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# ASYMPTOTICS OF INITIAL BOUNDARY VALUE PROBLEMS OF BIPOLAR HYDRODYNAMIC MODEL FOR SEMICONDUCTORS

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**Abstract** In this paper, we study the asymptotic behavior of the solutions to the bipolar hydrodynamic model with Dirichlet boundary conditions. It is shown that the initial boundary problem of the model admits a global smooth solution which decays to the steady state exponentially fast.
**Key Words** Bipolar hydrodynamic model; semiconductors; asymptotics; smooth solution.
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## 1. Introduction

We are concerned with the large time behavior of smooth solutions to the one-dimension Euler-Poisson(or hydrodynamic) model for semiconductors in the case of two carriers, i.e. electron and hole

$$n_t + (nu)_x = 0, \tag{1.1}$$

$$h_t + (hv)_x = 0, \tag{1.2}$$

$$(nu)_t + (nu^2 + p(n))_x = n\phi_x - \frac{nu}{\tau_n}, \tag{1.3}$$

$$(hv)_t + (hv^2 + q(h))_x = -h\phi_x - \frac{hv}{\tau_h}, \tag{1.4}$$

$$\phi_{xx} = n - h - d(x), \tag{1.5}$$

$(t, x) \in (0, \infty) \times (0, 1)$ where $(n, h)$ and $(u, v)$ are densities and velocities for electrons and holes, respectively. $j = nu$ and $k = hv$ stand for the electron and hole current densities. $\phi$ denotes the electrostatic potential and the doping profile $d(x)$ describes fixed charged background ions. $\tau_n$ and $\tau_h$ are the momentum relaxation times for electrons and holes, respectively. We assume $\tau_n = \tau_h = 1$ for convenience. To simplify the proof, we take $d(x)$ as a nonnegative constant $d$ and choose the typical form for pressure, namely:

$$p(n) = \frac{n^{\gamma_n}}{\gamma_n},\ \gamma_n > 1, \quad q(h) = \frac{h^{\gamma_h}}{\gamma_h},\ \gamma_h > 1. \tag{1.6}$$

The case with two different constants can be dealt with similarly.

Recently, the hydrodynamic model of semiconductors has attracted a lot of attention, because of its function to describe hot electron effects which are not accounted for in the classical drift-diffusion model. Rigorous results have been obtained in various papers. Most of them are concerned with the unipolar case, which only discusses the effect of the electron. However, there are few results on the bipolar case which is of more importance and physical meaning. Fang and Ito [1] show the existence of weak solutions to the system (1.1)-(1.5) in the transonic case using the viscosity argument. Natalini [2], Hsiao and Zhang [3] considered the relaxation limit problem from the bipolar hydrodynamic model to the drift-diffusion equations. Zhu and Hattori [4] showed the existence of the strong solutions to the Cauchy problem of (1.1)-(1.5) and discussed the asymptotic stability of the steady state solution, without the decay rate, when the doping profile is close to zero.

In the present paper, we will consider the initial boundary value problems for (1.1)-(1.5) with the following initial data

$$(n, h, j, k) = (n_0, h_0, j_0, k_0)(x), \quad x \in (0, 1) \tag{1.7}$$

and the density and potential Dirichlet boundary conditions

$$n(0, t) = n(1, t) = \bar{n}, \quad t \geq 0, \tag{1.8}$$

$$h(0, t) = h(1, t) = \bar{h}, \quad t \geq 0, \tag{1.9}$$

$$\phi(0, t) = \phi(1, t) = \bar{\phi}, \quad t \geq 0. \tag{1.10}$$

Here, $\bar{n}, \bar{h}$ and $\bar{\phi} > 0$, and $\bar{n} - \bar{h} = d$. This kind of boundary conditions is commonly used in physics of semiconductor devices.

The goal of this paper is to investigate the global existence of smooth solutions to (1.1)-(1.5) with (1.7)-(1.10) and the large time behavior of them when the initial data (1.7) are assumed to be perturbations of a steady state $(\hat{n}, \hat{h}, \hat{j}, \hat{k}, \hat{E})$ of (1.1)-(1.5) with $\hat{j} = \hat{k} = 0$ which satisfies:

$$\begin{aligned} &p(\hat{n})_x = \hat{n}\hat{\phi}_x, \\ &q(\hat{h})_x = -\hat{h}\hat{\phi}_x, \\ &\hat{\phi}_{xx} = \hat{n} - \hat{h} - d \end{aligned} \tag{1.11}$$

with the boundary condition

$$\begin{aligned} &\hat{n}(0) = \bar{n}, \\ &\hat{h}(0) = \bar{h}, \\ &\phi(0) = \phi(1) = \bar{\phi} \end{aligned}$$

from which we get $\hat{n}(1) = \bar{n}$ and $\hat{h}(1) = \bar{h}$. It is easy to prove that the above stationary problem admits uniquely the constant solution $(\bar{n}, \bar{h}, 0, 0, \bar{\phi})$.

Next, put

$$\psi_0 = n_0(x) - \bar{n}, \qquad \eta_0(x) = h_0(x) - \bar{h}.$$

The main result of this paper is the following theorem.

**Theorem 1** *Assume* $(\psi_0, \eta_0, j_0, k_0) \in H^2$. *Then there exists* $\varepsilon_0 > 0$*, such that if* $\|(\psi_0, \eta_0, j_0, k_0)\|_{H^2} \le \varepsilon_0$*, the IBVP (1.1)-(1.5)with (1.7)-(1.10) admits a global smooth solution* $(n, h, j, k, \phi)(x, t)$ *satisfying:*

$$\|(n - \bar{n}, h - \bar{h}, j, k, \phi - \bar{\phi})\|^2_{H^2} \le O(1)\|(\psi_0, \eta_0, j_0, k_0)\|^2_{H^2} \exp\{-\beta t\}, \quad t \ge 0$$

*with a positive constant* $\beta$.

