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HOMOCLINIC ORBIT IN A SIX DIMENSIONAL MODEL OF A PERTURBED HIGHER-ORDER NLS EQUATION

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Abstract In this paper, the perturbed higher-order NLS equation with periodic boundary condition is considered. The existence of the homoclinic orbits for the truncation equation is established by Melnikov analysis and geometric singular perturbation theory.

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1. Introduction

By using the reductive perturbation method, Kodama and Hasegawa proposed a higher-order nonlinear Schrödinger (HNLS) equation

$$\begin{aligned} &iq_t + \tfrac{1}{2}k_1 q_{xx} + l\ |q|^2 q \\ &\qquad = -i\varepsilon\left[-\tfrac{1}{6}k_2 q_{xxx} + h_1(q|q|^2)_x - h_2(|q|^2)_x q\right]. \end{aligned} \tag{1}$$

It can be used to describe the propagation of a femtosecond optical pulse in a monomode optical fiber.

In this paper, we consider the following perturbation HNLS equation

$$\begin{aligned} &iu_t + u_{xx} + (|u|^2 - 1)u \\ &\qquad = i\varepsilon\left[\alpha u + \beta_1 u_{xxx} + \beta_2(|u|^2 u)_x + \beta_3(|u|^2)_x u + \Gamma\right] \end{aligned} \tag{2}$$

with periodic boundary condition$u(x+2\pi,t) = u(x,t)$.Where $u = u(x,t)$ is a complex-value function of two real variables t and x, $\alpha, \beta_1, \beta_2, \beta_3$ and Γ are real parameters ($\alpha > 0, \Gamma > 0$), and $\varepsilon > 0$ is a small perturbation parameter. We adopt a three mode Fourier truncation and get a six dimensional ordinary differential equations. This equations will be considered and the persistence of the homoclinic orbits will be obtained by Melnikov's analysis together with the geometrical singular perturbation theory.

2. The Fourier Truncation of the Perturbation HNLS Equation

Suppose that the equation (2) have a solution with the following type

$$u(x,t) = \frac{1}{\sqrt{2}}a(t) + b(t)\cos x + c(t)\sin x. \tag{3}$$

where a, b, and c are complex. Inserting (3) into the perturbed HNLS equation (2) and neglecting the higher Fourier modes yields

$$\begin{aligned}
i\,\dot{a} &+ (\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 + \frac{1}{2}|c|^2 - 1)a + \frac{1}{2}(ab^* + a^*b)b + \frac{1}{2}(ac^* + a^*c)c \\
&= i\varepsilon[\alpha a + \frac{1}{2}\beta_3 b(ac^* + a^*c) - \frac{1}{2}\beta_3 c(ab^* + a^*b) + \sqrt{2}\Gamma] \\
i\,\dot{b} &+ (\frac{1}{2}|a|^2 + \frac{3}{4}|b|^2 + \frac{1}{4}|c|^2 - 2)b + \frac{1}{2}(ab^* + a^*b)a + \frac{1}{4}(bc^* + b^*c)c \\
&= i\varepsilon[\beta_2(\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 + \frac{1}{2}|c|^2)c + \frac{1}{2}(\beta_2 + \beta_3)(ac^* + a^*c)a \\
&\quad + \frac{1}{4}(\beta_2 + 2\beta_3)(bc^* + b^*c)b - \frac{1}{4}(\beta_2 + 2\beta_3)(|b|^2 - |c|^2)c] + i\varepsilon(\alpha b - \beta_1 c) \qquad (4) \\
i\,\dot{c} &+ (\frac{1}{2}|a|^2 + \frac{1}{4}|b|^2 + \frac{3}{4}|c|^2 - 2)c + \frac{1}{2}(ac^* + a^*c)a + \frac{1}{4}(bc^* + b^*c)b \\
&= -i\varepsilon[\beta_2(\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 + \frac{1}{2}|c|^2)b + \frac{1}{2}(\beta_2 + \beta_3)(ab^* + a^*b)a \\
&\quad + \frac{1}{4}(\beta_2 + 2\beta_3)(bc^* + b^*c)c + \frac{1}{4}(\beta_2 + 2\beta_3)(|b|^2 - |c|^2)b] + i\varepsilon(\alpha c + \beta_1 b).
\end{aligned}$$

