J.Partial Diff. Eqs. **12**(1999), 71–84 Vol.12, No.1

A HOPF BIFURCATION IN A GENERALIZED MCKEAN TYPE OF FREE BOUNDARY PROBLEM SATISFYING THE DIRICHLET BOUNDARY CONDITION *

YoonMee Ham

(Department of Mathematics, Kyonggi University, Suwon, 442-760, Korea)

(Received Dec. 10, 1997; revised July 15, 1998)

Abstract In this paper, we consider the free boundary problem satisfying the Dirichlet boundary condition. This problem is derived from the reaction diffusion equations with the generalized McKean reaction dynamics. We shall show a Hopf bifurcation occurs at some critical point τ when the stationary solution $(v^*(x), s^*)$ satisfies $1/3 < s^* < 1$.

Key Words Evolution equation; free boundary problem; Hopf bifurcation; internal layer solution; parabolic equation.

Classification 35R35, 35B32, 35B25, 35K22, 35K57, 58F14, 58F22.

1. Introduction

The well posedness and the Hopf bifurcation in a parabolic free boundary problem with the McKean reaction term are proved in [1]. In this paper, we consider the free boundary problem for the generalized McKean kinetics satisfying the Dirichlet boundary condition. We are dealing with the following problem.

$$\begin{cases} v_t = Dv_{xx} - (1+b)v + H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+ \\ v(0,t) = 0 = v(1,t) & \text{for } t > 0 \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1 \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0 \\ s(0) = s_0 \end{cases} \tag{1}$$

* This present work was supported by the Basic Science Research Institute Program, Ministry of Education, 1998, Project No. BSRI-98-1436.

where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega, \Omega = (0,1) \times (0,\infty)$. Here $H(y)$ is the Heaviside function. The velocity of the free boundary $s(t)$ is defined by

$$C : I \to \mathbf{R}, \quad I = \left(-\frac{a}{2}, \frac{1-a}{2}\right)$$

and the explicit form of C is given by

$$C(v) = \frac{2v - \frac{1-2a}{2}}{\sqrt{(\frac{1-a}{2} - v)(v + \frac{a}{2})}}, \quad 0 < a < 1$$

This problem has its origins in some work by P. Fife ([2], [3]) on asymptotic analysis of the dynamics of internal layers in reaction diffusion equations. The free boundary problem is an outgrowth of work done by M. Mimura, Y. Nishiura and their coworkers ([4–7]). These authors take as a starting point a system of two reaction diffusion equations

$$\varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v), \quad v_t = Dv_{xx} + g(u,v) \tag{2}$$

depending on two small parameters ε, τ. Here u and v measure the levels of two diffusing quantities. The functions u and v are assumed to satisfy Dirichlet boundary conditions at $x = 0, 1$. The functions f and g are assumed to be of bistable type, i.e., the equation $f = 0$ determines u as a triple valued function of v and the curves defined by $f = 0$, $g = 0$ have three points of intersection, which determine all of the interactions between u and v. The term bistable refers to the fact that these points of intersection correspond to equilibria of the system (2), two of which are stable, the third unstable.

When ε and τ are chosen to be very small, the system (2) models a situation in which the quantity measured by u reacts much faster than that measured by v (τ small), while at the same time u diffuses slower than v (ε small). The principal interest in systems like (2) comes from the fact that there exist families of stationary solutions parametrized by ε, which approach discontinuous functions of x as $\varepsilon \to 0$. When ε is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place within an x-interval of length $O(\varepsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called a layer – a boundary layer when it is adjacent to an endpoint of the interval or an internal layer when it is in the interior of the interval.

In 1981, Mimura, Tabata and Hosono ([4]) proved the existence of nontrival internal layer solutions to the stationary (time-independent) problem associated with (2). The question of the stability of these stationary layer solutions when ε is small was later dealt with in a pair of papers; one by Nishiura and Fujii ([5]) appearing in 1987 for the case where τ is large and the solution is asymptotically stable and the second in 1989 by Nishiura and Mimura ([6]) for the case where τ is small and there is a breakdown in the stability of the stationary solutions as τ approaches 0. In the latter paper, a particularly dramatic phenomenon occurs as the stationary solutions lose stability. The loss of stability results from a Hopf bifurcation and produces a kind of periodic oscillation in the location of the internal layers. (The amplitudes of the solutions also undergo a somewhat less pronounced periodic oscillation.) These periodic solutions are called "breathers" or "breathing solutions" because of the nature of the oscillation in the position of the internal layers.