**Remark** The result of this paper describes an interesting phenomenon in the bipolar hydrodynamic model for semiconductors: due to the reciprocal action between electrons and holes, the effect of elliptic mode is somehow weakened, on the other hand, the boundary effect becomes strong in the evolutionary carriers. Therefore, we can prove that the solutions to (1.1)-(1.5) and (1.7)-(1.10) tend to the steady state exponentially fast as $t$ goes to $\infty$.

Let us also mention that the result of this paper is the generalization of [5] by Li, Markowich and Mei, where the unipolar case is considered. However, new difficulties occur from the reciprocal action between electrons and holes in our bipolar case and more efforts are made to treat the coupled terms by electrons and holes.

## 2. The Proof of Theorem 1

Set

$$\psi = n - \bar{n}, \quad \eta = h - \bar{h}, \quad e = \phi - \bar{\phi}. \tag{2.1}$$

Then the new variables satisfy the systems

$$\psi_t + j_x = 0, \tag{2.2}$$

$$\eta_t + k_x = 0, \tag{2.3}$$

$$j_t + (p(\psi + \bar{n}) - p(\bar{n}))_x + j + (\frac{j^2}{\psi + \bar{n}})_x - (\psi + \bar{n})e_x = 0, \tag{2.4}$$

$$k_t + (q(\eta + \bar{h}) - q(\bar{h}))_x + k + (\frac{k^2}{\eta + \bar{h}})_x + (\eta + \bar{h})e_x = 0, \tag{2.5}$$

$$e_{xx} = \psi - \eta, \tag{2.6}$$

with the following initial-boundary conditions

$$\psi(0, t) = \psi(1, t) = 0, \quad t \ge 0, \tag{2.7}$$

$$\eta(0, t) = \eta(1, t) = 0, \quad t \ge 0, \tag{2.8}$$

$$e(0,t)=e(1,t)=0, \quad t\geq 0, \tag{2.9}$$

$$\psi(x,0)=\psi_0(x), \ \ \eta(x,0)=\eta_0(x), \tag{2.10}$$

$$j(x,0)=j_0(x), \ \ k(x,0)=k_0(x). \tag{2.11}$$

To prove Theorem 1, we first establish the a priori estimates.

**Lemma 2** *Suppose $(\psi,\eta,j,k)$ satisfies (2.2)-(2.11). There exist some positive constants $\varepsilon_1>0$ and $\alpha>0$ such that, for any $T>0$, if*

$$\sup_{0\leq t\leq T}\|(\psi,\eta,j,k)(t)\|_{H^2}\leq \varepsilon_1, \tag{2.12}$$

*then, for any $t\in[0,T]$,*

$$\|(\psi,\eta,j,k,e)\|_{H^2}^2\leq O(1)(\|(\psi_0,\eta_0,j_0,k_0)\|_{H^2}^2\exp(-\alpha t). \tag{2.13}$$

We assume that $\varepsilon_1$ is chosen so small that

$$0<n_-\leq \psi+\bar{n}\leq n_+, \ \ 0<h_-\leq \eta+\bar{h}\leq h_+$$

with $n_-,n_+,h_-$, and $h_+$ constants.

The proof of Lemma 2 consists of 5 steps. In Steps 1-3, we bound $e,j,k$ by $\psi,\eta$ in certain senses. And in Steps 4-5 we carry out the energy estimates on $\psi,\eta$ in detail.

**Step 1** For $e$, we have

$$\int_0^1 e_x^2\leq O(1)\int_0^1(\psi^2+\eta^2)dx, \ \ e_x^2\leq O(1)\int_0^1(\psi^2+\eta^2)dx, \tag{2.14}$$

$$\int_0^1 e_{xt}^2\leq O(1)\int_0^1(\psi_t^2+\eta_t^2)dx, \ \ e_{xt}^2\leq O(1)\int_0^1(\psi_t^2+\eta_t^2)dx, \ \ e_t^2\leq O(1)\int_0^1(\psi_t^2+\eta_t^2)dx. \tag{2.15}$$

To prove (2.14), we multiply (2.6) with $e$, and integrating over $(0,1)$ to yield, after integration by parts, that

$$\begin{aligned}\int_0^1 e_x^2dx &\leq \int_0^1|e(\psi-\eta)|dx\\ &\leq \int_0^1|e\eta|dx+\int_0^1|e\psi|dx\\ &\leq\left(\int_0^1 e^2dx\right)^{1/2}\left[\left(\int_0^1\eta^2dx\right)^{1/2}+\left(\int_0^1\psi^2dx\right)^{1/2}\right],\end{aligned}$$

with the help of (2.9) and Hölder inequality. Then Poincaré inequality gives the first one of (2.14).

On the other hand, by the integral mean value theorem, there exists a curve $x_1(t)$ satisfying $0<x_1(t)<1$ such that

$$e_x^2(x_1(t),t)=\int_0^1 e_x^2(x,t)dx,$$

thus

$$\begin{aligned} e_x^2(x,t) &= e_x^2(x_1(t),t) + 2\int_x^{x_1(t)} e_x e_{xx} dx \\ &\le \int_0^1 e_x^2 dx + 2\int_0^1 |e_x e_{xx}| dx \\ &\le 2\int_0^1 e_x^2 dx + \int_0^1 e_{xx}^2 dx \\ &\le O(1)\int_0^1 (\psi^2+\eta^2) dx, \end{aligned}$$

by (2.6) and Young inequality. The proof of (2.14) is completed.

Next, we prove (2.15). Differentiating (2.6) with respect to $t$ leads to

$$e_{xxt} = \psi_t - \eta_t. \tag{2.16}$$

Multiplying (2.16) by $e_t$, and integrating it over $[0,1]$, we get that

$$\int_0^1 e_{xt}^2 dx = \int_0^1 e_t(\eta_t - \psi_t) dx,$$

with the observation that $e_t(0,t) = e_t(1,t) = 0$. Similar to the above arguments, we have

$$\int_0^1 e_{xt}^2 \le O(1)\int_0^1 (\psi_t^2 + \eta_t^2) dx. \tag{2.17}$$

Similarly, the integral mean value theorem can also lead to the following two inequalities

$$e_{xt}^2 \le 2\int_0^1 e_{xt}^2 dx + \int_0^1 e_{xxt}^2 dx, \quad e_t^2 \le 2\int_0^1 e_t^2 dx + \int_0^1 e_{xt}^2 dx. \tag{2.18}$$

The other two estimates in (2.15) follow from (2.16), (2.17), (2.18) and Poincaré inequality immediately.