From (4) the unperturbed equations are obtained by setting $\varepsilon = 0$

$$\begin{aligned}
i\,\dot{a} &+ (\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 + \frac{1}{2}|c|^2 - 1)a + \frac{1}{2}(ab^* + a^*b)b + \frac{1}{2}(ac^* + a^*c)c = 0 \\
i\,\dot{b} &+ (\frac{1}{2}|a|^2 + \frac{3}{4}|b|^2 + \frac{1}{4}|c|^2 - 2)b + \frac{1}{2}(ab^* + a^*b)a + \frac{1}{4}(bc^* + b^*c)c = 0 \\
i\,\dot{c} &+ (\frac{1}{2}|a|^2 + \frac{1}{4}|b|^2 + \frac{3}{4}|c|^2 - 2)c + \frac{1}{2}(ac^* + a^*c)a + \frac{1}{4}(bc^* + b^*c)b = 0.
\end{aligned} \tag{5}$$

By inspection, we see that the unperturbed equations are invariant under the following coordinate transformations

$$(a,b,c) \to (-a,b,c); (a,b,c) \to (a,-b,-c). \tag{6a,6b}$$

We want to describe the invariant manifold structure and phase space geometry of (5), we also want ultimately to utilize the generalized Melnikov theory described in [1]. For these purpose, we rewrite the equations (4) in the appropriate form by introducing the following coordinate transformation

$$\begin{aligned}
a &= \rho(t)\exp\{i\theta(t)\} \\
b &= [x_1(t) + ix_2(t)]\exp\{i\theta(t)\} \\
c &= [y_1(t) + iy_2(t)]\exp\{i\theta(t)\},
\end{aligned} \tag{7}$$

and Let $I=\frac{1}{2}(\rho^2+x^2+y^2)$. In these coordinates the perturbed equations (4) become

$$
\begin{aligned}
\dot{I}=&2\varepsilon\alpha I+\sqrt{2}\varepsilon\Gamma\sqrt{2I-x^2-y^2}\cos\theta\\
\dot{\theta}=&I-1+x_1^2+y_1^2+\varepsilon\beta_3(x_2y_1-x_1y_2)-\frac{1}{\rho}\sqrt{2}\varepsilon\Gamma\sin\theta\\
\dot{x}_1=&x_2+\frac{3}{4}x_1^2x_2-\frac{1}{4}x_2^3+\frac{5}{4}x_2y_1^2+\frac{1}{4}x_2y_2^2-\frac{1}{2}(x_1y_1+x_2y_2)y_2\\
&+\varepsilon(\alpha x_1-\beta_1y_1)+\varepsilon\{\beta_2y_1I+\rho^2(\beta_2+\beta_3)y_1+\frac{1}{2}(\beta_2+2\beta_3)(x_1y_1+x_2y_2)x_1\\
&-\frac{1}{4}(\beta_2+2\beta_3)(x^2-y^2)y_1\}+\varepsilon\beta_3(x_2y_1-x_1y_2)x_2-\frac{1}{\rho}\sqrt{2}\varepsilon\Gamma x_2\sin\theta\\
\dot{x}_2=&(2I-1)x_1-\frac{3}{4}x_1x_2^2-\frac{7}{4}x_1^3-\frac{9}{4}x_1y_1^2-\frac{5}{4}x_1y_2^2+\frac{1}{2}(x_1y_1+x_2y_2)y_1\\
&+\varepsilon(\alpha x_2-\beta_1y_2)+\varepsilon\{\beta_2y_2I+\frac{1}{2}(\beta_2+2\beta_3)(x_1y_1+x_2y_2)x_2\\
&-\frac{1}{4}(\beta_2+2\beta_3)(x^2-y^2)y_2\}-\varepsilon\beta_3(x_2y_1-x_1y_2)x_1+\frac{1}{\rho}\sqrt{2}\varepsilon\Gamma x_1\sin\theta\\
\dot{y}_1=&y_2+\frac{3}{4}y_1^2y_2-\frac{1}{4}y_2^3+\frac{5}{4}x_1^2y_2+\frac{1}{4}x_2^2y_2-\frac{1}{2}(x_1y_1+x_2y_2)x_2\\
&+\varepsilon(\alpha y_1+\beta_1x_1)-\varepsilon\{\beta_2x_1I+\rho^2(\beta_2+\beta_3)x_1+\frac{1}{2}(\beta_2+2\beta_3)(x_1y_1+x_2y_2)y_1\\
&+\frac{1}{4}(\beta_2+2\beta_3)(x^2-y^2)x_1\}+\varepsilon\beta_3(x_2y_1-x_1y_2)y_2-\frac{1}{\rho}\sqrt{2}\varepsilon\Gamma y_2\sin\theta\\
\dot{y}_2=&(2I-1)y_1-\frac{3}{4}y_1y_2^2-\frac{7}{4}y_1^3-\frac{9}{4}x_1^2y_1-\frac{5}{4}x_2^2y_1+\frac{1}{2}(x_1y_1+x_2y_2)x_1\\
&+\varepsilon(\alpha y_2+\beta_1x_2)-\varepsilon\{\beta_2x_2I+\frac{1}{2}(\beta_2+2\beta_3)(x_1y_1+x_2y_2)y_2\\
&+\frac{1}{4}(\beta_2+2\beta_3)(x^2-y^2)x_2\}-\varepsilon\beta_3(x_2y_1-x_1y_2)y_1+\frac{1}{\rho}\sqrt{2}\varepsilon\Gamma y_1\sin\theta,
\end{aligned}
\tag{8}
$$