In this paper, we are interested in the singular limit $\varepsilon \downarrow 0$ of the system (2). In this case, an analysis of the layer solutions suggests that the layer of width $O(\varepsilon)$ converges to an interfacial curve $x = s(t)$ in x, t-space as $\varepsilon \downarrow 0$. An analysis of the dynamics of this process has been shown (See for example [6, 7]) to lead a free boundary problem consisting of the initial-boundary value problem

$$\begin{cases} v_t^{\pm} = Dv_{xx}^{\pm} + g(h^{\pm}(v), v) & \text{for } (x,t) \in \Omega_{\pm} \\ v^-(0,t) = 0 = v^+(1,t) & \text{for } t > 0 \\ v^-(s(t),t) = v^+(s(t),t) & \text{for } t > 0 \\ v_x^-(s(t),t) = v_x^+(s(t),t) & \text{for } t > 0 \\ v^{\pm}(x,0) = v_0(x) & \end{cases} \tag{3}$$

together with an initial value problem for the interface

$$\frac{ds}{dt} = \frac{1}{\tau} C(v(s(t),t)), \quad t > 0; \quad s(0) = s_0 \tag{4}$$

Here Ω^-, Ω^+ have the same meaning introduced earlier in the problem (1). The function $C(v)$ in (4), which specifies the evolution of the interface $s(t)$, is determined from the first equation in (2) by using asymptotic techniques. Details in the derivation of (3), (4) from (2) can be found in the references [2, 6, 8].

In the present work, we establish the occurrence of a Hopf bifurcation as $\tau \downarrow 0$ in the free boundary problem (3), (4). This free boundary problem comes from the problem (2) where the reaction terms f and g are of the generalized type investigated by McKean [9], namely

$$f(u,v) = \begin{cases} -u - v & \text{for } u < -\dfrac{a}{2} \\ u - v - a & \text{for } -\dfrac{a}{2} < u < \dfrac{1-a}{2} \\ -u - v + 1 & \text{for } u > \dfrac{1-a}{2} \end{cases}$$

and

$$g(u,v) = u - bv$$

Since we assume that f and g are of the bistable type, b must satisfy $b > \dfrac{1+a}{1-a}$. The velocity of the interface, $C(v)$ may be represented by $h^+(v)+h^-(v)-2h^0(v)$ but it must be defined in the interval $(v_{\min}, v_{\max})$ where $v_{\max} = \dfrac{1-a}{2}$ and $v_{\min} = -\dfrac{a}{2}$. Therefore, it can be normalized by

$$C(v) = \frac{2v - \frac{1-2a}{2}}{\sqrt{(\frac{1-a}{2} - v)(v + \frac{a}{2})}}$$

The corresponding picture of the nullclines is illustrated in Figure 1.

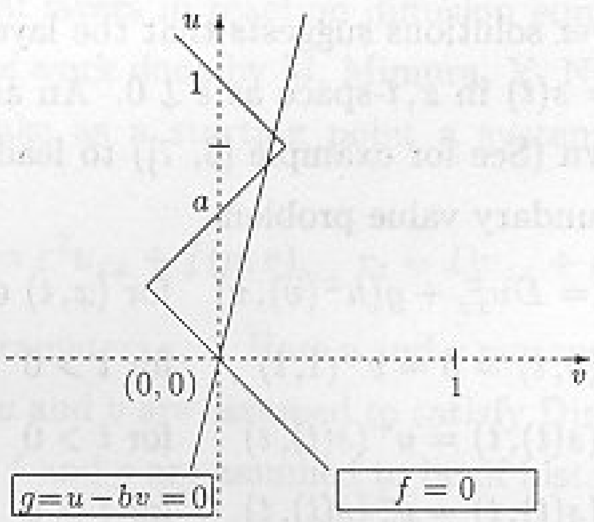

Fig.1 Generalized McKean reaction terms.

In this paper, we will establish the existence of the Hopf bifurcation described above by an application of the implicit function theorem along the lines of the results in [10]. In order to apply the implicit function theorem, we require more regularity of the solution than that obtained in the paper by Hilhorst, Nishiura and Mimura cited earlier [7]. Our approach to the problem of well-posedness and to the Hopf bifurcation is to write (1) in the form of an abstract evolution equation on a Banach space which is the product of a function space and an interval of real numbers. Once we have done this, we are able to apply standard results from the theory of nonlinear evolution equations (See for instance [11]) to show well-posedness of the problem and, more importantly, give an analysis of the Hopf bifurcation.

