate Solutions

Let $\{\phi_j\}_{j=1}^{\infty}$ be an normal orthogonal basis of the space $H_{per}^2(\Omega)$, and for each j, ω_j is a eigenvector of the operator A. Let μ_j be the eigenvalue of the operator A corresponding to ϕ_j, $j = 1, 2, \cdots$. For any number N, we denote that $H_N = span\{\phi_1, \phi_2, \cdots, \phi_N\}$.

Definition 3.1(approximate solution) *Let $f \in C\left(\omega, H_{per}^1(\Omega)\right)$, for any number N and a group function $(d_{1N}(t), d_{2N}(t), \cdots d_{NN}(t))$ where undermined functions $d_{kN}(t)$ $(k = 1, 2, \cdots, N)$ of variable $t \in R^+$ belong to $C^1(\omega, \mathcal{C})$ and $\mathcal{C}$ is the set of all complex numbers, the function $u_N(t) = \sum_{k=1}^{N} d_{kN}(t)\phi_k \in C(\omega, H_N(\Omega))$ is called an approximate solution of* (2.1) (2.2) *if it satisfies the system as follows*

$$(u_{Nt} + Au_N, \phi_j) = (N(u_N) + f, \phi_j), \quad j = 1, 2, \cdots N, \tag{3.1}$$

$$u_N(\cdot, t) = u_N(\cdot, t + \omega). \tag{3.2}$$

In order to prove that (3.1) (3,2) has an approximate solution, we definite an image $\mathcal{F}$ in $C^1(\omega, H_N(\Omega))$. For any $v_N(t) = \sum_{k=1}^{N} a_{kN}(t)\phi_k \in C^1(\omega, H_N(\Omega))$, the differential equation system

$$(u_{Nt} + Au_N, \phi_j) = (N(v_N) + f, \phi_j), \quad j = 1, 2, \cdots N, \tag{3.3}$$

is an linear ordinary differential equation system about$(d_{1N}(t), d_{2N}(t), \cdots, d_{NN}(t))$. By the theory of ordinary differential equation, there exist a unique $(d_{1N}(t), d_{2N}(t), \cdots, d_{NN}(t))$ and the image $\mathcal{F}$:$v_N \to u_N$ is a continuous and compact image from the space $C^1(\omega, H_N(\Omega))$ to itself. For the existence of approximate solution of the problem (1.1) (1.3), it is sufficient to prove that the image $\mathcal{F}$ has a fixed point in $C^1(\omega, H_N(\Omega))$. We prove the existence of fixed point of the image $\mathcal{F}$ by Leray-Schauder fixed point theorem. For this purpose, we introduce operator $\mathcal{F}_\lambda$ with parameter λ. Similar to the image $\mathcal{F}$, the image $\mathcal{F}_\lambda : v_N - u_N$ is defined by the equation system as follows

$$(u_{Nt} + Au_N, \phi_j) = (\lambda N(v_N) + f, \phi_j), \quad j = 1, 2, \cdots N,$$

where $0 \le \lambda \le 1$. For $\lambda \in [0, 1]$, $\mathcal{F}_\lambda$ also is a continuous and compact image from $C^1(\omega, H_N)$ to itself. It is obvious that $\mathcal{F}_1 = \mathcal{F}$. As $\lambda = 0$, the equations system (2.1), (2.2) has a unique solution $\widetilde{u}_N(t) = \left(\widetilde{d}_{1N}(t), \widetilde{d}_{2N}(t), \cdots \widetilde{d}_{NN}(t)\right) \in C^1(\omega, H_N)$. Hence $\mathcal{F}_0$ has a unique fixed point $\widetilde{u}_N(t)$ in $C^1(\omega, H_N)$. For the existence of fixed point of the image $\mathcal{F}$, using to Leray-Schauder fixed point theorem, we need only to prove that if the equation $\mathcal{F}_\lambda u_N = u_N$ has a solution $u_N(t)$, it must satisfy the inequality as follows

$$\sup_{0 \le t \le \omega} \|u_N(t)\| \le K_1,$$

where K_1 is a positive constant which is independent of λ and depends only on ρ, γ, μ, σ, ω, L and f.

4. A Priori Estimate

Lemma 4.1 *Assume that* $f \in C\left(\omega, L^2_{per}(\Omega)\right)$, *if* $\mathcal{F}_\lambda u_N = u_N$, $0 \le \lambda \le 1$, *then there exists a positive constant* K_1, *such that*

$$\sup_{0\le t\le\omega} \|u_N(t)\| \le K_1. \tag{4.1}$$

Proof We have $\mathcal{F}_\lambda u_N = u_N$, i.e.

$$(u_{Nt} + Au_N, \phi_j) = (\lambda N(u_N) + f, \phi_j), \ j = 1, 2, \cdots N. \tag{4.2}$$

Multiply each equation system (3.1) by d^*_{kN} and sum up over j from $j = 1$ to N to obtain

$$(u_{Nt} + Au_N, u_N) = (\lambda N(u_N) + f, u_N), \tag{4.3}$$

i.e.

$$(u_{Nt} - [(1+i\gamma)\Delta + d]\, u_N, u_N) = \left(\lambda\left((\rho - d)\, u_N - (1+i\mu)\,|u_N|^{2\sigma} u_N\right) + f, u_N\right).$$