where $x^2=x_1^2+x_2^2, y^2=y_1^2+y_2^2$ and $\rho=\sqrt{2I-x_1^2-x_2^2-y_1^2-y_2^2}$. Under the coordinate transformation (7) the unperturbed equations (5) become

$$
\begin{aligned}
\dot{x}_1=&\frac{\partial H}{\partial x_2}; &\qquad \dot{x}_2=&-\frac{\partial H}{\partial x_1};\\
\dot{y}_1=&\frac{\partial H}{\partial y_2}; &\qquad \dot{y}_2=&-\frac{\partial H}{\partial y_1};\\
\dot{I}=&0; &\qquad \dot{\theta}=&-\frac{\partial H}{\partial I}.
\end{aligned}
\tag{9}
$$

Where H is the following energy integration

$$\begin{aligned} H = & -\frac{1}{2}I^2 + I - (x_1^2 + y_1^2)I + \frac{1}{2}x_1^2 + \frac{7}{16}x_1^4 + \frac{9}{8}x_1^2y_1^2 + \frac{3}{8}x_1^2x_2^2 \\ & + \frac{5}{8}x_1^2y_2^2 + \frac{1}{2}y_1^2 + \frac{7}{16}y_1^4 + \frac{3}{8}y_1^2y_2^2 + \frac{5}{8}x_2^2y_1^2 + \frac{1}{2}(x_2^2 + y_2^2) \\ & - \frac{1}{16}x_2^4 + \frac{1}{8}x_2^2y_2^2 - \frac{1}{16}y_2^4 - \frac{1}{4}(x_1y_1 + x_2y_2)^2. \end{aligned} \tag{10}$$

Hence, when $\varepsilon = 0$ the unperturbed system is an integrable Hamiltonian system.

3. The unperturbed integrable structure

In order to show that (9) has the invariant manifold structure described in the general theory [1] we must consider the (x_1, x_2, y_1, y_2) component of (9) which we rewrite below

$$\begin{aligned} \dot{x}_1 = & x_2 + \frac{3}{4}x_1^2x_2 - \frac{1}{4}x_2^3 + \frac{5}{4}x_2y_1^2 + \frac{1}{4}x_2y_2^2 - \frac{1}{2}(x_1y_1 + x_2y_2)y_2 \\ \dot{x}_2 = & (2I-1)x_1 - \frac{3}{4}x_1x_2^2 - \frac{7}{4}x_1^3 - \frac{9}{4}x_1y_1^2 - \frac{5}{4}x_1y_2^2 + \frac{1}{2}(x_1y_1 + x_2y_2)y_1 \\ \dot{y}_1 = & y_2 + \frac{3}{4}y_1^2y_2 - \frac{1}{4}y_2^3 + \frac{5}{4}x_1^2y_2 + \frac{1}{4}x_2^2y_2 - \frac{1}{2}(x_1y_1 + x_2y_2)x_2 \\ \dot{y}_2 = & (2I-1)y_1 - \frac{3}{4}y_1y_2^2 - \frac{7}{4}y_1^3 - \frac{9}{4}x_1^2y_1 - \frac{5}{4}x_2^2y_1 + \frac{1}{2}(x_1y_1 + x_2y_2)x_1. \end{aligned} \tag{11}$$