Before we proceed to the main results of the paper, we first point out a particular problem which arises in the formulation of (1) as an abstract evolution equation. Briefly stated, the idea is to write (1) in the form

$$\frac{d(v,s)}{dt} + A(v,s) = F(v,s), \quad (v,s)(0) = (v_0(\cdot), s_0)$$

of a differential equation in a Banach space X of the form $\tilde{X} = X \times I$, where X is a space of functions and I is a real interval. For the problem (1) this could be done, for instance, by identifying the operator A, represented in matrix form, as

$$A := \begin{pmatrix} -D\dfrac{d^2}{dx^2} + b + 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the nonlinear operator F by

$$F(v,s) = \begin{pmatrix} F_1(v(\cdot,t), s(t)) \\ F_2(v(\cdot,t), s(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) \\ \dfrac{1}{\tau} C(v(s(t),t)) \end{pmatrix}$$

The Dirichlet boundary conditions are incorporated in the definition of the Banach space X.

The difficulty comes from the fact that the nonlinear forcing term $F(v,s)$ contains a Heaviside function in its first component. The combination of this jump discontinuity and the nature of the dependence of v on s in the second component of F makes it impossible to find a function space of the form $X = L_p$, $1 \le p \le \infty$ such that F satisfies a Lipschitz condition on $\tilde{X} \subset X \times \mathrm{R}$.

In next section, Section 2, a change of variables is given which regularizes the problem (1) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In Section 3, we show that as τ decreases, a Hopf bifurcation occurs at a critical value of τ_c.

2. Regularization, Existence, Uniqueness and Dependence on Initial Conditions

We now examine a free boundary value problem depending on a parameter $\nu \in \mathrm{R}$, $\nu = 1/\tau$ of the form

$$\begin{cases} v_t + Av = H(x-s) & (x \in (0,1)\backslash\{s\}, t > 0) \\ s'(t) = \nu C(v(s(t),t)) & (t > 0) \\ v(x,0) = v_0(x), s(0) = s_0 \end{cases} \tag{F}$$

Here A is the operator $Av = -v_{xx} + (1+b)^2 v$ together with Dirichlet boundary conditions $v(0) = v(1) = 0$. Note that by a rescaling of t in (1) we can always achieve that $D = 1$. For the purposes of the results in this section, A can also be any other invertible second order operator. For the application of semigroup theory to (F), we choose the space

$$X := L_2((0,1)) \quad \text{with norm } \|\cdot\|_2$$

The operator A can be considered as a densely defined operator

$$\begin{cases} A : D(A) \subset_{\text{dense}} X \to X \\ D(A) := \{v \in H^{2,2}((0,1)) : v(0) = v(1) = 0\} \end{cases}$$

Definition 2.1 *We call (v,s) a solution of* (F), *if it satisfies the following natural properties: There exists $T > 0$ such that $v(x,t)$ is defined for $(x,t) \in [0,1] \times [0,T)$, $s(t) \in (0,1)$ and $v(s(t),t) \in I$ for $t \in [0,T)$,*

a) *$v(\cdot,t) \in C^1([0,1])$ for $t > 0$ with $v(0,t) = v(1,t) = 0$,*
b) *$s \in C^0([0,T)) \cap C^1((0,T))$ with $s(0) = s_0 \in (0,1)$,*
c) *$(Av)(x,t)$ and $v_t(x,t)$ exist for $x \in (0,1)\backslash\{s(t)\}$ and $t \in (0,T)$,*
d) *$t \mapsto v(\cdot,t) \in C^0([0,T), X)$ with $v(\cdot,0) = v_0 \in X$ and*
e) *v and s solve the differential equation for $t \in (0,T)$ and $x \in (0,1)\backslash\{s(t)\}$.*

For fixed s satisfying Definition 2.1, the map $t \mapsto H(\cdot - s(t))$ is locally Hölder-continuous into X on $(0,T)$, so by standard results for parabolic problems (See e.g. [11]) we obtain from the first equation in (F) that the following regularity holds for v:

Proposition 2.2 *If (v,s) is a solution of* (F), *then $v(\cdot,t) \in D(A)$ and the map $t \mapsto v(\cdot,t)$ is in $C^0([0,T),X) \cap C^1((0,T),X)$.*

An existence proof for (F) can be obtained along these lines (See [7]), but it is impossible to get differentiable dependence on initial conditions this way, because the right hand side $H(\cdot - s)$ is not regular enough, and it is this differentiability that is needed for an application of the Hopf bifurcation theorem. To remedy this difficulty, we decompose v in (F) into a part u, which is a solution to a more regular problem, and a part g, which is explicitly known in terms of the Green's function G of the operator A.

Proposition 2.3 *Let $G : [0,1]^2 \to \mathbf{R}$ be a Green's function of the operator A. Define $g : [0,1]^2 \to \mathbf{R}$*

$$g(x,s) := \int_s^1 G(x,y)dy = A^{-1}(H(\cdot - s))(x)$$

and $\gamma : [0,1] \to \mathbf{R}$

$$\gamma(s) := g(s,s)$$

Then $g(\cdot,s) \in D(A)$ for all s, $\dfrac{\partial g}{\partial s}(x,s) = -G(x,s)$ is in $H^{1,\infty}((0,1)\times(0,1))$, and $\gamma \in C^\infty([0,1])$.