**Step 2** For $j$, we have

$$\int_0^1 j^2 dx \le O(1)\Big(\exp\{-c_0 t\}\int_0^1 j_0^2 dx + \int_0^1 (\psi_t^2 + \psi^2 + \eta^2) dx\Big), \tag{2.19}$$

$$j^2 \le O(1)\Big(\exp\{-c_0 t\}\int_0^1 j_0^2 dx + \int_0^1 (\psi_t^2 + \psi^2 + \eta^2) dx\Big), \tag{2.20}$$

$$\int_0^1 j_t^2 dx \le O(1)\Big(\exp\{-c_0 t\}\int_0^1 j_0^2 dx + \int_0^1 (\psi_t^2 + \psi_x^2 + \psi^2 + \eta^2) dx\Big), \tag{2.21}$$

with $c_0 > 0$ a constant.

Let us prove (2.19) first.

Now, multiplying (2.4) with $j$ and integrating it over (0,1), one has, by (2.7) and integration by parts, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 j^2 dx\Big) + \int_0^1 j^2 dx \\ &\quad = -\frac{j^3}{\bar n}\Big|_0^1 + \int_0^1 (\psi+\bar n) e_x j dx + \int_0^1 \Big[p(\psi+\bar n) - p(\bar n) + \frac{j^2}{\psi+\bar n}\Big] j_x dx \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

The $I_1, I_2$ and $I_3$ can be estimated as follows:

$$\begin{aligned}
I_1 &\le \int_0^1 \left|\left(\frac{j^3}{\bar n}\right)_x\right| dx = \int_0^1 \left|\frac{3j^2 j_x}{\bar n}\right| dx \le O(\varepsilon_1)\int_0^1 j^2 dx,\\
I_2 &\le \tfrac{1}{2}\int_0^1 j^2 + O(1)\int_0^1 (\psi^2+\eta^2),\\
I_3 &\le \int_0^1 \left|\psi_t\left[p(\psi+\bar n)-p(\bar n)+\frac{j^2}{\psi+\bar n}\right]\right|\\
&\le O(\varepsilon_1)\int_0^1 j^2 + O(1)\int_0^1 (\psi^2+\psi_t^2).
\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\Big(\int_0^1 j^2 dx\Big) + b\int_0^1 j^2 \le O(1)\int_0^1 (\psi^2+\psi_t^2+\eta^2)dx,$$

for a positive constant $b$, independent of $t$ and $T$, provided $\varepsilon_1$ is small enough.

Integrating the above inequality over $[0,t]$ gives

$$\int_0^1 j^2 dx \le \exp\{-c_0 t\}\int_0^1 j_0^2 dx + O(1)(1-\exp\{-c_0 t\})\int_0^1 (\psi^2+\eta^2+\psi_t^2)dx.$$

with a constant $c_0>0$, which implies (2.19).

(2.20) and (2.21) can be proved by (2.19) and the following relations

$$\begin{aligned}
j^2 &\le \int_0^1 j^2 dx + 2\int_0^1 |j_x j| dx\\
&\le 2\int_0^1 j^2 dx + \int_0^1 \psi_t^2 dx
\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}
j_t^2 &\le O(1)\left\{\left[p(\psi+\bar n)-p(\bar n)+\frac{j^2}{\psi+\bar n}\right]_x\right\}^2 + O(1)(j^2+(\psi+\bar n)^2 e_x^2)\\
&\le O(1)\left(j^2+\psi_x^2+\psi_t^2+\psi^2+\eta^2\right)
\end{aligned}\tag{2.23}$$

in view of (2.4) and (2.14). Thus, the proof of (2.19)-(2.21) is finished.

By the similar procedure, we get the following estimates for $k$.

**Step 3** For $k$, we have

$$\int_0^1 k^2 dx \le O(1)\Big(\exp\{-c_1 t\}\int_0^1 k_0^2 dx + \int_0^1 (\eta_t^2+\psi^2+\eta^2)dx\Big),\tag{2.24}$$

$$k^2 \le O(1)\Big(\exp\{-c_1 t\}\int_0^1 k_0^2 dx + \int_0^1 (\eta_t^2+\psi^2+\eta^2)dx\Big),\tag{2.25}$$

$$\int_0^1 k_t^2 \le O(1)\Big(\exp\{-c_1 t\}\int_0^1 k_0^2 dx + \int_0^1 (\eta_t^2+\eta_x^2+\psi^2+\eta^2)dx\Big),\tag{2.26}$$

with $c_1 > 0$ a constant.

**Step 4**

$$\int_0^1 (\psi_t^2 + \eta_t^2 + \psi_x^2 + \eta_x^2 + \psi^2 + \eta^2)dx \le O(1)\|(\psi_0, \eta_0, j_0, k_0)\|_{H^2}^2 \exp\{-\beta_1 t\} \tag{2.27}$$

$$\int_0^1 (e^2 + e_x^2 + e_{xx}^2)dx \le O(1)\|(\psi_0, \eta_0, j_0, k_0)\|_{H^2}^2 \exp\{-\beta_1 t\} \tag{2.28}$$

with $\beta_1 > 0$ a constant, provided that $\varepsilon_1$ is small enough.

We prove (2.27) now.