Note that (11) has a fixed point at $(x_1, x_2, y_1, y_2) = (0, 0, 0, 0)$ for all values of I, this is a result of the symmetry given by (6b). A simple linear stability analysis shows that $(x_1, x_2, y_1, y_2) = (0, 0, 0, 0)$ is a saddle point for $I > \frac{1}{2}$. Moreover, an examination of the level set of the Hamiltonian that contains the origin, i.e.,

$$\{(x_1, x_2, y_1, y_2) \mid H(x_1, x_2, y_1, y_2, I) - H(0, 0, 0, 0, I) = 0\}$$

shows that for each I in this range the origin has a pair of symmetric homoclinic orbits. Interpreting these results in the full six-dimensional phase space, the set

$$M_0 = \{x_1 = x_2 = y_1 = y_2 = 0, I_1 < I < I_2, \theta \in [0, 2\pi]\}$$

is a two-dimensional invariant manifold under the flow generated by (9)(where I_1 and I_2 are given constants).

In calculating the Melnikov functions it will be important to have analytical expressions for the homoclinic orbits of (11) that connect the origin as a function of I. For $\frac{1}{2} < I < 4$, the hyperbolic fixed point $(0, 0, 0, 0)$ for the system (11) has two dimensional stable and unstable manifolds. These two manifolds intersect into a two-dimensional homoclinic manifold.

Proposition 2.1 *For any point (I, θ) ($\frac{1}{2} < I < 4, \theta \in [0, 2\pi]$), the homoclinic manifold has the following form:*

(1) *If* $x_1 \neq 0$ *or* $x_2 \neq 0$*, then for any* $k \in R$*,*

$$\begin{aligned} x_1(t) =& r(t)\cos\varphi(t) \\ x_2(t) =& r(t)\sin\varphi(t) \\ y_1(t) =& kr(t)\cos\varphi(t) \\ y_2(t) =& kr(t)\sin\varphi(t). \end{aligned}$$

(2) If $x_1 = x_2 = 0$, then

$$\begin{aligned} x_1(t) =& x_2(t) = 0 \\ y_1(t) =& r(t)\cos\varphi(t) \\ y_2(t) =& r(t)\sin\varphi(t). \end{aligned}$$

Where

$$\begin{aligned} r^2 =& \frac{8I(1+\cos 2\varphi) - 8}{(1+k^2)(3+4\cos 2\varphi)}, \\ \tan\varphi =& \lambda \tanh(-\lambda t) \end{aligned}$$

and $\lambda = \sqrt{2I-1}$.

Proof One would notice that the eigenfunction of the fixed point $(0,0,0,0)$ for system (11) is

$$F(\lambda) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 1-2I & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 1-2I & \lambda \end{vmatrix}$$

The eigenvalues are $\lambda_1 = \lambda_3 = \sqrt{2I-1}$, $\lambda_2 = \lambda_4 = -\sqrt{2I-1}$ for $\frac{1}{2} < I$, and the eigenvectors are

$$\begin{aligned} v_1 =& (f_1(I), f_2(I), 0, 0)^T \\ v_2 =& (f_1(I), -f_2(I), 0, 0)^T \\ v_3 =& (0, 0, f_1(I), f_2(I))^T \\ v_4 =& (0, 0, f_1(I), -f_2(I))^T. \end{aligned}$$

So the local unstable manifold near $(0,0,0,0)$ is the combination of the two vectors v_1 and v_3, i. e., $\{v|v = c_1v_1 + c_3v_3\}$. If $c_1 \neq 0$, we have $y_1 = kx_1, y_2 = kx_2$ (where $k = \frac{c_3}{c_1} \in R$).

We would also notice that for any $k \in R$, manifold $\{(x_1,x_2,y_1,y_2)|y_1 = kx_1, y_2 = kx_2\}$ is invariant for system (11). The system restricted on the invariant plane is

$$\begin{aligned} \dot{x}_1 =& x_2 + \frac{3}{4}(1+k^2)x_1^2x_2 - \frac{1}{4}(1+k^2)x_2^3 \\ \dot{x}_2 =& (2I-1)x_1 - \frac{3}{4}(1+k^2)x_1x_2^2 - \frac{7}{4}(1+k^2)x_1^3. \end{aligned} \tag{12}$$

It is also a Hamilton system with energy function

$$H_0 = -\frac{1}{2}I^2 + I - (I - \frac{1}{2})x_1^2 + \frac{1}{2}x_2^2 + \frac{7}{16}(1+k^2)x_1^4 - \frac{1}{16}(1+k^2)x_2^4 + \frac{3}{8}(1+k^2)x_1^2x_2^2. \tag{13}$$

Simple analysis gives that the point $(0,0)$ is a hyperbolic fixed point, there is a homoclinic orbit connecting the fixed point. Now we give the explicit form of the homoclinic orbit.