Proof Everything follows from the fact that G is in $H^{1,\infty}$ and C^∞ on either $\{x \le y\}$ or $\{x \ge y\}$, and that $H(\cdot - s) \in L^2$.

Using these preliminary observations, we decompose a solution (v, s) of (F) into two parts by defining

$$u(t)(x) := v(x,t) - g(x, s(t))$$

Then

$$\begin{cases} u'(t)(x) + (Au(t))(x) = \nu C(u(t)(s(t)) + \gamma(s(t))) \cdot G(x, s(t)) \\ s'(t) = \nu C(u(t)(s(t)) + \gamma(s(t))) \end{cases}$$

We denote the space $X \times \mathbf{R}$ by $\widetilde{X}$ and define

$$\begin{cases} D(\widetilde{A}) := D(A) \times \mathbf{R} \\ \widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \to \widetilde{X}, \widetilde{A}(u,s) := (Au, 0) \end{cases}$$

This system can be written as an abstract evolution equation with initial conditions

$$\begin{cases} \dfrac{d}{dt}(u,s) + \widetilde{A}(u,s) = \nu f(u,s) \\ (u,s)(0) = (u(0), s(0)) = (u_0, s_0) \end{cases} \tag{R}$$

The nonlinear forcing term f is

$$f(u,s) = \begin{pmatrix} \nu C(u(t)(s(t)) + \gamma(s(t))) \cdot G(x, s(t)) \\ \nu C(u(t)(s(t)) + \gamma(s(t))) \end{pmatrix}$$

and defined on the set $W := \{(u,s) \in C^1([0,1]) \times (0,1) : u(s) + \gamma(s) \in I\} \subset_{\text{open}} C^1([0,1]) \times \mathbf{R}$ as follows

$$\begin{cases} f : W \to X \times \mathbf{R}, f(u,s) := f_2(u,s) \cdot (f_1(s), 1), \quad \text{where} \\ f_1 : (0,1) \to X, f_1(s)(x) := G(x,s), \quad \text{and} \\ f_2 : W \to \mathbf{R}, f_2(u,s) := C(u(s) + \gamma(s)) \end{cases}$$

The advantage of (R) over (F) is, that the right hand side of (R) is one step more regular than that of (F) since it involves $G(x, s)$ instead of $H(x - s)$. More precisely, we can show the following:

Lemma 2.4 *The functions $f_1 : (0,1) \to X$, $f_2 : W \to \mathbf{R}$ and $f : W \to \widetilde{X}$ are continuously differentiable with derivatives given by*

$$f_1'(s) = \frac{\partial G}{\partial y}(\cdot, s)$$

$$Df_2(u,s)(\widehat{u}, \widehat{s}) = C'(u(s) + \gamma(s)) \cdot (u'(s)\widehat{s} + \gamma'(s)\widehat{s} + \widehat{u}(s))$$

$$Df(u,s)(\widehat{u}, \widehat{s}) = f_2(u,s) \cdot (f_1'(s), 0) \cdot \widehat{s} + Df_2(u,s)(\widehat{u}, \widehat{s})) \cdot (f_1(s), 1)$$

We can now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0,1]$ of A and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha), \quad \widetilde{X}^\alpha := D(\widetilde{A}^\alpha); \quad \widetilde{X}^\alpha = X^\alpha \times \mathrm{R}$$

For this we need to find an $\alpha \in (0,1)$ such that $X^\alpha \subset C^1([0,1])$ because $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is continuously differentiable. Theorem 1.6.1 in [11], for example, ensures that this is the case for $\alpha > 3/4$. Standard applications of theorems for existence, uniqueness and dependence on initial conditions (cf. [11]) together with the starting regularity of solutions to (F) (Proposition 2.2), as well as the regularity of the functions g and γ (Proposition 2.3) then give the following result:

Theorem 2.5 i) *For any $1 > \alpha > 3/4$, $(u_0, s_0) \in W \cap \widetilde{X}^\alpha$ and $\nu \in \mathrm{R}$ there exists a unique solution*

$$(u,s)(t) = (u,s)(t; u_0, s_0, \nu)$$

of (R). *The solution operator*

$$(u_0, s_0, \nu) \mapsto (u,s)(t; u_0, s_0, \nu)$$

is continuously differentiable from $\widetilde{X}^\alpha \times \mathrm{R}$ into $\widetilde{X}^\alpha$ for $t > 0$. The functions $v(x,t)$

$$v(x,t) := u(t)(x) + g(x, s(t))$$

and s then satisfy (F) *with $v(\cdot, 0) \in X^\alpha$, $v(s_0, 0) \in I$.*

ii) *If (v,s) is a solution of* (F) *for some $\nu \in \mathrm{R}$ with initial condition $v_0 \in X^\alpha$, $1 > \alpha > 3/4$, $s_0 \in (0,1)$, $v_0(s_0) \in I$, then $(u_0, s_0) := (v_0 - g(\cdot, x_0), s_0) \in \widetilde{X}^\alpha \cap W$ and*

$$(v(\cdot, t), s(t)) = (u,s)(t; u_0, s_0, \nu) + (g(\cdot, s(t)), 0)$$

where $(u,s)(t; u_0, s_0, \nu)$ is the unique solution of (R).

iii) *For any $1 > \alpha > 3/4$, $\nu \in \mathrm{R}$, $(v_0, s_0) \in U := \{(v,s) \in X^\alpha \times (0,1) : v(s) \in I\}$ the problem* (F) *has a unique solution*

$$(v(x,t), s(t)) = (v,s)(x, t; v_0, s_0, \nu)$$

Additionally, the mapping

$$(v_0, s_0, \nu) \mapsto (v,s)(\cdot, t; v_0, s_0, \nu)$$

is continuously differentiable from $X^\alpha \times \mathrm{R}^2$ into $X^\alpha \times \mathrm{R}$.

3. Stationary Solutions and Hopf Bifurcation

In this section, we shall show the Hopf bifurcation occurs for some μ. The stationary problem, corresponding to (R) is given by

$$Au^* = 1/\tau G(x,s^*)C(u^*(s^*)+\gamma(s^*)), \quad u^*(0) = 0 = u^*(1)$$
$$0 = 1/\tau C(u^*(s^*)+\gamma(s^*))$$

for $(u^*, s^*) \in D(\widetilde{A}) \cap W$. The function $\gamma(s) = \int_s^1 G(s,y)dy$ then becomes

$$\gamma(s) = \frac{\sinh(1+b)s}{(1+b)^2 \sinh(1+b)}(\cosh(1+b)(1-s)-1)$$

For $\tau \neq 0$, this system is equivalent to the pair of equations

$$u^* = 0, \quad C(\gamma(s^*)) = 0 \tag{5}$$

We thus obtain

Proposition 3.1 *If* $0 < \dfrac{1-2a}{4} < \dfrac{2\sinh^3\frac{(1+b)}{3}}{(1+b)^2\sinh(1+b)}$, *then* (R) *has a unique stationary solution* $(0, s^*)$ *for all* $\tau \neq 0$ *in* $s^* \in (0, \frac{1}{3})$ *or* $(\frac{1}{3}, 1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0,s^*)(\widehat{u},\widehat{s}) = 8(\widehat{u}(s^*)+\gamma'(s^*)\widehat{s})\cdot(G(\cdot,s^*),1)$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of* (F) *for* $\tau \neq 0$ *with*

$$v^*(x) = g(x, s^*)$$

Proof We note $C(r) = 0$ if and only if $r = \dfrac{1-2a}{4}$. Since $\gamma'(s) > 0$ for $0 < s < \dfrac{1}{3}$ and $\gamma'(s) < 0$ for $\dfrac{1}{3} < s < 1$, the equation (5) is solvable with $s^* \in (0,1)$ if and only if $\gamma(0) < \dfrac{1-2a}{4} < \gamma\left(\dfrac{1}{3}\right)$ or $\gamma\left(\frac{1}{3}\right) < \dfrac{1-2a}{4} < \gamma(1)$. We note that $\gamma\left(\dfrac{1}{3}\right) = \dfrac{2\sinh^3\frac{1+b}{3}}{(1+b)^2\sinh(1+b)}$. Thus, there exists a unique solution s^* in $\left(0, \dfrac{1}{3}\right)$ or $\left(\dfrac{1}{3}, 1\right)$.

The formula for $Df(0, s^*)$ follows from Lemma 2.4 and the relation $C'((1-2a)/4) = 8$. The corresponding steady state (v^*, s^*) for (F) is obtained by using Theorem 2.5.

Since $C'((1-2a)/4) = 8$, we define the new parameter $\mu = 8/\tau$. In order to show the occurrence of Hopf bifurcations at some μ^* in (R), we must show the stationary solution $(u^*(x), s^*, \mu^*)$ is a Hopf point and then there is a periodic solution near the stationary point by the Hopf bifurcation theorem in [10] and [1]. We introduce the definition of Hopf points.