Taking the real part in two sides of the resulting identity yields

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \|\nabla u_N\|^2 - d\,\|u_N\|^2 = \lambda(\rho - d)\,\|u_N\|^2 - \lambda\int |u_N|^{2\sigma+2} + Re\,(f, u_N).$$

By Young's inequality and

$$Re\,(f, u_N) \le \varepsilon\,\|u_N\|^2 + k_1(\varepsilon)\,\|f\|^2, \tag{4.4}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \|\nabla u_N\|^2 - d\,\|u_N\|^2 + \lambda\int |u_N|^{2\sigma+2} \\ = (\lambda(\rho - d))\,\|u_N\|^2 + k_1(\varepsilon)\,\|f\|^2.\end{aligned} \tag{4.5}$$

As

$$\|u_N\|^2 \le \frac{1}{(\rho - d)}\int |u_N|^{2\sigma+2} + c, \tag{4.6}$$

(4.5)+$\lambda\times$(4.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \|\nabla u_N\|^2 - d\,\|u_N\|^2 \le k_1(\varepsilon)\,\|f\|^2. \tag{4.7}$$

Considering the periodicity of u_N and integrating (4.7) over $[0, \omega]$, we have

$$-d\int_0^\omega \|u_N\|^2\, dt \le k_1(\varepsilon)\,\|f\|^2\,\omega.$$

By the middle value theorem, there exists $t^* \in [0, \omega]$, such that

$$\|u_N(t_*)\|^2 \leq -\frac{k_1(\varepsilon)}{d}\|f\|^2.$$

Integrating (4.7) over(0, ω) from t_* to $t_* + \omega$ $t_* \in [0, \omega]$ yields

$$\begin{aligned}\|u_N\|^2 &\leq 2k_1(\varepsilon)\|f\|^2\omega + \|u_N(t_*)\|^2 \\ &\leq 2k_1(\varepsilon)\|f\|^2\omega - \frac{k_1(\varepsilon)}{d}\|f\|^2 \\ &= K_1.\end{aligned}$$

Then (4.1) holds, i.e. there exists a positive costant $K_1 = K_1(\rho, \sigma, \mu, \gamma, \omega, f, L)$, which is independent of λ and N, such that the inequality (4.1) holds. By Leray-Schauder fixed point theorem, $\mathcal{F}$ has a fixed point. Here we have the theorem below.

Theorem 4.2 *Let $f \in C\left(\omega, L^2_{per}(\Omega)\right)$, for any number N,* (2.1) (2.2) *has an approximate solution $u_N(t) \in C^1(\omega, H_N(\Omega))$.*

5. Priori Estimate of Derivative

We have proved that (2.1) (2.2) has a sequence of approximate solutions $\{u_N\}_{N=1}^{\infty}$. For proving the existence of the strong periodic estimate, it is needed to prove that the sequence is convergent and the limit is a solution of (2.1) (2.2). For this purpose, we need some priori estimate about derivative of $u_N(t)$.

Lemma 5.1 *If $\mathcal{F}u_N = u_N$, we have*

$$\begin{aligned}&\frac{1}{2(1+\sigma)}\frac{d}{dt}\int |u_N|^{2\sigma+2} \\ &\quad\leq -\frac{1}{2}\int |u_N|^{4\sigma+2} - \frac{1}{4}\int |u_N|^{2\sigma-2}\left((1+2\sigma)\left|\nabla|u_N|^2\right|^2\right. \\ &\quad\quad \left.- 2\gamma\sigma\nabla|u_N|^2 \cdot i\left(u_N\nabla u_N^* - u_N^*\nabla u_N\right) + |u_N\nabla u_N^* - u_N^*\nabla u_N|^2\right) \\ &\quad\quad + Re\int_\Omega f\cdot|u_N|^{2\sigma}u_N^* + c. \end{aligned} \tag{5.1}$$

Proof By (3.1), we have

$$\left(u_{Nt} + Au_N, |u_N|^{2\sigma}u_N\right) = \left(N(u_N) + f, |u_N|^{2\sigma}u_N\right).$$

Taking real part of the resulting identity, we find that

$$\begin{aligned}\frac{1}{2(1+\sigma)}\frac{d}{dt}\int_\Omega |u_N|^{2\sigma+2} = \rho\int_\Omega |u_N|^{2\sigma+2} + Re(1+i\gamma)\int_\Omega \Delta u_N|u_N|^{2\sigma}u_N^* \\ - \int_\Omega |u_N|^{4\sigma+2} + \int_\Omega f\cdot|u_N|^{2\sigma}u_N^*.\end{aligned} \tag{5.2}$$