Let $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$, then

$$\begin{aligned} \dot{r} &= \frac{1}{2}[2Ir - (1+k^2)r^3]\sin 2\varphi, \\ \dot{\varphi} &= -1 + I(1+\cos 2\varphi) - (1+k^2)r^2(\frac{3}{4} + \cos 2\varphi). \end{aligned} \tag{14}$$

The Hamilton energy function (13) becomes

$$H_0 = -\frac{1}{2}I^2 + I + (1+k^2)r^4(\frac{3}{16} + \frac{1}{4}\cos 2\varphi) - r^2[\frac{1}{2}I + \frac{1}{2}(I\cos 2\varphi - 1)].$$

Then

$$r^2 = \frac{8I(1+\cos 2\varphi) - 8}{(1+k^2)(3+4\cos 2\varphi)} \tag{15}$$

and

$$\dot{\varphi} = 1 - I(1+\cos 2\varphi).$$

For $\frac{1}{2} < I < 4$ with initial condition $\varphi(t=0) = 0$, we have

$$\tan\varphi = \lambda\tanh(-\lambda t). \tag{16}$$

where $\lambda = \sqrt{2I-1}$. So we get

$$r^2 = \frac{8\lambda^2}{(1+k^2)[(4-I)\cosh(2\lambda t) + 3 + I]}. \tag{17}$$

Then the proposition is true.

Next we let $\xi = \theta + \varphi$, then

$$\dot{\xi} = \dot{\theta} + \dot{\varphi} = I - 1 + \frac{1}{8}(1+k^2)r^2. \tag{18}$$

Using (17) and (18) with the initial condition $\xi(t=0) = \xi_0$ we have

$$\xi(t) = \frac{1}{\sqrt{7}(1+k^2)}\tanh^{-1}[\frac{\lambda}{\sqrt{7}}\tanh(\lambda t)] + (I-1)t + \xi_0. \tag{19}$$

The unperturbed vector field restricted to M_0 is given by

$$\begin{aligned} \dot{I} &= 0 \\ \dot{\theta} &= I - 1. \end{aligned} \tag{20}$$

The dynamics described by (20) is quite simple; all trajectories lie on periodic orbits except at $I=1$. At $I=1$ the frequency ($\dot{\theta}$) vanishes, which results in a circle of fixed points. Thus we have a resonance and we will often refer to $I=1$ as the resonant I level or value.

Using (16) and (19) at $I=1$ gives

$$\begin{aligned}\theta(-\infty) =&\xi_0 - \frac{\pi}{4} - \frac{1}{\sqrt{7}(1+k^2)}\tanh^{-1}(\frac{1}{\sqrt{7}})\\ \theta(\infty) =&\xi_0 + \frac{\pi}{4} + \frac{1}{\sqrt{7}(1+k^2)}\tanh^{-1}(\frac{1}{\sqrt{7}}) \qquad (21)\\ \Delta\theta =&\theta(\infty)-\theta(-\infty)\\ =&\frac{\pi}{2} + \frac{2}{\sqrt{7}(1+k^2)}\tanh^{-1}(\frac{1}{\sqrt{7}}). \qquad (22)\end{aligned}$$

4. The Persistence of the Normally Hyperbolic Invariant Manifold

In this section, we will list some results about the existence of the normal hyperbolic invariant manifold and its stable and unstable manifolds. First, for $\varepsilon=0$, M_0 is a normal hyperbolic invariant manifold.

For $\varepsilon \neq 0$, we have that $x_1=x_2=y_1=y_2=0$ is invariant under the perturbed system (8). Thus, the set

$$M_\varepsilon = \{(x,y,I,\theta) \mid x=y=0, \frac{1}{2}<I<4, \theta\in[0,2\pi]\} \qquad (23)$$

is an invariant manifold for the perturbed problem (here we denote (x_1,x_2) and (y_1,y_2) by x and y respectively). However, there is an important, general difference in the behavior of trajectories in M_0 and M_ε. Since $\dot{I}\neq 0$ in the perturbed problem, M_ε must be considered as an invariant manifold with boundary. This means that trajectories in M_ε may leave M_ε but only by crossing the boundary of M_ε. In this case one can show that, for ε sufficiently small, there exists locally invariant manifolds of the perturbed problem, denoted $W^s_{loc}(M_\varepsilon)$ and $W^u_{loc}(M_\varepsilon)$, that can be represented as graphs over the local unperturbed stable and unstable manifolds, $W^s_{loc}(M_0)$ and $W^u_{loc}(M_0)$, respectively. Moreover, these manifolds are as differentiable as the vector field. We define the global stable and unstable manifolds of M_ε, denote $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$, respectively, as follows: let $\Phi_t(\cdot)$ denote the flow generated by (8),then