Differentiating (2.4) with respect to $x$ and using (2.2) and (2.6), we obtain

$$\psi_{tt} + \psi_t + \psi_x e_x + (\psi + \bar{n})(\psi - \eta) - \Big[\frac{j^2}{\psi + \bar{n}} + p(\psi + \bar{n}) - p(\bar{n})\Big]_{xx} = 0. \tag{2.29}$$

Multiplying (2.29) with $\psi$ and integrating over $[0,1]$, one has, by (2.7) and integrating by parts, that

$$\begin{aligned}\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}\psi^2 + \psi\psi_t\Big)dx - \int_0^1 \psi_t^2 dx + \int_0^1 (\psi_x \psi e_x + \psi^3 - \psi^2\eta)dx + \bar{n}\int_0^1 (\psi^2 - \psi\eta)dx \\ + \int_0^1 \Big(\frac{j^2}{\psi + \bar{n}} + p(\psi + \bar{n}) - p(\bar{n})\Big)_x \psi_x dx = 0.\end{aligned} \tag{2.30}$$

It is easy to show that $\int_0^1 (\psi^3 - \psi^2\eta)dx$ is bounded by $O(\varepsilon_1)\int_0^1 \psi^2$. By the Hölder inequality and (2.14), we get

$$\int_0^1 \psi_x \psi e_x dx \ge -O(\varepsilon_1)(\int_0^1 e_x^2)^{1/2}(\int_0^1 \psi^2)^{1/2} dx \ge -O(\varepsilon_1)\int_0^1 (\psi^2 + \eta^2)dx. \tag{2.31}$$

By (2.2), we have

$$\begin{aligned}\int_0^1 \Big(\frac{j^2}{\psi + \bar{n}}\Big)_x \psi_x dx &= \int_0^1 \Big[\frac{2jj_x\psi_x}{\psi + \bar{n}} - \frac{j^2\psi_x^2}{(\psi + \bar{n})^2}\Big]dx \\ &\ge -O(\varepsilon_1)\int_0^1 (\psi_t^2 + \psi_x^2)dx.\end{aligned} \tag{2.32}$$

In view of (2.7), by Poincaré inequality, we have

$$\begin{aligned}\int_0^1 (p(\psi + \bar{n}) - p(\bar{n}))_x \psi_x dx &= \int_0^1 p'(\psi + \bar{n})\psi_x^2 dx \\ &\ge O(1)\int_0^1 (\psi^2 + \psi_x^2)dx.\end{aligned} \tag{2.33}$$

Thus, (2.30)-(2.33) imply

$$\begin{aligned}\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}\psi^2 + \psi\psi_t\Big)dx - \int_0^1 \psi_t^2 dx + a_1\int_0^1 (\psi^2 + \psi_x^2)dx + \bar{n}\int_0^1 (\psi^2 - \psi\eta)dx \\ \le O(\varepsilon_1)\int_0^1 (\psi_t^2 + \eta^2)dx.\end{aligned} \tag{2.34}$$

where $a_1$ is a positive constant. Differentiating (2.5) with respect to $x$, by the alike process as above, one has

$$\frac{d}{dt}\int_0^1\Big(\frac{1}{2}\eta^2+\eta\eta_t\Big)dx-\int_0^1\eta_t^2dx+a_2\int_0^1(\eta^2+\eta_x^2)dx+\bar{h}\int_0^1(\eta^2-\psi\eta)dx$$
$$\le O(\varepsilon_1)\int_0^1(\eta_t^2+\psi^2)dx. \tag{2.35}$$

where $a_2$ is a positive constant. Next, multiplying (2.29) by $\psi_t$ and integrating it over $[0,1]$, we have, using $\psi_t(0,t)=\psi_t(1,t)=0$, that

$$\frac{d}{dt}\int_0^1\frac{1}{2}\Big(\psi_t^2+\bar{n}\psi^2\Big)dx+\int_0^1\psi_t^2dx-\bar{n}\int_0^1\eta\psi_tdx+\int_0^1(\psi-\eta)\psi\psi_tdx+\int_0^1e_x\psi_x\psi_tdx$$
$$+\int_0^1\Big(\frac{j^2}{\psi+\bar{n}}\Big)_x\psi_{tx}dx+\int_0^1\Big(p(\psi+\bar{n})-p(\bar{n})\Big)_x\psi_{tx}dx=0. \tag{2.36}$$

It is easy to see that $\int_0^1(\psi-\eta)\psi\psi_tdx$ is bounded by $O(\varepsilon_1)\int_0^1(\psi^2+\psi_t^2)dx$, and $\int_0^1e_x\psi_x\psi_tdx$ by $O(\varepsilon_1)\int_0^1(\psi^2+\psi_t^2)dx$, with the help of (2.14). The last two integrals in (2.36) are estimated as follows.

$$\begin{aligned}\int_0^1[p(\psi+\bar{n})-p(\bar{n})]_x\psi_{tx}dx&=\int_0^1p'(\psi+\bar{n})\psi_x\psi_{tx}dx\\&=\frac{d}{dt}\int_0^1\frac{1}{2}p'(\psi+\bar{n})\psi_x^2-\frac{1}{2}\int_0^1p''(\psi+\bar{n})\psi_t\psi_x^2dx\\&\ge\frac{d}{dt}\int_0^1\frac{1}{2}p'(\psi+\bar{n}))\psi_x^2-O(\varepsilon_1)\int_0^1\psi_x^2dx.\end{aligned} \tag{2.37}$$

By (2.2) and (2.21), we get

$$\begin{aligned}\int_0^1\Big(\frac{j^2}{\psi+\bar{n}}\Big)_x\psi_{tx}dx&=-\int_0^1\frac{2\psi_tj\psi_{tx}}{\psi+\bar{n}}dx-\int_0^1\frac{j^2\psi_x\psi_{tx}}{(\psi+\bar{n})^2}dx\\&=-\frac{d}{dt}\int_0^1\frac{1}{2}\frac{j^2\psi_x^2}{(\psi+\bar{n})^2}dx+\int_0^1\frac{jj_t\psi_x^2}{(\psi+\bar{n})^2}dx-\int_0^1\frac{j^2\psi_x^2}{(\psi+\bar{n})^3}\psi_tdx\\&\quad+\int_0^1\frac{\psi_t^2j_x}{\psi+\bar{n}}dx-\int_0^1\frac{\psi_t^2j\psi_x}{(\psi+\bar{n})^2}ddx\\&\ge-\frac{d}{dt}\int_0^1\frac{1}{2}\frac{j^2\psi_x^2}{(\psi+\bar{n})^2}dx-O(\varepsilon_1)\int_0^1(\psi_x^2+\psi_t^2+\psi^2+\eta^2)dx\\&\quad-O(\varepsilon_1)\exp\{-c_0t\}\int_0^1j_0^2dx.\end{aligned} \tag{2.38}$$