Since

$$|u|^2|\nabla u|^2 = \frac{1}{4}\left|\nabla|u|^2\right|^2 + \frac{1}{4}|u\nabla u^* - u^*\nabla u|^2, \tag{5.3}$$

the second term in the right side of (5.2) is changed into

$$\begin{aligned}
&Re\,(1+i\gamma)\int_\Omega \Delta u_N\,|u_N|^{2\sigma}\,u_N^*\\
&\quad=-\,Re\,(1+i\gamma)\int_\Omega |\nabla u_N|^2\,|u_N|^{2\sigma}-Re\,(1+i\gamma)\int_\Omega \sigma\,|u_N|^{2\sigma-2}\,u_N^*\nabla u_N\nabla\,|u_N|^2\\
&\quad=-\int_\Omega |\nabla u_N|^2\,|u_N|^{2\sigma}-\frac{\sigma}{2}\int_\Omega |u_N|^{2\sigma-2}\left|\nabla\,|u_N|^2\right|^2\\
&\qquad+\frac{1}{2}\gamma\sigma\int_\Omega |u_N|^{2\sigma-2}\,\nabla\,|u_N|^2\cdot i\,(u_N\nabla u_N^*-u_N^*\nabla u_N)\\
&\quad=-\frac{1}{4}\int_\Omega |u_N|^{2\sigma-2}\,((1+2\sigma)\left|\nabla\,|u_N|^2\right|^2-2\gamma\sigma\nabla\,|u_N|^2\cdot i\,(u_N\nabla\overline{u_N}-u_N^*\nabla u_N)\\
&\qquad+|u_N\nabla u_N^*-u_N^*\nabla u_N|^2).
\end{aligned}\tag{5.4}$$

The first term in the right side of (5.2) is dudeced by Young's inequality and Lemma 3.1

$$\begin{aligned}
\rho\int_\Omega |u_N|^{2\sigma+2}=&\rho\int_\Omega |u_N|^{2\sigma+1}\cdot|u_N|\\
\le&\frac{1}{2}\int_\Omega |u_N|^{4\sigma+2}+\frac{1}{2}\rho^2\int_\Omega |u_N|^2\le\frac{1}{2}\int_\Omega |u_N|^{4\sigma+2}+c.
\end{aligned}\tag{5.5}$$

We infer to Lemma 5.1 from (5.2), (5.4) and (5.5). And thus the proof is completed.

Lemma 5.2 *Assume that* $2\sigma+2\le\dfrac{2\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2}-1}$ *holds, if* $\mathcal{F}u_N=u_N$, *then we have*

$$\sup_{0\le t\le\omega}\|u_N\|_{2\sigma+2}^{2\sigma+2}\le K_2,\tag{5.6}$$

where $\|\cdot\|_p$ *is the norm of* $L^p(\Omega)$, K_2 *is a positive constant which is independent of* λ *and dependent on* $\rho,\ \gamma,\ \mu,\ \sigma,\ \omega,\ \ L$ *and* f.

Proof Direct calculation yields

$$\begin{aligned}
\frac{1}{2\sigma+2}\frac{d}{dt}\|u_N\|_{2\sigma+2}^{2\sigma+2}=&Re\int |u|^{2\sigma}\,u_N^*u_{Nt}dx\\
=&Re\int |u_N|^{2\sigma}\,u_N^*\left(\rho u_N+(1+i\gamma)\Delta u_N-(1+i\mu)\,|u_N|^{2\sigma}\,u_N\right)\\
=&\rho\,\|u_N\|_{2\sigma+2}^{2\sigma+2}-\|u_N\|_{4\sigma+2}^{4\sigma+2}+Re\int(1+i\gamma)\,|u_N|^{2\sigma}\,u_N^*\Delta u_N\\
&+Re\int_\Omega f\cdot|u_N|^{2\sigma}\,u_N^*.
\end{aligned}\tag{5.7}$$

First, we have

$$Re\int_\Omega f\cdot|u_N|^{2\sigma}\,u_N^*\le\frac{1}{2}\,\|u_N\|_{4\sigma+2}^{4\sigma+2}+\frac{1}{2}\,\|f\|^2\tag{5.8}$$

and

$$\begin{aligned}
& Re\int(1+i\gamma)\left|u_N\right|^{2\sigma}u_N^*\Delta u_N \\
&\quad= -Re\,(1+i\gamma)\int\left|\nabla u_N\right|\left|u_N\right|^{2\sigma} - Re\,(1+i\gamma)\int\sigma\left|u_N\right|^{2\sigma-2}u_N^*\nabla u_N\nabla\left|u_N\right|^2 \\
&\quad= -Re\,(1+i\gamma)\int\left|\nabla u_N\right|^2\left|u_N\right|^{2\sigma} - \frac{\sigma}{2}Re\,(1+i\gamma)\int\left|u_N\right|^{2\sigma-2}\left|\nabla\left|u_N\right|^2\right|^2 \\
&\qquad+ \frac{1}{2}\gamma\sigma Re\,(1+i\gamma)\int\left|u_N\right|^{2\sigma-2}\nabla\left|u_N\right|^2\cdot i\left(u_N\nabla u_N^* - u_N^*\nabla u_N\right) \\
&\quad= -\frac{1}{4}\int\left|u_N\right|^{2\sigma-2}\left((2\sigma+1)\left|\nabla\left|u_N\right|^2\right|^2 - 2\sigma\gamma\nabla\left|u_N\right|^2\cdot i\left(u_N\nabla u_N^* - u_N^*\nabla u_N\right)\right. \\
&\qquad\left.+\left|u_N\nabla u_N^* - u_N^*\nabla u_N\right|^2\right). \qquad (5.9)
\end{aligned}$$