$$\begin{aligned}W^s(M_\varepsilon) &= \bigcup_{t\le 0}\Phi_t(W^s_{loc}(M_\varepsilon)\cap U^\delta)\\ W^u(M_\varepsilon) &= \bigcup_{t\ge 0}\Phi_t(W^u_{loc}(M_\varepsilon)\cap U^\delta).\end{aligned}$$

Where U^δ is a δ neighborhood of M_0. A more detailed description of the perturbed stable and unstable manifolds can be found in [1].

Next we study the dynamics on M_ε near resonance. The perturbed vector field (8) restricted to M_ε is given by

$$\begin{aligned}\dot{I}&=2\varepsilon[\Gamma\sqrt{I}\cos\theta+\alpha I]\\ \dot{\theta}&=I-1-\frac{\varepsilon\Gamma}{\sqrt{I}}\sin\theta.\end{aligned}\tag{24}$$

Let $I=1+\sqrt{2\varepsilon\Gamma}h, \tau=\sqrt{2\varepsilon\Gamma}t$, the equations (24) can be written as

$$\begin{aligned}h'&=\cos\theta+\frac{\alpha}{\Gamma}+\eta(\frac{\alpha}{\Gamma}+\frac{1}{2}\cos\theta)h+O(\eta^2)\\ \theta'&=h-\frac{1}{2}\eta\sin\theta+O(\eta^2).\end{aligned}\tag{25}$$

where the prime denotes differentiation with respect to τ and $\eta=\sqrt{2\varepsilon\Gamma}$. Since we will be interested mainly in the dynamics near the resonance we will restrict the domain of M_ε to an annulus containing the resonance. More precisely, the region of interest on M_ε is defined as follows:

$$\mathcal{A}_\varepsilon=\{(x,y,h,\theta)\mid x=y=0,|h|<C,\theta\in[0,2\pi]\}$$

where C is an $O(1)$ constant chosen sufficiently large to contain the resonance structures.

For $\eta=0$ the equations (25) reduce to

$$\begin{aligned}h'&=\cos\theta+\frac{\alpha}{\Gamma}\\ \theta'&=h.\end{aligned}\tag{26}$$

A simple analysis shows:

(1) The system (26) is a Hamiltonian system with Hamilton energy function

$$\mathcal{H}=-\frac{h^2}{2}+\sin\theta+\frac{\alpha}{\Gamma}\theta.\tag{27}$$

(2) The system (26) has two fixed points: a center p_0 and a saddle q_0, their coordinates are given by

$$\begin{aligned}p_0&=(h_{p_0},\theta_{p_0})=(0,\pi-arc\cos\frac{\alpha}{\Gamma})\\ q_0&=(h_{q_0},\theta_{q_0})=(0,\pi+arc\cos\frac{\alpha}{\Gamma}).\end{aligned}\tag{28}$$

From an application of the implicit function theorem and standard phase plane results, for η sufficiently small and $0<\frac{\alpha}{\Gamma}<1$, p_0 becomes a sink, denoted p_ε, q_0 remains a saddle, denoted q_ε, and the homoclinic orbit breaks with a branch of the unstable manifold of q_ε falling into p_ε. We emphasize here that

$$\begin{aligned}p_\varepsilon&=p_0+O(\varepsilon)=p_0+O(\eta^2)\\ q_\varepsilon&=q_0+O(\varepsilon)=q_0+O(\eta^2).\end{aligned}$$

For the stable and unstable manifolds of $\mathcal{A}_\varepsilon$, we have a fibers theorem which is similar to the theorem 7.52 in ref.[2]. Moreover, we can construct the fiber representations of $W^u(q_\varepsilon)$ and $W^s(q_\varepsilon)$.