From (2.36)-(2.38), it follows

$$\frac{d}{dt}\int_0^1\Big[\psi_t^2+\bar{n}\psi^2+p'(\psi+\bar{n})\psi_x^2-\frac{j^2\psi_x^2}{(\psi+\bar{n})^2}\Big]dx+2\int_0^1\psi_t^2dx-2\bar{n}\int_0^1\eta\psi_tdx$$
$$\le O(\varepsilon_1)\int_0^1(\psi_x^2+\psi^2+\eta^2+\psi_t^2)dx+O(\varepsilon_1)\exp\{-c_0t\}\int_0^1j_0^2dx. \tag{2.39}$$

The corresponding estimate for $\eta$ is

$$\frac{d}{dt}\int_0^1\Big[\eta_t^2+\bar{h}\eta^2+q'(\eta+\bar{h})\eta_x^2-\frac{k^2\eta_x^2}{(\eta+\bar{h})^2}\Big]dx+2\int_0^1\eta_t^2dx-2\bar{h}\int_0^1\psi\eta_t dx$$
$$\le O(\varepsilon_1)\int_0^1(\eta_x^2+\eta^2+\psi^2+\eta_t^2)dx+O(\varepsilon_1)\exp\{-c_0t\}\int_0^1 k_0^2dx. \tag{2.40}$$

$\bar{h}\times(2.39)+\bar{n}\times(2.40)$ gives

$$\frac{d}{dt}\int_0^1\Big\{\bar{h}\Big[\psi_t^2+\bar{n}\psi^2+p'(\psi+\bar{n})\psi_x^2-\frac{j^2\psi_x^2}{(\psi+\bar{n})^2}\Big]+\bar{n}\Big[\eta_t^2+\bar{h}\eta^2+p'(\eta+\bar{h})\eta_x^2$$
$$-\frac{k^2\eta_x^2}{(\eta+\bar{h})^2}\Big]-2\bar{n}\bar{h}\psi\eta\Big\}dx+2\bar{h}\int_0^1\psi_t^2dx+2\bar{n}\int_0^1\eta_t^2dx$$
$$\le O(\varepsilon_1)\int_0^1(\psi_x^2+\psi^2+\eta_x^2+\eta^2+\psi_t^2+\eta_t^2)dx+O(\varepsilon_1)\exp\{-c_0t\}\int_0^1(k_0^2+j_0^2)dx. \tag{2.41}$$

Thus, $\bar{h}\times(2.34)+\bar{n}\times(2.35)+(2.41)$ yields

$$\frac{d}{dt}\int_0^1\Big\{\bar{h}\Big[\frac{1}{2}\psi^2+\psi\psi_t+\psi_t^2+\bar{n}\psi^2+p'(\psi+\bar{n})\psi_x^2-\frac{j^2\psi_x^2}{(\psi+\bar{n})^2}\Big]$$
$$+\bar{n}\Big[\frac{1}{2}\eta^2+\eta\eta_t+\eta_t^2+\bar{h}\eta^2+q'(\eta+\bar{h})\eta_x^2-\frac{k^2\eta_x^2}{(\eta+\bar{h})^2}\Big]$$
$$-2\bar{n}\bar{h}\psi\eta\Big\}dx+a_3\int_0^1(\psi^2+\psi_x^2+\psi_t^2+\eta^2+\eta_x^2+\eta_t^2)dx$$
$$\le O(\varepsilon_1)\exp\{-c_0t\}\int_0^1(k_0^2+j_0^2)dx. \tag{2.42}$$

with $a_3>0$ a constant if $\varepsilon_1$ is small enough. By Young inequality and (2.12), we can show that

$$c_2\left(\psi^2+\psi_x^2+\psi_t^2+\eta^2+\eta_x^2+\eta_t^2\right)$$
$$\le\bar{h}\Big[\frac{1}{2}\psi^2+\psi\psi_t+\psi_t^2+\bar{n}\psi+p'(\psi+\bar{n})\psi_x^2-\frac{4j^2\psi_x^2}{(\psi+\bar{n})^2}\Big]$$
$$+\bar{n}\Big[\frac{1}{2}\eta^2+\eta\eta_t+\eta_t^2+\bar{h}\eta+q'(\eta+\bar{h})\eta_x^2-\frac{4k^2\eta_x^2}{(\eta+\bar{h})^2}\Big]-2\bar{n}\bar{h}\psi\eta$$
$$\le c_1\left(\psi^2+\psi_x^2+\psi_t^2+\eta^2+\eta_x^2+\eta_t^2\right) \tag{2.43}$$

for some positive constants $c_2\le c_1$, provided $\varepsilon_1$ is small enough.

This combined with (2.41) implies (2.27). And (2.28) follows from (2.27), (2.14), (2.6) and Poincaré inequality.

**Step 5**

$$\int_0^1(\psi_t^2+\psi_{xt}^2+\psi_{tt}^2+\psi_{xx}^2+\eta_t^2+\eta_{xt}^2+\eta_{tt}^2+\eta_{xx}^2)dx\le O(\varepsilon_1)\|(\psi_0,\eta_0)\|_{H^2}^2\exp\{-\beta_2t\} \tag{2.44}$$

$$\int_0^1 (e_t^2 + e_{xt}^2 + e_{xxt}^2)dx \le O(\varepsilon_1)\|(\psi_0, \eta_0)\|_{H^2}^2 \exp\{-\beta_2 t\} \tag{2.45}$$

with $\beta_2 > 0$ a constant, provided that $\varepsilon_1$ is small enough.