The integrand in the last term in (5.9) is a guadratic form in these quantities that will be nonnegative provided the matrix

$$\begin{pmatrix} 2\sigma+1 & \gamma\sigma \\ \gamma\sigma & 1 \end{pmatrix}$$

is nonnegative define, i. e. whenever $2\sigma+2\le\dfrac{2\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2}-1}$. In this case, neglecting the third term of (5.7), we have

$$\frac{1}{2\sigma+2}\frac{d}{dt}\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2} + \frac{1}{2}\left\|u_N\right\|_{4\sigma+2}^{4\sigma+2} \le \rho\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2} + \frac{1}{2}\left\|f\right\|_2^2. \qquad (5.10)$$

By Young's inequality with ε

$$\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2} \le \frac{1}{3\rho}\left\|u_N\right\|_{4\sigma+2}^{4\sigma+2} + c \qquad \left(\int\left|u_N\right|^{2\sigma+2} \le \frac{1}{3\rho}\int\left|u_N\right|_{4\sigma+2}^{4\sigma+2} + c\right)$$

and then

$$\frac{1}{\sigma+1}\frac{d}{dt}\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2} \le -\rho\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2} + \left\|f\right\|_2^2 + c. \qquad (5.11)$$

Considering the periodicity of u_N and integrating (5.11) over $[0,\omega]$, we obtain

$$\rho\int_0^\omega\left\|u_N\right\|_{2\sigma+2}^{2\sigma+2}dt \le \left(\left\|f\right\|^2 + c\right)\omega. \qquad (5.12)$$

By the middle value theorem, we have $t^*\in[0,\omega]$, such that

$$\left\|u_N\left(t_*\right)\right\|_{2\sigma+2}^{2\sigma+2} \le \frac{\left\|f\right\|^2 + c}{\rho}. \qquad (5.13)$$

Integrating (5.11) over$(0,\ \omega)$ from t_* to $t_*+\omega\ \ t_*\in[0,\omega]$ yields

$$\begin{aligned}\|u_N\|_{2\sigma+2}^{2\sigma+2} &\le (\|f\|^2+c)\omega+\|u_N(t_*)\|_{2\sigma+2}^{2\sigma+2}\\ &\le (\|f\|^2+c)\omega+\frac{\|f\|^2+c}{\rho}\\ &= K_2,\end{aligned}$$

i.e. there exists a positive costant $K_2=K_2(\rho,\sigma,\mu,\gamma,\omega,f,L)$, which is independent of λ and N, such that the inequality (5.6) holds.

Lemma 5.3 *Under the assumptions of* (A), *let* $f\in C\left(\omega,L^2_{per}(\Omega)\right)$, *if* $\mathcal{F}u_N=u_N$, *then there exists a positive constant* K_3 *such that*

$$\sup_{0\le t\le\omega}\left(\|\nabla u\|^2+\|u\|_{L^{2\sigma+2}}\right)\le K_3, \tag{5.14}$$

where K_3 *is a positive constant which is independent of* λ *and depends only on* γ, μ, σ, ω,L,ω *and* f, t_3 *depends on the data* R *when* $\|u_0\|\le R$.

Proof By (3.1), we have

$$(u_{Nt}+Au_N,\Delta u_N)=(N(u_N)+f,\Delta u_N). \tag{5.15}$$

Taking the real part of the resulting identity, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2+\|\Delta u_N\|^2 =&\rho\|\nabla u_N\|^2+Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}u_N\Delta u_N^*\\ &+\int_\Omega f\cdot\Delta u_N.\end{aligned} \tag{5.16}$$

Due to (5.4), we have

$$\begin{aligned}&Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}u_N\Delta u_N^*\\ &\quad= -Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}|\nabla u_N|^2-Re(1+i\mu)\sigma\int_\Omega |u_N|^{2\sigma-2}u_N\nabla u_N^*\nabla|u_N|^2\\ &\quad=-\int_\Omega |u_N|^{2\sigma}|\nabla u_N|^2-\frac{\sigma}{2}\int_\Omega |u_N|^{p-2}\left|\nabla|u_N|^2\right|^2\\ &\qquad+\frac{1}{2}\mu\sigma\int_\Omega |u_N|^{p-2}\nabla|u_N|^2\cdot i(u_N\nabla u_N^*-u_N^*\nabla u_N)\\ &\quad=-\frac{1}{4}\int_\Omega |u_N|^{2\sigma-1}((1+2\sigma)\left|\nabla|u_N|^2\right|^2-2\mu\sigma\nabla|u_N|^2\cdot i(u_N\nabla u_N^*-u_N^*\nabla u_N)\\ &\qquad+|u_N\nabla u_N^*-u_N^*\nabla u_N|^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla u_N\right\|^2+\left\|\Delta u_N\right\|^2\\
&\qquad=\rho\left\|\nabla u_N\right\|^2-\frac{1}{4}\int_\Omega\left|u_N\right|^{2\sigma-1}\left((1+2\sigma)\left|\nabla\left|u_N\right|^2\right|^2\right.\\
&\qquad\quad\left.-2\mu\sigma\nabla\left|u_N\right|^2\cdot i\left(u_N\nabla u_N^*-u_N^*\nabla u_N\right)+\left|u_N\nabla u_N^*-u_N^*\nabla u_N\right|^2\right)\\
&\qquad\quad+\int_\Omega f\cdot\Delta u_N. \qquad (5.17)
\end{aligned}$$