5. The Persistence of the Homoclinic Orbits

The perturbed system (8) can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial x_2} + \varepsilon g^{x_1}, \\
\dot{x}_2 &= -\frac{\partial H}{\partial x_1} + \varepsilon g^{x_2}, \\
\dot{y}_1 &= \frac{\partial H}{\partial y_2} + \varepsilon g^{y_1}, \\
\dot{y}_2 &= -\frac{\partial H}{\partial y_1} + \varepsilon g^{y_2}, \\
\dot{I} &= 0 + \varepsilon g^{I}, \\
\dot{\theta} &= -\frac{\partial H}{\partial I} + \varepsilon g^{\theta}.
\end{aligned} \tag{29}$$

Where H have been given in (10). $(g^{x_1}, g^{x_2}, g^{y_1}, g^{y_2}, g^I, g^\theta)^T$ is defined by (8) and have the following representation

$$\begin{pmatrix} g^{x_1} \\ g^{x_2} \\ g^{y_1} \\ g^{y_2} \\ g^I \\ g^\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial H_1}{\partial x_2} \\ -\frac{\partial H_1}{\partial x_1} \\ \frac{\partial H_1}{\partial y_2} \\ -\frac{\partial H_1}{\partial y_1} \\ \frac{\partial H_1}{\partial \theta} \\ -\frac{\partial H_1}{\partial I} \end{pmatrix} + \alpha \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ 2I \\ 0 \end{pmatrix} + \begin{pmatrix} g_1^{x_1} \\ g_1^{x_2} \\ g_1^{y_1} \\ g_1^{y_2} \\ g_1^I \\ g_1^\theta \end{pmatrix}. \tag{30}$$

Where $H_1 = \sqrt{2}\Gamma\sqrt{2I - x_1^2 - x_2^2 - y_1^2 - y_2^2}\sin\theta$ and $(g_1^{x_1}, g_1^{x_2}, g_1^{y_1}, g_1^{y_2}, g_1^I, g_1^\theta)^T$ consists of terms that have β_j $(j = 1, 2, 3)$ coefficients in (8).

To show the existence of the homoclinic orbit for the system (29) we use two steps to analysis this problem. First, we use Melnikov's method to compute the distance of $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$. By this method we will show that the unstable manifold of the fixed point in M_ε is in $W^s(M_\varepsilon)$. The second step we show that the unstable manifold will be intersect to the stable fiber of the stable manifold of the fixed point in M_ε.

Now we discuss the distance of $W^u(q_\varepsilon)$ and $W^s(\mathcal{A}_\varepsilon \subset M_\varepsilon)$. From the higher dimensional Melnikov theory in [1] we known that for any point in the homoclinic manifold, denote $(x_1, x_2, y_1, y_2, I, \theta)$, the normal vector of the point is $\vec{n} = (\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \frac{\partial H}{\partial I}, 0)$ and the Melnikov function is given by

$$M(\theta_0) = \int_{-\infty}^{\infty} < \vec{n}, (g^{x_1}, g^{x_2}, g^{y_1}, g^{y_2}, g^I, g^\theta) > (q^h(t, I = 1, \theta_0))dt. \tag{31}$$

Where $q^h(t, I=1, \theta_0)$ is a homoclinic orbit in the homoclinic manifold which pass the point $(x_1, x_2, y_1, y_2, I, \theta) = (0, 0, 0, 0, 1, \theta_0)$. It is easy to show that the function $M(\theta_0)$ is independent in $\beta_j (j = 1, 2, 3)$. Hence,in the homoclinic manifold we have

$$
\begin{aligned}
&< \vec{n}, (g^{x_1}, g^{x_2}, g^{y_1}, g^{y_2}, g^I, g^\theta) > (q^h(t, I=1, \theta)) \\
&= - \dot{x}_2 \left(\frac{\partial H_1}{\partial x_2} + \alpha x_1\right) + \dot{x}_1 \left(-\frac{\partial H_1}{\partial x_1} + \alpha x_2\right) - \dot{y}_2 \left(\frac{\partial H_1}{\partial y_2} + \alpha y_1\right) \\
&\quad + \dot{y}_1 \left(-\frac{\partial H_1}{\partial y_1} + \alpha y_2\right) - \dot{\theta} \left(\frac{\partial H_1}{\partial \theta} + 2\alpha\right) \\
&= - \frac{dH_1}{dt} + \alpha(1+k^2)(\dot{x}_1 x_2 - x_1 \dot{x}_2) - 2\alpha \dot{\theta} . \qquad (32)
\end{aligned}
$$