Definition 3.2 *Under the assumptions of Proposition 3.1, define (for $1 \geq \alpha > 3/4$) the operator $B \in L(\widetilde{X}^\alpha, \widetilde{X})$*

$$B := \frac{1}{8} Df(0, s^*)$$

We then define $(0, s^*, \mu^*)$ to be a Hopf point *for* (R) *if and only if there exists an $\varepsilon_0 > 0$ and a C^1-curve*

$$(-\varepsilon_0 + \mu^*, \mu^* + \varepsilon_0) \mapsto (\lambda(\mu), \phi(\mu)) \in \mathbf{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendate for $-\widetilde{A} + \mu B$ with

i) $(-\widetilde{A} + \mu B)(\phi(\mu)) = \lambda(\mu)\phi(\mu)$, $(-\widetilde{A} + \mu B)\overline{\phi(\mu)} = \overline{\lambda(\mu)}\,\overline{\phi(\mu)}$;

ii) $\lambda(\mu^*) = i\beta$ *with* $\beta > 0$;

iii) $\mathrm{Re}\,(\lambda) \neq 0$ *for all* $\lambda \in \sigma(-\widetilde{A} + \mu^* B) \backslash \{\pm i\beta\}$;

iv) $\mathrm{Re}\,\lambda'(\mu^*) \neq 0$ *(transversality).*

We now check (R) for Hopf points. First we have to solve the eigenvalue problem

$$-\widetilde{A}(u, s) + \mu B(u, s) = \lambda(u, s)$$

where $v = u - G(\cdot, s^*)$. This problem is equivalent to

$$\begin{aligned} (A + \lambda)u &= \mu \cdot (\gamma'(s^*)s + u(s^*)) \cdot G(\cdot, s^*) \\ \lambda s &= \mu \cdot (\gamma'(s^*)s + u(s^*)) \end{aligned} \tag{6}$$

We now shall show that there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (6) with $\beta > 0$ for some μ^* in order for $(0, s^*, \mu^*)$ to be a Hopf point. As a first result, we show the $(0, s^*, \mu^*)$ satisfies the condition of (i) and (ii) in Definition 3.2.

Lemma 3.3 *Assume that the operator $-\widetilde{A} + \mu^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues for $\mu^* \in \mathbf{R}^+$. Suppose that ϕ^* be the (normalized) eigenfunction corresponding to the eigenvalue $i\beta$. Then there exists a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$.*

The proof is similar to the proof of Theorem 3.4 in [1].

In order to show the transversality condition, we use some equations from the proof of Theorem 10 in [1].

The equation (6) to λ being an eigenvalue of $-\widetilde{A} + \mu B$ with eigenfunction $(u, 1)$ is equivalent to the following equation $E(u, \lambda, \mu) = 0$ where

$$E : D(A)_{\mathbf{C}} \times \mathbf{C} \times \mathbf{R} \to X_{\mathbf{C}} \times \mathbf{C}$$

by

$$E(u, \lambda, \mu) := ((A + \lambda)u - \mu(\gamma'(s^*) + u(s^*)) \cdot G(\cdot, s^*), \lambda - \mu(\gamma'(s^*) + u(s^*)))$$

satisfying $E(\psi_0, i\beta, \mu^*) = 0$ where $\phi^* := (\psi_0, 1) \in D(A) \times \mathbf{R}$ is the (normalized) eigenfunction corresponding to the eigenvalue $i\beta$ and shows that E is continuously differentiable. The following theorem gives that $(0, s^*, \mu^*)$ is a Hopf point for (R):

Theorem 3.4 *Assume that for $\mu^* \in \mathbf{R}^+$, the operator $-\widetilde{A} + \mu^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues. Then $(0, s^*, \mu^*)$ is a Hopf point for* (R).

Proof It remains to be shown that the transversality condition $\operatorname{Re} \lambda'(\mu^*) \neq 0$ holds. Let $\phi(\mu) = (\psi(\mu), 1)$. Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ implies that

$$D_{(u,\lambda)} E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) = (\gamma'(s^*) + \psi_0(\mu^*)(s^*)) \cdot (G(\cdot, s^*), 1)$$

This means that the function $\widehat{u} := \psi'(\mu^*)$ and $\widehat{\lambda} := \lambda'(\mu^*)$ satisfy the equations

$$(A + i\beta)\widehat{u} - \mu^* \widehat{u}(s^*) G(\cdot, s^*) + \widehat{\lambda}\psi_0 = (\gamma'(s^*) + \psi_0(\mu^*)(s^*)) G(\cdot, s^*) \tag{7}$$

and

$$-\mu^* \widehat{u}(s^*) + \widehat{\lambda} = \gamma'(s^*) + \psi_0(\mu^*)(s^*) \tag{8}$$