We prove (2.44) first. Differentiating (2.29) with respect to $t$, we have

$$\begin{aligned}&\psi_{ttt} + \psi_{tt} + \bar{n}(\psi_t - \eta_t) + \psi_{xt}e_x + \psi_x e_{xt} + (\psi(\psi - \eta))_t \\ &\qquad - \Big[\frac{j^2}{\psi + \bar{n}} + p(\psi + \bar{n}) - \psi(\bar{n})\Big]_{xxt} = 0.\end{aligned} \tag{2.46}$$

Multiplying (2.46) with $\psi_t$ and integrating it over $[0,1]$, and using $\psi_t(0,t) = \psi_t(1,t) = 0$, we get

$$\begin{aligned}&\frac{d}{dt}\int_0^1 (\frac{1}{2}\psi_t^2 + \psi_t\psi_{tt})dx - \int_0^1 \psi_{tt}^2 dx + \bar{n}\int_0^1 (\psi_t^2 - \psi_t\eta_t)dx + \int_0^1 (\psi_{xt}e_x + \psi_x e_{xt})\psi_t dx \\ &\quad + \int_0^1 \Big[\frac{j^2}{\psi + \bar{n}} + p(\psi + \bar{n}) - p(\bar{n})\Big]_{xt}\psi_{xt}dx + \int_0^1 (\psi(\psi - \eta))_t\psi_t dx = 0.\end{aligned} \tag{2.47}$$

The last integral of the above equality is bounded by $O(\varepsilon_1)\int_0^1 (\psi^2 + \eta^2 + \psi_t^2)$. The other integrals in that equality can be treated as follows.

By (2.6) and (2.15), we have

$$\begin{aligned}\int_0^1 (\psi_{xt}e_x + \psi_x e_{xt})\psi_t dx &= \int_0^1 \psi_x e_{xt}\psi_t dx - \frac{1}{2}\int_0^1 \psi_t^2 e_{xx} \\ &\ge -O(\varepsilon_1)\int_0^1 (\psi_t^2 + \eta_t^2)dx.\end{aligned} \tag{2.48}$$

By (2.2) and (2.21), we have

$$\begin{aligned}&\int_0^1 \Big[\frac{j^2}{\psi + \bar{n}}\Big]_{xt}\psi_{xt}dx \\ &\quad = \int_0^1 \Big[\frac{2j_t j_x}{\psi + \bar{n}} + \frac{2jj_{xt}}{\psi + \bar{n}} - \frac{2jj_x\psi_t}{(\psi + \bar{n})^2} - \frac{2jj_t\psi_x}{(\psi + \bar{n})^2} - \frac{j^2\psi_{xt}}{(\psi + \bar{n})^2} + \frac{2j^2\psi_x\psi_t}{(\psi + \bar{n})^3}\Big]\psi_{xt}dx \\ &\quad \ge -O(\varepsilon_1)\int_0^1 (\psi^2 + \eta^2 + \psi_t^2 + \psi_x^2 + \psi_{xt}^2 + \psi_{tt}^2)dx \\ &\qquad - O(\varepsilon_1)\exp\{-c_0 t\}\int_0^1 j_0^2 dx.\end{aligned} \tag{2.49}$$

$$\begin{aligned}\int_0^1 [p(\psi + \bar{n}) - p(\bar{n})]_{xt}\psi_{xt}dx &= \int_0^1 p'(\psi + \bar{n})\psi_{xt}^2 dx + \int_0^1 p''(\psi + \bar{n})\psi_t\psi_x\psi_{xt}dx \\ &\ge \int_0^1 p'(\psi + \bar{n})\psi_{xt}^2 - O(\varepsilon_1)\int_0^1 (\psi_t^2 + \psi_{xt}^2)dx.\end{aligned} \tag{2.50}$$

Thus (2.47)-(2.50) imply

$$\frac{d}{dt}\int_0^1 (\frac{1}{2}\psi_t^2 + \psi_t\psi_{tt})dx + \int_0^1 p'(\psi + \bar{n})\psi_{xt}^2 dx - \int_0^1 \psi_{tt}^2 dx + \bar{n}\int_0^1 (\psi_t^2 - \psi_t\eta_t)dx$$

$$\le O(\varepsilon_1)\int_0^1(\psi^2+\eta^2+\psi_t^2+\psi_x^2+\psi_{xt}^2+\psi_{tt}^2)dx$$
$$+O(\varepsilon_1)\exp\{-c_0t\}\int_0^1 j_0^2dx. \tag{2.51}$$

The corresponding estimates for $\eta$ are

$$\frac{d}{dt}\int_0^1(\frac{1}{2}\eta_t^2+\eta_t\eta_{tt})dx+\int_0^1 q'(\eta+\bar h)\eta_{xt}^2dx-\int_0^1\eta_{tt}^2dx+\bar h\int_0^1(\eta_t^2-\psi_t\eta_t)dx$$
$$\le O(\varepsilon_1)\int_0^1(\psi^2+\eta^2+\psi_t^2+\eta_x^2+\eta_{xt}^2+\eta_{tt}^2)dx$$
$$+O(\varepsilon_1)\exp\{-c_0t\}\int_0^1 k_0^2dx. \tag{2.52}$$

Multiplying (2.46) by $\psi_{tt}$ and integrating it over $[0,1]$, we have

$$\frac{d}{dt}\int_0^1\frac{1}{2}(\psi_{tt}^2+\bar n\psi_t^2)dx+\int_0^1\psi_{tt}^2dx-\bar n\int_0^1\eta_t\psi_{tt}dx+\int_0^1(\psi_{xt}e_x+\psi_xe_{xt})\psi_{tt}dx$$
$$+\int_0^1\Big[\frac{j^2}{\psi+\bar n}+p(\psi+\bar n)-p(\bar n)\Big]_{xt}\psi_{xtt}dx+\int_0^1\Big[\psi(\psi-\eta)\Big]_t\psi_{tt}dx=0. \tag{2.53}$$

It is easy to see that the last integral in (2.53) is bounded by $O(\varepsilon_1)\int_0^1(\psi_t^2+\eta_t^2+\psi_{tt}^2)dx$. Next, we estimate the other integrals.