We now integrate (32) around the unperturbed heteroclinic orbit at $I = 1$ that approaches q_0 asymptotically as $t \to -\infty$. It is clear that the first term and the third term in (32) can be integrated directly to give

$$
- \int_{-\infty}^{\infty} \frac{dH_1}{dt} dt = -2\Gamma[\sin(\theta_{q_0} + \Delta\theta) - \sin\theta_{q_0}], \qquad (33)
$$

$$
- \int_{-\infty}^{\infty} 2\alpha \dot{\theta} \, dt = -2\alpha\Delta\theta. \qquad (34)
$$

We now examine the second term in (32). Since $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$, then

$$
\dot{x}_1 x_2 - x_1 \dot{x}_2 = -r^2 \dot{\varphi} .
$$

Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty} (\dot{x}_1 x_2 - x_1 \dot{x}_2) dt &= - \int_{-\infty}^{\infty} r^2 \dot{\varphi} \, dt \\
&= - \frac{8}{1+k^2} \int_{\varphi(-\infty)}^{\varphi(+\infty)} \frac{\cos 2\varphi}{3 + 4\cos 2\varphi} d\varphi \\
&= - \frac{2}{1+k^2} \Delta\varphi + \frac{6}{1+k^2} \int_{\varphi(-\infty)}^{\varphi(+\infty)} \frac{d\varphi}{3 + 4\cos 2\varphi} \\
&= - \frac{2}{1+k^2} \Delta\varphi - \frac{6}{\sqrt{7}(1+k^2)} \tanh^{-1}\left(\frac{\sqrt{7}}{4}\right). \qquad (35)
\end{aligned}
$$

Where $\Delta\varphi = \varphi(+\infty) - \varphi(-\infty) = \frac{\pi}{2}$.

Using (33), (34) and (35), the Melnikov function becomes

$$
\begin{aligned}
M(\alpha, \Gamma, k; \theta_{q_0}) = &- 2\Gamma[\sin(\theta_{q_0} + \Delta\theta) - \sin\theta_{q_0}] \\
&- \frac{2\alpha}{\sqrt{7}(1+k^2)} \tanh^{-1}\left(\sqrt{\frac{1}{7}}\right) - \frac{6\alpha}{\sqrt{7}} \tanh^{-1}\left(\frac{\sqrt{7}}{4}\right). \qquad (36)
\end{aligned}
$$

Where $\theta_{q_0} = \pi + \arccos\frac{\alpha}{\Gamma}$. and $\Delta\theta = \frac{\pi}{2} + \frac{2}{\sqrt{7}(1+k^2)} \tanh^{-1}\left(\sqrt{\frac{1}{7}}\right)$.

Following the theory developed in [1, 3], in order to show that there exists an orbit homoclinic to q_ε we must first show that the Melnikov function has a simple zero. This condition is a sufficient condition for the existence of an orbit that is asymptotic to q_ε as $t \to -\infty$ and asymptotic to an orbit in $\mathcal{A}_\varepsilon$ as $t \to +\infty$. Further, in order to verify that the unstable manifold $W^u(q_\varepsilon)$ is intersect to the stable fiber of the stable manifold of the fixed point in $\mathcal{A}_\varepsilon$, we define

$$\Delta\mathcal{H} = \mathcal{H}(0, \theta_b) - \mathcal{H}(0, \pi + \arccos\frac{\alpha}{\Gamma}),$$

where $\mathcal{H}$ is given by (27). Hence, the location that the unstable manifold of q_ε returns to $\mathcal{A}_\varepsilon$ is given by the solution of the following equation

$$\Delta\mathcal{H} = \frac{\alpha}{\Gamma}(\theta_b - \theta_{q0}) + \sin\theta_b - \sin\theta_{q0} = 0. \tag{37}$$

Where θ_b is called as "take off angle".

By the above discussion we can get the following theorem for the existence of homoclinic orbit for the saddle point q_ε.

Theorem 5.1 *Choosing the parameters such that $M(\alpha, \Gamma, k; \theta_{q0})$ has simple zero with parameters and*

$$\frac{\alpha}{\Gamma}(\theta_b - \theta_{q0}) + \sin\theta_b - \sin\theta_{q0} = 0$$

take value throughout an $O(1)$ interval at a zero point of the Melnikov function. Then for ε sufficiently small, there are homoclinic orbits connecting to q_ε.

Remark (1) Taking the similar discussion in [4, 6] we may show that the conditions of the theorem 5.1 can be satisfied for the appropriate parameters α and Γ.

(2) By the same discussion we can get the existence of homoclinic orbit for the fixed point p_ε.

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