Taking a linear combination of (5.17) and (5.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}V_\delta\left(u_N(t)\right)+\left\|\Delta u_N\right\|^2+\frac{\delta^2}{2}\int\left|u_N\right|^{4\sigma+2}\\
&\qquad\le\rho\left\|\nabla u_N\right\|^2-\frac{1}{4}\int|u|^{2\sigma-2}\left((1+2\sigma)\left(1+\delta^2\right)\left|\nabla\left|u_N\right|^2\right|^2\right.\\
&\qquad\quad+2\sigma\left(\gamma\delta^2-\mu\right)\nabla\left|u_N\right|^2\cdot i\left(u_N\nabla u_N^*-u_N^*\nabla u_N\right)\\
&\qquad\quad\left.+\left(1+\delta^2\right)\left|u_N\nabla u_N^*-u_N^*\nabla u_N\right|^2\right)+\int_\Omega f\cdot\Delta u_N+c. \qquad (5.18)
\end{aligned}$$

Remark : $\|\nabla u_N\|^2+\dfrac{\delta^2}{1+\sigma}\displaystyle\int_\Omega|u_N|^{2\sigma+2}dx$ a "perturbation" of the NLS energy function, where $\delta>0$ is to be chosen, will help to considerably improve the region for which the global existence of solution can be shown. The technical reason why this particular combination yields better results than separate estimates on H^1 and $L^{2\sigma+2}$ norms is that the problematic term on H^1 and $L^{2\sigma+2}$ estimates have the same function dependence on the solution u and therefore taking the linear combination and optimization in δ provides for partial cancellation of this term.

The integrand in the last term in (5.18) is a guadratic form in these quantities that will be nonnegative provided the matrix

$$\begin{pmatrix}(1+2\sigma)\left(1+\delta^2\right) & \sigma\left(\gamma\delta^2-\mu\right)\\ \sigma\left(\gamma\delta^2-\mu\right) & \left(1+\delta^2\right)\end{pmatrix}$$

is nonnegative define, i. e.

$$\sigma<\frac{1}{\sqrt{1+(\dfrac{\mu-\gamma\delta^2}{1+\delta^2})^2}-1}.$$

In this case, neglecting the last term (3.18), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla u_N\right\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega\left|u_N\right|^{2\sigma+2}dx\right)+\left\|\Delta u_N\right\|^2+\frac{\delta^2}{2}\int_\Omega\left|u_N\right|^{4\sigma+2}dx\\
&\qquad\le\rho\left\|\nabla u_N\right\|^2+\int_\Omega f\cdot\Delta u_N dx+c\\
&\qquad\le\rho\left\|\nabla u_N\right\|^2+\frac{1}{2}\left\|\Delta u_N\right\|^2+\frac{1}{2}\|f\|^2+c.
\end{aligned}$$

We get

$$\frac{d}{dt}\left(\|\nabla u_N\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N|^{2\sigma+2}\,dx\right)+\|\Delta u_N\|^2+\delta^2\int_\Omega |u_N|^{4\sigma+2}\,dx$$
$$\leq 2\rho\,\|\nabla u_N\|^2+\|f\|^2+c. \tag{5.19}$$

By Lemma 5.1 and Young's inequality we have

$$\begin{aligned}(2\rho+1)\,\|\nabla u_N\|^2&=(2\rho+1)\,(-\Delta u_N,u_N)\leq(2\rho+1)\,\|\Delta u_N\|\,\|u_N\|\\&\leq(2\rho+1)\,c\,\|\Delta u_N\|\leq\|\Delta u_N\|^2+\frac{1}{4}\,(2\rho+1)^2\,c^2\end{aligned}$$

and

$$\begin{aligned}\frac{\delta^2}{2\,(1+\sigma)}\int_\Omega |u_N|^{2\sigma+2}&=\frac{\delta^2}{2\,(1+\sigma)}\int_\Omega |u_N|^{2\sigma+1}\,|u_N|\\&\leq\delta^2\int_\Omega |u_N|^{4\sigma+2}+c\int_\Omega |u_N|^2\leq\delta^2\int_\Omega |u_N|^{4\sigma+2}+cK_1^2.\end{aligned}$$

Therefore, we give from (5.19)

$$\frac{d}{dt}\left(\|\nabla u_N\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N|^{2\sigma+2}\,dx\right)+\left(\|\nabla u_N\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N|^{2\sigma+2}\,dx\right)$$
$$\leq\|f\|^2+cK_1^2-\frac{1}{4}\,(2\rho+1)^2\,c^2+c=k. \tag{5.20}$$

Considering the periodicity of u_N and integrating (5.20) from 0 to ω as follows, we have

$$\int_0^\omega\left(\|\nabla u_N\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N|^{2\sigma+2}\,dx\right)dt\leq k\omega.$$

By the middle value theorem, there exist $t^*\in[0,\omega]$ such that

$$\|\nabla u_N\,(t_*)\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N\,(t_*)|^{2\sigma+2}\,dx\leq k.$$

Integrating (5.20) from t_* to t, $\ t\in[t_*,t_*+\omega]$, we have as follows

$$\begin{aligned}&\|\nabla u_N\,(t)\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N\,(t)|^{2\sigma+2}\,dx\\&\qquad\leq k\omega+\|\nabla u_N\,(t^*)\|^2+\frac{\delta^2}{1+\sigma}\int_\Omega |u_N\,(t^*)|^{2\sigma+2}\,dx\\&\qquad\leq k\omega+k=K_3.\end{aligned}$$

Therefore, there exists a constant K_3 which only depends on $\rho,\ \gamma,\ \mu,\ \sigma,\ \omega,\ \ L$ and f, such that

$$\sup_{0\leq t\leq\omega}\left(\|\nabla u\|^2+\|u\|_{L^{2\sigma+2}}\right)\leq K_3.$$

This completes the proof of the lemma.