By (2.14) and (2.15), we get

$$\int_0^1(\psi_{xt}e_x+\psi_xe_{xt})\psi_{tt}dx\le O(\varepsilon_1)\int_0^1(e_{xt}^2+\psi_{tt}^2)dx+\frac{1}{2}\int_0^1|e_x|(\psi_{xt}^2+\psi_{tt}^2)dx$$
$$\le O(\varepsilon_1)\int_0^1(\psi_t^2+\eta_t^2+\psi_{xt}^2+\psi_{tt}^2)dx. \tag{2.54}$$

By (2.2), we have

$$\int_0^1\Big[\frac{j^2}{\psi+\bar n}\Big]_{xt}\psi_{xtt}dx=\int_0^1\Big[\frac{2j\psi_{tt}}{\psi+\bar n}+\frac{2j_t\psi_t}{\psi+\bar n}-\frac{2j\psi_t^2}{(\psi+\bar n)^2}\Big]\psi_{xtt}dx$$
$$+\int_0^1\Big[\frac{2j^2\psi_t}{(\psi+\bar n)^3}-\frac{2jj_t}{(\psi+\bar n)^2}\Big]\psi_x\psi_{xtt}dx-\int_0^1\frac{j^2\psi_{xt}}{(\psi+\bar n)^2}dx$$
$$\doteq I_4+I_5+I_6. \tag{2.55}$$

Using (2.24) and $\psi_{tt}(0,t)=\psi_{tt}(1,t)=0$, we have

$$I_4=-\int_0^1\Big[\frac{2j}{\psi+\bar n}\Big]_x\psi_{tt}^2dx+\int_0^1\Big[\frac{2j\psi_t^2}{(\psi+\bar n)^2}-\frac{2j_t\psi_t}{\psi+\bar n}\Big]_x\psi_{tt}dx$$
$$\ge-O(1)\int_0^1\psi_{tt}^2(\psi_x+j_x)dx-O(1)\int_0^1\psi_{tt}\psi_{xt}(j_t+\psi_t)dx$$

$$
\begin{aligned}
&- O(1)\int_0^1 \psi_{tt}(\psi_t^2+\psi_t\psi_x+j_x^2)dx \\
\geq &- O(\varepsilon_1)\int_0^1 (\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx, \qquad (2.56)
\end{aligned}
$$

$$
\begin{aligned}
I_5 =&\frac{d}{dt}\int_0^1 \Big[\frac{2j^2\psi_t}{(\psi+\bar n)^3}-\frac{2jj_t}{(\psi+\bar n)^2}\Big]\psi_{xt}\psi_x dx \\
&+\int_0^1 \Big[\frac{2jj_t}{(\psi+\bar n)^2}-\frac{2j^2\psi_t}{(\psi+\bar n)^3}\Big]_t\psi_{xt}\psi_x dx+\int_0^1 \Big[\frac{2jj_t}{(\psi+\bar n)^2}-\frac{2j^2\psi_t}{(\psi+\bar n)^3}\Big]\psi_{xt}^2 dx \\
\geq&\frac{d}{dt}\int_0^1 \Big[\frac{2jj_t}{(\psi+\bar n)^2}-\frac{2j^2\psi_t}{(\psi+\bar n)^3}\Big]\psi_{xt}\psi_x dx \\
&- O(\varepsilon_1)\int_0^1 (\psi_{xt}^2+\psi_x^2+j_{tt}^2+\psi_{tt}^2+j_t^2)dx, \qquad (2.57)
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 =&-\frac{d}{dt}\int_0^1 \frac{j^2\psi_{xt}^2}{2(\psi+\bar n)^2}dx+\int_0^1 \frac{jj_t\psi_{xt}^2}{(\psi+\bar n)^2}dx-\int_0^1 \frac{\psi_t j^2\psi_{xt}^2}{(\psi+\bar n)^3}dx \\
\geq&-\frac{d}{dt}\int_0^1 \frac{j^2\psi_{xt}^2}{2(\psi+\bar n)^2}dx-O(\varepsilon_1)\int_0^1 (\psi_t^2+\psi_x^2+\psi^2+\eta^2+\psi_{xt}^2)dx \\
&-O(\varepsilon_1)\exp\{-c_0t\}\int_0^1 j_0^2dx. \qquad (2.58)
\end{aligned}
$$

Thus (2.55) – (2.58) imply

$$
\begin{aligned}
\int_0^1 \Big[\frac{j^2}{\psi+\bar n}\Big]_{xt}\psi_{xtt}dx \geq&\frac{d}{dt}\int_0^1 \Big[\frac{2j^2\psi_t}{(\psi+\bar n)^3}-\frac{2jj_t}{(\psi+\bar n)^2}-\frac{j^2\psi_{xt}^2}{2(\psi+\bar n)^2}\Big]dx \\
&-O(\varepsilon_1)\int_0^1 (\psi^2+\eta^2+\psi_x^2+\psi_t^2+\psi_{xt}^2+\psi_{tt}^2+j_{tt}^2)dx \\
&-O(\varepsilon_1)\exp\{-c_0t\}\int_0^1 j_0^2dx. \qquad (2.59)
\end{aligned}
$$

By $\psi_{tt}(0,t)=\psi_{tt}(1,t)=0$, we have

$$
\begin{aligned}
&\int_0^1 [p(\psi+\bar n)-p(\bar n)]_{xt}\psi_{xxt}dx \\
&\quad=\int_0^1 [p''(\psi+\bar n)\psi_t\psi_x+p'(\psi+\bar n)\psi_{xt}]\psi_{xtt}dx \\
&\quad=\frac{d}{dt}\int_0^1 \frac12 p'(\psi+\bar n)\psi_{xt}^2dx-\int_0^1 p'''(\psi+\bar n)\psi_t\psi_x^2\psi_{tt}dx-\frac12\int_0^1 p''(\psi+\bar n)\psi_t\psi_{xt}^2 \\
&\qquad-\int_0^1 p''(\psi+\bar n)\psi_{tx}\psi_x\psi_{tt}dx-\int_0^1 p''(\psi+\bar n)\psi_t\psi_{xx}\psi_{tt}dx
\end{aligned}
$$

$$\geq \frac{d}{dt}\int_0^1 \frac{1}{2}p'(\psi+\bar{n})\psi_{xt}^2 dx - O(\varepsilon_1)\int_0^1 (\psi_t^2+\psi_{xx}^2+\psi_{xt}^2+\psi_{tt}^2)dx. \tag{2.60}$$