Putting (8) into (7) and using $\psi_1 := \psi_0 - G(\cdot, s^*)$, as before, we obtain

$$(A + i\beta)\widehat{u} + \widehat{\lambda}\psi_1 = 0 \tag{9}$$

Multiplying $\overline{\widehat{u}}$ by (9) and integrating it then we obtain

$$\begin{aligned} -\overline{\widehat{u}(s^*)} &= \int_0^1 (A + i\beta)\psi_1 \overline{\widehat{u}} = \int_0^1 \psi_1 \overline{(A + i\beta)\widehat{u}} + \int_0^1 2i\beta \overline{\widehat{u}} \psi_1 \\ &= -\overline{\widehat{\lambda}} \int_0^1 |\psi_1|^2 + 2i\beta \int_0^1 \psi_1 \overline{\widehat{u}} \end{aligned} \tag{10}$$

Furthermore, multiply $(A + i\beta)\overline{\widehat{u}}$ to (9) and integrate it, then

$$\int_0^1 ((A^2 - \beta^2)\widehat{u}\overline{\widehat{u}} + 2i\beta A \widehat{u}\overline{\widehat{u}}) = \widehat{\lambda}\overline{\widehat{u}(s^*)}$$

and integrate (9) after multiplied by $\overline{(A + i\beta)\widehat{u}}$, then

$$\int_0^1 ((A^2 + \beta^2)\widehat{u}\overline{\widehat{u}} - 2i\beta\widehat{\lambda}\psi_1 \overline{\widehat{u}}) = \widehat{\lambda}\overline{\widehat{u}(s^*)}$$

Subtract one from the other, we now obtain

$$2\beta^2 \int_0^1 \widehat{u}\overline{\widehat{u}} = 2i\beta \int_0^1 (A\widehat{u}\overline{\widehat{u}} + \widehat{\lambda}\psi_1 \overline{\widehat{u}}) \tag{11}$$

Multiplying $\widehat{\lambda}$ by (10) and subtracting it to (11), and using the fact $\mu^* \int_0^1 |\psi_1|^2 = 1$ (See (16) in [1]), we have

$$2\beta^2 \int_0^1 \widehat{u}\overline{\widehat{u}} + \widehat{\lambda}\overline{\widehat{u}(s^*)} = 2i\beta \int_0^1 A\widehat{u}\overline{\widehat{u}} + \frac{|\widehat{\lambda}|^2}{\mu^*}$$

From the fact $E(\psi_0, i\beta, \mu^*) = 0$ and (8), the above equation follows

$$2\beta^2 \int_0^1 \widehat{u}\overline{\widehat{u}} + \frac{|\widehat{\lambda}|^2}{\mu^*} + i\beta\frac{\widehat{\lambda}}{\mu^{*2}} = 2i\beta \int_0^1 A\widehat{u}\overline{\widehat{u}} + \frac{|\widehat{\lambda}|^2}{\mu^*}$$

and hence the real part of the $\widehat{\lambda} = \lambda'(\mu^*)$ is given by

$$\mathrm{Re}\,(\lambda'(\mu^*)) = 2\mu^{*2} \int_0^1 A\widehat{u}\overline{\widehat{u}} = 2\mu^{*2} \int_0^1 |A^{1/2}\widehat{u}|^2$$

Hence the transversality condition holds for all $\mu^* > 0$. Therefore, by the Hopf-bifurcation theorem in [1], there exists a family of periodic solutions which bifurcates from the stationary solution as μ passes μ^*.

As a final result we will now show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point, thus μ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 3.5 *There exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (6) with $\beta > 0$ for a unique critical point $\mu^* > 0$ in order for $(0, s^*, \mu^*)$ to be a Hopf point with $\frac{1}{3} < s^* < 1$.*

Proof To do this, we have only to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$\begin{aligned} (A + i\beta)u &= \mu \cdot (\gamma'(s^*) + u(s^*)) \cdot G(\cdot, s^*) \\ i\beta &= \mu \cdot (\gamma'(s^*) + u(s^*)) \end{aligned}$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$\begin{aligned} (A + i\beta)v &= -\delta_{s^*} \\ i\beta &= \mu \cdot (\gamma'(s^*) + G(s^*, s^*) + v(s^*)) \end{aligned} \tag{12}$$

(Actually, this is just the eigenvalue problem for the formal linearization of (F) about (v^*, s^*).)