Lemma 5.4 *Under the assumption* $(\mathbf{A})$, *let* $f \in C\left(\omega, H^1_{per}(\Omega)\right)$, *if* $\mathcal{F}u_N = u_N$, *then there exists a positive constant* K_4 *such that*

$$\sup_{0\le t\le\omega} \|\Delta u_N\|^2 \le K_4, \tag{5.21}$$

where K_4 *is only dependent on* $\rho,\ \gamma,\ \mu,\ \sigma,\ \omega,\ \ L$ *and* f.

Proof By (3.1), we have

$$\left(u_{Nt} + Au_N, \Delta^2 u_N\right) = \left(N(u_N) + f, \Delta^2 u_N\right). \tag{5.22}$$

Taking real part of the resulting identity yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u_N\|^2 =& \rho\|\Delta u_N\|^2 - \|\nabla\Delta u_N\|^2\\ &- Re(1+i\mu)\int_\Omega |u_N|^{2\sigma} u_N \Delta^2 u_N^* + Re\int_\Omega f\cdot\Delta^2 u_N dx.\end{aligned} \tag{5.23}$$

We estimate each term of (5.23). First, we have

$$\begin{aligned}&Re(1+i\mu)\int_\Omega |u_N|^{2\sigma} u_N\Delta^2 u_N\\ &= - Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}\nabla u_N\nabla\Delta u_N^* - Re(1+i\mu)\int_\Omega \nabla|u_N|^{2\sigma} u_N\nabla\Delta u_N^*\\ &= Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}\nabla u_N\nabla\Delta u_N^*\\ &\quad + Re(1+i\mu)\int_\Omega |u_N|^{2\sigma-2}\left(u_N\nabla u_N^* + u_N^*\nabla u_N\right)u_N\nabla\Delta u_N^*.\end{aligned} \tag{5.24}$$

We estimate the first term of the right side of (5.24)

$$\begin{aligned}&Re(1+i\mu)\int_\Omega |u_N|^{2\sigma}\nabla u_N\nabla\Delta u_N^*\\ &\le \sqrt{1+\mu^2}\int_\Omega |u_N|^{2\sigma}|\nabla u_N||\nabla\Delta u_N|\\ &\le \sqrt{1+\mu^2}\|u_N\|_{8\sigma}^{2\sigma}\|\nabla u_N\|_{L^4}\|\nabla\Delta u_N\|\\ &\le \frac{1}{12}\|\nabla\Delta u_N\|^2 + 3(1+\mu)\|\nabla u_N\|_{L^4}^2\|u_N\|_{8\sigma}^{4\sigma}\\ &\le \frac{1}{12}\|\nabla\Delta u_N\|^2 + 3(1+\mu)\|\nabla\Delta u_N\|^{\frac{3}{2}}\|\nabla u_N\|^{\frac{1}{2}}\|u_N\|_{8\sigma}^{4\sigma}\\ &\le \frac{1}{12}\|\nabla\Delta u_N\|^2 + \frac{1}{12}\|\nabla\Delta u_N\|^2 + c\|\nabla u_N\|^2\left(\|u_N\|_{8\sigma}^{8\sigma}\right)^2\\ &\le \frac{1}{6}\|\nabla\Delta u_N\|^2 + c.\end{aligned} \tag{5.25}$$

The above result comes from Lemma 5.1 and 5.3. Noting that

$$2\sigma + 2 \le \frac{2\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2}-1},$$

hence we choose

$$\sigma \leq \min \left\{ \frac{\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2}-1} - 1, \frac{1}{4} \frac{\sqrt{1+\gamma^2}}{\sqrt{1+\gamma^2}-1} \right\}. \tag{5.26}$$

Similarly, we can estimate the last term of the right side of (5.14)

$$Re\,(1+i\mu) \int |u_N|^{2\sigma-2} \left(u_N \nabla u_N^* + u_N^* \nabla u_N\right) u_N \nabla \Delta u_N^*) \leq \frac{1}{6} \left\|\nabla \Delta u_N\right\|^2 + c. \tag{5.27}$$

$$Re \left(f, \Delta^2 u_N\right) \leq \varepsilon \left\|\Delta u_N\right\|^2 + k\,(\varepsilon) \left\|\Delta f\right\|^2. \tag{5.28}$$

By (5.23)-(5.25), (5.27) and (5.28), we get

$$\frac{1}{2} \frac{d}{dt} \left\|\Delta u_N\right\|^2 + \left\|\nabla \Delta u_N\right\|^2 \leq (\rho + \varepsilon) \left\|\Delta u_N\right\|^2 + k\,(\varepsilon) \left\|\Delta f\right\|^2 + c.$$