On the other hand, we differentiate (2.4) with respect to $t$, and have with the help of (2.14), (2.15), (2.19) and (2.21), that:

$$\begin{aligned}\int_0^1 j_{tt}^2 dx \leq & O(1)\int_0^1 \Big[\frac{j^2}{\psi+\bar{n}} + p(\psi+\bar{n}) - p(\bar{n})\Big]_{xt} dx \\ & + O(1)\int_0^1 (j^2+\psi^2+\psi_t^2+e_{xt}^2+e_x^2)dx \\ \leq & O(1)\int_0^1 (\psi^2+\eta^2+\psi_t^2+\eta_t^2+\psi_{xt}^2 \\ & +\psi_{tt}^2)dx + O(1)\exp\{-c_0 t\}\int_0^1 j_0^2 dx. \end{aligned} \tag{2.61}$$

Thus, (2.53), (2.54), (2.59), (2.60) and (2.61) give

$$\begin{aligned} & \frac{d}{dt}\int_0^1 \Big[\frac{1}{2}(\psi_{tt}^2+\bar{n}\psi_t^2) - \frac{2jj_t}{(\psi+\bar{n})^2} + \frac{2j^2\psi_t}{(\psi+\bar{n})^3} - \frac{j^2\psi_{xt}^2}{2(\psi+\bar{n})^2}\Big]dx \\ & + \int_0^1 \psi_{tt}^2 dx - \bar{n}\int_0^1 \eta_t\psi_{tt} dx \\ \leq & O(\varepsilon_1)\int_0^1 (\psi^2+\eta^2+\psi_x^2+\psi_t^2+\psi_{xt}^2+\psi_{tt}^2)dx \\ & + O(\varepsilon_1)\exp\{-c_0 t\}\int_0^1 j_0^2 dx. \end{aligned} \tag{2.62}$$

The same procedure gives:

$$\begin{aligned} & \frac{d}{dt}\int_0^1 \Big[\frac{1}{2}(\eta_{tt}^2+\bar{h}\eta_t^2) - \frac{2kk_t}{(\eta+\bar{h})^2} + \frac{2k^2\eta_t}{(\eta+\bar{h})^3} - \frac{k^2\eta_{xt}^2}{2(\eta+\bar{h})^2}\Big]dx \\ & + \int_0^1 \eta_{tt}^2 dx - \bar{h}\int_0^1 \psi_t\eta_{tt} dx \\ \leq & O(\varepsilon_1)\int_0^1 (\psi^2+\eta^2+\eta_x^2+\eta_t^2+\eta_{xt}^2+\eta_{tt}^2)dx \\ & + o(\varepsilon_1)\exp\{-c_0 t\}\int_0^1 k_0^2 dx. \end{aligned} \tag{2.63}$$

By(2.27), $\bar{h}\times(2.51)+\bar{n}\times(2.53)+2\bar{h}\times(2.62)+2\bar{n}\times(2.63)$ leads to

$$\begin{aligned} & \frac{d}{dt}\int_0^1 \bar{h}\Big[\frac{1}{2}\psi_t^2+\psi_t\psi_{tt}+\psi_{tt}^2+\bar{n}\psi_t^2+\frac{4jj_t}{(\psi+\bar{n})^2}-\frac{4j^2\psi_t}{(\psi+\bar{n})^3}-\frac{j^2\psi_{xt}^2}{(\psi+\bar{n})^2}\Big]dx \\ & \quad + \frac{d}{dt}\int_0^1 \bar{n}\Big[\frac{1}{2}\eta_t^2+\eta_t\eta_{tt}+\eta_{tt}^2+\bar{h}\eta_t^2+\frac{4kk_t}{(\eta+\bar{h})^2}-\frac{4k^2\eta_t}{(\eta+\bar{h})^3}-\frac{k^2\eta_{xt}^2}{(\eta+\bar{h})^2}\Big]dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{d}{dt}\int_0^1 2\bar{n}\bar{h}\psi_t\eta_t dx+\bar{h}\int_0^1(\psi_{tt}^2+p'(\psi+\bar{n})\psi_{xt}^2)dx+\bar{n}\int_0^1(\eta_{tt}^2+q'(\eta+\bar{h})\eta_{xt}^2)dx\\
\le& O(\varepsilon_1)\|(\psi_0,\eta_0,j_0,k_0)\|_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\})\\
&+o(\varepsilon_1)\int_0^1(\psi_{tt}^2+\eta_{tt}^2+\psi_{xt}^2+\eta_{xt}^2)dx. \qquad (2.64)
\end{aligned}
$$

In addition, from (2.29), we have

$$\int_0^1 \psi_{xx}^2 dx \le O(1)\int_0^1(\eta^2+\psi_x^2+\psi_{xt}^2+\psi_{tt}^2+e_x^2+j^2)dx. \qquad (2.65)$$

The similar estimate for $\eta_{xx}$ is

$$\int_0^1 \eta_{xx}^2 dx \le O(1)\int_0^1(\psi^2+\eta_x^2+\eta_{xt}^2+\eta_{tt}^2+e_x^2+k^2)dx. \qquad (2.66)$$

By the same argument as (2.43), using(2.64)-(2.66), we can prove (2.44). It is easy to see that (2.45) follows from (2.6), (2.15) and (2.7).

**Proof of Theorem 1**

Based on Lemma 2, the proof of Theorem 1 is standard. In fact, combining the standard theory of existence and uniqueness of local (in time) solutions ( see, for instance Majda [6]) with the estimates (2.13), we can extend the local solution by the usual continuation arguments and show that the estimates (2.13) hold globally( see, for instance Hsiao and Luo [7]) if the perturbation $\|(\psi_0,\eta_0,j_0,k_0)\|_{H^2}$ is sufficiently small.

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