Now the first equation in (12) has, for fixed $\beta \geq 0$, the unique solution $v = -G_\beta(\cdot, s^*)$, where G_β is the Green's function for the operator $A + i\beta$. We are thus left with having to solve the complex valued equation

$$i\beta = \mu \cdot (\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*))$$

Since $\gamma'(s^*)+G(s^*,s^*)$ is real valued, this is equivalent to the real valued system

$$\gamma'(s^*)+G(s^*,s^*)-\operatorname{Re}G_\beta(s^*,s^*)=0 \tag{13}$$

$$\mu\cdot\operatorname{Im}G_\beta(s^*,s^*)+\beta=0 \tag{14}$$

Since the equation (13) does not depend on μ, it suffices to find a unique solution $\beta>0$ of (13), from this β the unique $\mu^*>0$ can then be calculated by using (14), provided $\operatorname{Im}G_\beta(s^*,s^*)$ is negative. By Lemma 12 in [12], the expression $\operatorname{Re}G_\beta(s^*,s^*)$ is strictly decreasing in $\beta\in\mathbf{R}^+$ with

$$\operatorname{Re}G_0(s^*,s^*)=G(s^*,s^*),\quad \lim_{\beta\to\infty}\operatorname{Re}G_\beta(s^*,s^*)=0$$

and $\operatorname{Im}G_\beta(s^*,s^*)<0$ for any $\beta>0$. Since $\gamma'(s^*)+G(s^*,s^*)>0$ and $\gamma'(s^*)<0$ for $\frac{1}{3}<s^*<1$, a unique solution (β,μ^*) of (13) and (14) with $\beta>0$ and $\mu^*>0$ exists for $\frac{1}{3}<s^*<1$.

The following theorem summarizes what we have proved for the free boundary problem with the Dirichlet boundary condition:

Theorem 3.6 *Assume that* $0<\frac{1-2a}{4}<\frac{2\sinh^3\frac{1+b}{3}}{(1+b)^2\sinh(1+b)}$, *so that* (R), *respectively* (F), *has a unique stationary solution* (u^*,s^*) *where* $u^*=0$ *and* $\frac{1}{3}<s^*<1$ *respectively* (v^*,s^*), *for all* $\mu>0$. *Then there exists a unique* μ^* *such that the linearization* $-\tilde{A}+\mu^*B$ *has a purely imaginary pair of eigenvalues. The point* $(0,s^*,\mu^*)$ *is then a Hopf point for* (R) *and there exists a* C^0*-curve of nontrivial periodic orbits for* (R), (F), *respectively, bifurcating from* $(0,s^*,\mu^*)$, (v^*,s^*,μ^*), *respectively.*

Acknowledgements The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year in 1997.

References

[1] Ham Y.M. Schaaf R. and Thompson R., A Hopf bifurcation in a parabolic free boundary problem, *J. of Comput. Appl. Math.*, **52** (1994), 305–324.

[2] Fife P., Current topics in reaction-diffusion systems, in Nonequilibrium Cooperative Phenomena in Physics and Related Fields, M.G. Velarde, ed. Plenum Press, New York, (1984), 371–441

[3] Fife P., Dynamics of internal layers and diffusive interfaces, CMBS-NSF Regional Convergence Series in Applied Mathematics, **53** Philadelphia SIAM 1988.

[4] Mimura M., Tabata M. and Hosono Y., Multiple solutions of two-point boundary value problems for Neumann type with a small parameter, *SIAM J. Math. Anal.*, **11** (1981), 613–633.

[5] Nishiura Y. and Fujii H., Stability of singularly perturbed solutions to systems of reaction-diffusion equations, *SIAM J. Math. Anal.*, **18** (1987), 1726–1770.

[6] Nishiura Y. and Mimura M., Layer oscillations in reaction-diffusion systems, *SIAM J. Appl. Math.*, **49** (1989), 481–514.

[7] Hilhorst D, Nishiura Y. and Mimura M., A free boundary problem arising in some reacting-diffusing system, *Proc. Royal Soc. Edinburgh*, **118A** (1991), 355–378.

[8] Keener J.P., A geometrical theory for spiral waves in excitable media, *SIAM J. Appl. Math.*, (1986), 1039–1056.

[9] McKean H.P., Nagumo's equation, *Adv. Math.*, **4** (1975), 209–223.

[10] Crandall M.G. and Rabinowitz P.H., The Hopf bifurcation theorem in infinite dimensions, *Arch. Rational Mech. Anal.*, **67** (1978), 53–72.

[11] Henry D., Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics **840**, Springer-Verlag, New York-Berlin, 1981.

[12] Smoller J.A., Shock Waves and Reaction-Diffusion Equations, Grundlehren **258**, Springer-Verlag, New York-Berlin, 1983.

[13] Ham Y.M., A Hopf bifurcation in a parabolic free boundary problem with double layers, *J. of partial Differential Equation*, **9** (1996) 277–288.