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THE MOTION OF SURFACE WITH CONSTANT NEGATIVE GAUSSIAN CURVATURE*

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Abstract In this paper, an approach is suggested to consider the relation between the integrable equation and the motion of surface with constant negative Gaussian curvature.

Key Words Integrable equations; motion of surface; surface with constant negative Gaussian curvature.

Classification 35Q53, 53A05, 53C42.

1. Introduction

There are many interesting phenomena which are related to dynamics of the motion of surface, those equations had been considered in [1-4]. In [4], the authors considered that the motion of the constant negative curvature is specified by a single evolution equation for the angle between the asymptotic coordinate line, they showed that one choice for the normal velocity of the surface ($w = \theta_x$) leads to the modified KdV equation for the surface evolution, but they do not know whether more complicated evolution (e.g., $w = \theta_x + \theta_y$) can give rise to integrable equations.

As we know, in the case of evolution of curves, a geometrical formulation gives the evolution equation for the curvature and torsion of curve. The system contains many integrable equations[5-9], our interest is to get the corresponding formulation which gives a compact description of the motion of surface from the view point of mathematics.

In [10], Bobenko reformulates the classical theory of surface in a form familiar to the soliton theory which makes possible an application of the analytical methods of this theory to integrable cases. We find that this formulation is convenient to be used for considering the relation between the integrable equation and the motion of surface. In this paper, we extend this method to consider the motion of surface with constant

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negative curvature, we can not only answer the question mentioned above, but also give explicit expression of the motion of the surface. In Section 2, we deduce the five fundamental equations of the motion of surface from thirteen equations, which are deduced from the compatibility conditions. In Section 3, we take $\omega = \alpha\theta_x + \beta\theta_y$ and give one soliton solution of the evolution equation, then we deduce the explicit expression of the motion of surface. To keep the reasonable paragraph in this paper, the multisoliton solution of the evolution equation and the motion of positive constant Gaussian curvature surface and the motion of constant mean curvature of surface will be published elsewhere.

2. The Five Fundamental Equations of the Motion of Surface

By using the notation in [10], we identify a 3-dimensional Euclidean space with the space imaginary $\mathrm{Im}H$.

$$X = -i\sum_{j=1}^{3} X_j\sigma_j \in \mathrm{Im}H \longleftrightarrow X = (X_1, X_2, X_3) \in \mathbf{R}^3 \tag{2.1}$$

σ_j $(j = 1, 2, 3)$ are Pauli matrices. The scalar product of vector in terms of matrices is then

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{Tr}\, XY \tag{2.2}$$

Let us consider the surface $F = (F_1, F_2, F_3)$ with constant negative Gaussian curvature $k = -1$, we use the asymptotic coordinate line, and by the scalar transform $(x, y) \to (\lambda x, y/\lambda)$, the fundamental forms are as follows

$$I = \langle dF, dF\rangle = \lambda^2 dx^2 + 2\cos\theta dxdy + \frac{1}{\lambda^2}dy^2 \tag{2.3}$$

$$II = -\langle dF, dN\rangle = 2\langle F_{xx}, N\rangle dxdy = 2\sin\theta dxdy \tag{2.4}$$

where θ is the angle between the asymptotic line and

$$F_x = -i\lambda\phi\begin{pmatrix} 0 & e^{-i\theta\frac{1}{2}} \\ e^{i\theta\frac{1}{2}} & 0\end{pmatrix}\phi, F_y = -i\frac{1}{\lambda}\phi\begin{pmatrix} 0 & e^{i\theta\frac{1}{2}} \\ e^{-i\theta\frac{1}{2}} & 0\end{pmatrix}\phi, N = -i\phi^{-1}\sigma_3\phi \tag{2.5}$$

and $\phi = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a}\end{pmatrix}$ satisfies the equations

$$\phi_x = U\phi, \quad U = i\begin{pmatrix} \frac{\theta_x}{4} & -\frac{\lambda}{2}e^{-i\frac{\theta}{2}} \\ -\frac{\lambda}{2}e^{i\frac{\theta}{2}} & -\frac{\theta_x}{4}\end{pmatrix} \tag{2.6}$$

$$\phi_y = V\phi, \quad V = i\begin{pmatrix} -\frac{\theta_y}{4} & \frac{1}{2\lambda}e^{i\frac{\theta}{2}} \\ \frac{1}{2\lambda}e^{-i\frac{\theta}{2}} & \frac{\theta_y}{4}\end{pmatrix} \tag{2.7}$$

We assume that the velocity

$$F_t = uF_x + vF_y + wN = -i\phi^{-1}\begin{pmatrix} w & \bar{g} \\ g & -w \end{pmatrix}\phi$$

$$g = \left(\lambda u + \frac{v}{\lambda}\right)\cos\frac{\theta}{2} + i\left(\lambda u - \frac{v}{\lambda}\right)\sin\frac{\theta}{2} \tag{2.8}$$

In this section, we shall prove the theorem.

Theorem 1 *If the motion of the surface with the constant negative Gaussian curvature $k=-1$, then u, v, w and θ should satisfy the five fundamental equations,*

$$\theta_{xy} = \sin\theta \tag{2.9}$$

$$w_{xy} = w\cos\theta \tag{2.10}$$

$$2v_x\sin\theta + w_{xx} - \lambda^2 w - \theta_x w_x\cot\theta - \lambda^2\theta_x w_y\csc\theta = 0 \tag{2.11}$$

$$2u_y\sin\theta + w_{yy} - \frac{1}{\lambda^2}w - \theta_y w_y\cot\theta + \frac{1}{\lambda^2}\theta_y w_x\csc\theta = 0 \tag{2.12}$$

$$\dot{\theta} = 2w + (\theta_x u + \theta_y v) - (\lambda^2 u_y + \frac{1}{\lambda^2}v_x)\sin\theta, \dot{\theta} = \frac{\partial\theta}{\partial t} \tag{2.13}$$

First of all, we assume that (where w_j to be defined)

$$\phi_t = W\phi, \quad W = i\begin{pmatrix} w_3 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 \end{pmatrix} \tag{2.14}$$

It is easy to prove the following lemma.