By Lemma 5.3 we give

$$\left\|\Delta u_N\right\|^2 = \left(-\nabla \Delta u_N, \nabla u_N\right) \leq \left\|\nabla \Delta u_N\right\| \left\|\nabla u_N\right\| \leq c \left\|\nabla \Delta u_N\right\|.$$

By Young's inequality we have

$$2\,(\rho + \varepsilon) \left\|\Delta u_N\right\|^2 \leq 2c\,(\rho + \varepsilon) \left\|\nabla \Delta u_N\right\| \leq \left\|\nabla \Delta u_N\right\|^2 + (\rho + \varepsilon)^2 c^2.$$

Hence, we have

$$\frac{d}{dt} \left\|\Delta u_N\right\|^2 + 2(\rho + \varepsilon) \left\|\Delta u_N\right\|^2 \leq 2k\,(\varepsilon) \left\|\Delta f\right\|^2 + 2c - 2\,(\rho + \varepsilon)^2 c^2 = k'. \tag{5.29}$$

We consider the periodicity of u_N, integrating (5.29) from 0 to ω as follows

$$\int_0^\omega \left\|\Delta u_N\right\|^2 dt \leq k'\omega.$$

By the middle value theorem, there exist $t^* \in [0, \omega]$ such that

$$\left\|\Delta u_N\,(t^*)\right\|^2 \leq k'.$$

Integrating (5.29) from t^{**} to $t + \omega$ $t \in [0, \omega]$, we have as follows

$$\left\|\Delta u_N\,(t)\right\|^2 \leq K_4 \omega + \left\|\Delta u_N\,(t^{**})\right\|^2 = k'\omega + k' = K_4.$$

Therefore, there exists a constant $K_4\,(\rho,\ \gamma, \mu, \sigma, L, \omega, f)$ such that

$$\sup_{0 \leq t \leq \omega} \left\|\Delta u\right\|^2 \leq K_4$$

which concludes Lemma 5.4.

Lemma 5.5 *Let* $f \in C\left(\omega, H_{per}^1(\Omega)\right)$, *if* $\mathcal{F}u_N = u_N$, *then there exists a positive constant* K_5 *such that*

$$\sup_{0\le t\le\omega} \|u_{Nt}\|^2 \le K_5\ . \tag{5.30}$$

Proof By (3.1), we have

$$(u_{Nt} + Au_N, \phi_j) = (N(u_N) + f, \phi_j),\ j = 1, 2, \cdots N. \tag{5.31}$$

Multiply each equation systems (5.31) by d_{kN}^* and sum up over j from $j = 1$ to N to obtain

$$(u_{Nt} + Au_N, u_{Nt}) = (N(u_N) + f, u_{Nt}). \tag{5.32}$$

Taking real part of the resulting identity, we obtain that

$$\|u_{Nt}\|^2 \le \sqrt{1+\gamma^2}\,\|\Delta u_N\|\,\|u_{Nt}\| + \|N(u_N) + f + du_N\|\,\|u_{Nt}\|. \tag{5.33}$$

By Agmon inequality

$$\|u\|_\infty \le c\,\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\,\|u\|_{H^2(\Omega)}^{\frac{1}{2}} \le \overline{K}, \qquad \forall u \in H^2(\Omega)$$

and the definition of $N(u_N)$, we have

$$\begin{aligned}\|N(u_N) + f + du_N\| &= \left\|\rho u_N - (1+i\mu)\,|u_N|^{2\sigma} u + f\right\| \\ &\le \rho\overline{K} + \sqrt{1+\mu^2}\,\overline{K}^{2\sigma+1} + \|f\| \le K'.\end{aligned}$$

Therefore, by the inequality (5.33), Lemma 5.4 and Young's inequality, there exists a positive constant K_5 such that

$$\sup_{0\le t\le\omega} \|u_{Nt}\|^2 \le K_5,$$

where K_5 is only dependent on $\rho,\ \gamma, \mu, \sigma, L, f$. This completes the proof of Lemma 5.5.

6. Main Theorem

Using Lemmas 5.1−5.5 and existence of approximate solution, we finally get the main theorem.

Theorem 6.1 (Existence of periodic solution) *Under the assumption* (**A**), *let* $f \in C\left(\omega, H_{per}^1(\Omega)\right)$ *then the complex Ginzburg-Landau equation for 3-D has a unique solution*

$$u(t) \in C\left([0,\omega); H_{per}^2(\Omega)\right) \cap C^1\left([0,\omega); L_{per}^2(\Omega)\right).$$

Proof By Lemmas 5.1−5.5, we can choose a subsequence $\{u_{N_k}(t)\}$ from the sequence $\{u_N(t)\}$ such that

$$u_{N_k}(t) \to u(t),\ \ (k\to\infty) \text{ weakly in } C\left([0,\omega); H_{per}^2(\Omega)\right)$$

$$u_{N_k}(t) \to u(t), \quad (k \to \infty) \text{ strong in } C\left([0,\omega); H^1_{per}(\Omega)\right)$$

$$u_{N_k}(t) \to u(t), \quad (k \to \infty) \text{ weakly in } C^1\left([0,\omega); L^2_{per}(\Omega)\right)$$

Next, we estimate nonlinear term$(1+i\mu)\left|u_{N_k}\right|^{2\sigma} u_{N_k}$

$$\begin{aligned}
&\left\|(1+i\mu)\ \left|u_{N_k}\right|^{2\sigma} u_{N_k} - (1+i\mu)\left|u\right|^{2\sigma} u\right\| \\
&\quad \le \sqrt{1+\mu^2}\left\|\left(\left|u_{N_k}\right|^{2\sigma} - |u|^{2\sigma}\right) u_{N_k} + |u|^{2\sigma}\left(u_{N_k} - u\right)\right\| \\
&\quad \le \sqrt{1+\mu^2}\left\|f'(\xi)\left(u_{N_k} - u\right) u_{N_k} + |u|^{2\sigma}\left(u_{N_k} - u\right)\right\| \qquad \left(f(s) = s^{2\sigma}\right) \\
&\quad \le \sqrt{1+\mu^2}\left\|f'(\xi)\right\|_\infty \left\|u_{N_k}\right\|_\infty \left\|\left(u_{N_k} - u\right)\right\| + \sqrt{1+\mu^2}\left\||u|^{2\sigma}\right\|_\infty \left\|\left(u_{N_k} - u\right)\right\| \\
&\quad \le c\left\|\left(u_{N_k} - u\right)\right\| \to 0, \qquad \left(N_k \to \infty\right).
\end{aligned}$$

Then we obtain

$$N\left(u_{N_k}\right) \to N(u), \qquad k \to \infty, \qquad \text{uniformly in time.}$$

Taking $k \to \infty$ from (3.1), we have

$$\left(u_t + Au, \phi_j\right) = \left(N(u) + f, \phi_j\right), \ j = 1, 2, \cdots.$$

By using the density of $\{\phi_j; j = 1, 2, \cdots\}$ in $L^2(\Omega)$, it follows that: $\forall \omega \in L^2(\Omega)$,

$$\left(u_t + Au, \omega\right) = \left(N(u) + f, \omega\right).$$

Finally, we prove uniqueness of solution.

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the problem (3.1), (3.2). Subtracting the equations for $u_1(x,t)$ and $u_2(x,t)$ we get

$$\begin{aligned}
\frac{1}{2}\partial_t \left\|\left(u_1 - u_2\right)\right\|_2^2 = &- \rho\left\|\left(u_1 - u_2\right)\right\|_2^2 - \left\|\Delta\left(u_1 - u_2\right)\right\|_{2,}^2 \\
&- Re(1+i\mu)\int\left(\left|u_1\right|^{2\sigma} u_1 - \left|u_2\right|^{2\sigma} u_2\right)\left(u_1 - u_2\right).
\end{aligned} \tag{6.1}$$

Taking the inner product of (6.1) with $u_1 - u_2$, and taking real part of the resulting identity yield

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|u_1 - u_2\right\|_2^2 = &- \rho\left\|u_1 - u_2\right\|_{2,}^2 - \left\|\nabla(u_1 - u_2)\right\|_{2,}^2 \\
&- Re(1+i\mu)\int\left(\left|u_1\right|^{2\sigma} u_1 - \left|u_2\right|^{2\sigma} u_2\right)\left(u_1 - u_2\right)^*
\end{aligned} \tag{6.2}$$

Noting

$$\left|\left|u_1\right|^{2\sigma} u_1 - \left|u_2\right|^{2\sigma} u_2\right| \le (2\sigma + 1)\sup\left(\left|u_1\right|^{2\sigma}, \left|u_2\right|^{2\sigma}\right)\left|u_1 - u_2\right|,$$

by Höder's inequality and Agmom's inequality, the above inequality can be estimated as follows

$$\begin{aligned}\int \left(|u_1|^{2\sigma} u_1 - |u_2|^{2\sigma} u_2\right)(u_1-u_2)&\\ &\le(2\sigma+1)\int \sup\left(|u_1|^{2\sigma}, |u_2|^{2\sigma}\right)|u_1-u_2|^2\\ &\le c(2\sigma+1)\left(\|u_1\|^{2\sigma}_{L^\infty}, \|u_2\|^{2\sigma}_{L^\infty}\right)\|u_1-u_2\|^2_{2,}\\ &\le c(2\sigma+1)\left(\|u_1\|_{H^2}, \|u_2\|_{H^2}\right)\|u_1-u_2\|^2_2 . \end{aligned} \tag{6.3}$$

Thus we have from (6.2), (6.3)

$$\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|_2^2+\|\nabla u_1-u_2\|_2^2=(c(2\sigma+1)-\rho)\left(\|u_1\|_{H^2}, \|u_2\|_{H^2}\right)\|u_1-u_2\|_2^2 .$$

It follows from Gronwall's inequality that

$$\|u_1(x,t)-u_2(x,t)\|_2^2\le e^{\widetilde{k}T}\|u_1(0)-u_2(0)\|_2^2 \quad t\in[0,T],$$

where $\widetilde{k}=(c(2\sigma+1)-\rho)\left(\|u_1\|_{H^2}, \|u_2\|_{H^2}\right)$. Namely,

$$\|u_1(x,t)-u_2(x,t)\|\longrightarrow 0, \quad \text{as} \quad \|u_1(0)-u_2(0)\|\longrightarrow 0.$$

This completes the proof of the theorem.

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