Lemma 1 *From the condition $F_{xt} = F_{tx}$ and $F_{yt} = F_{ty}$, we get the equations (2.15)-(2.17) and (2.18)-(2.20) respectively,*

$$\sin\frac{\theta}{2}\left(\frac{\dot{\theta}}{4} + w_3\right) = \sin\frac{\theta}{2}\left(\frac{1}{2}w + \frac{1}{2}\theta_x u\right) - \frac{1}{2\lambda}\left(\lambda u_x + \frac{v_x}{\lambda}\right)\cos\frac{\theta}{2} \tag{2.15}$$

$$\cos\frac{\theta}{2}\left(\frac{\dot{\theta}}{4} + w_3\right) = \cos\frac{\theta}{2}\left(\frac{1}{2}w + \frac{1}{2}\theta_x u\right) + \frac{1}{2\lambda}\left(\lambda u_x - \frac{v_x}{\lambda}\right)\sin\frac{\theta}{2} \tag{2.16}$$

$$\cos\frac{\theta}{2}w_2 - \sin\frac{\theta}{2}w_1 = -\frac{1}{2\lambda}w_x - \frac{1}{2\lambda}v\sin\theta \tag{2.17}$$

$$\sin\frac{\theta}{2}\left(\frac{\dot{\theta}}{4} - w_3\right) = \sin\frac{\theta}{2}\left(\frac{1}{2}w + \frac{1}{2}\theta_y v\right) - \frac{\lambda}{2}\left(\lambda u_y + \frac{v_y}{\lambda}\right)\cos\frac{\theta}{2} \tag{2.18}$$

$$\cos\frac{\theta}{2}\left(\frac{\dot{\theta}}{4} - w_3\right) = \cos\frac{\theta}{2}\left(\frac{1}{2}w + \frac{1}{2}\theta_y v\right) - \frac{\lambda}{2}\left(\lambda u_y - \frac{v_y}{\lambda}\right)\sin\frac{\theta}{2} \tag{2.19}$$

$$\cos\frac{\theta}{2}w_2 + \sin\frac{\theta}{2}w_1 = -\frac{\lambda}{2}w_y - \frac{\lambda}{2}u\sin\theta \tag{2.20}$$

From $\cos\frac{\theta}{2}\cdot(2.15) - \sin\frac{\theta}{2}\cdot(2.16)$, it follows that

$$\lambda u_x + \frac{1}{\lambda}v_x\cos\theta = 0 \tag{2.21}$$

From $\sin\frac{\theta}{2}\cdot(2.15)+\cos\frac{\theta}{2}\cdot(2.16)$, it follows that

$$\frac{\dot\theta}{4}+w_3=\frac{1}{2}\omega+\frac{\theta_x u}{2}-\frac{1}{2}\frac{v_x}{\lambda^2}\sin\theta \tag{2.22}$$

From $\cos\frac{\theta}{2}\cdot(2.18)-\sin\frac{\theta}{2}\cdot(2.19)$, it follows that

$$\lambda u_y\cos\theta+\frac{1}{\lambda}v_y=0 \tag{2.23}$$

From $\sin\frac{\theta}{2}\cdot(2.18)+\cos\frac{\theta}{2}\cdot(2.19)$, it follows that

$$\frac{\dot\theta}{4}-w_3=\frac{1}{2}\omega+\frac{\theta_y v}{2}-\frac{1}{2}\lambda^2 u_y\sin\theta \tag{2.24}$$

Solving (2.17) and (2.20), we get

$$w_1=-\frac{1}{2}\Big(\lambda u-\frac{v}{\lambda}\Big)\cos\frac{\theta}{2}+\frac{1}{4}\Big(\frac{1}{\lambda}w_x-\lambda w_y\Big)\csc\frac{\theta}{2} \tag{2.25}$$

$$w_2=-\frac{1}{2}\Big(\lambda u+\frac{v}{\lambda}\Big)\sin\frac{\theta}{2}-\frac{1}{4}\Big(\frac{1}{\lambda}w_x+\lambda w_y\Big)\sec\frac{\theta}{2} \tag{2.26}$$

solving (2.22) and (2.24) we get (2.13) and

$$w_3=\frac{1}{4}\Big[\Big(\lambda^2 u_y-\frac{1}{\lambda^2}v_x\Big)\sin\theta+(\theta_x u-\theta_y v)\Big] \tag{2.27}$$

We have proved Lemma 2.

Lemma 2 *The expression* (2.15)–(2.20) *can be rewritten as* (2.21), (2.23), (2.13) *and* (2.25)–(2.27).

Lemma 3 *From the condition* $\phi_{xy}=\phi_{yx}$ *we get* (2.9) *and from the condition* $\phi_{xt}=\phi_{tx}$, $\phi_{yt}=\phi_{ty}$, *it yields the equations* (2.28)–(2.30) *and* (2.31)–(2.33) *respectively.*

Proof The equation (2.9) is reduced in [4]. By direct computation, we get

$$\frac{\dot\theta_x}{4}-w_{3x}+\lambda\Big(w_2\cos\frac{\theta}{2}-w_1\sin\frac{\theta}{2}\Big)=0 \tag{2.28}$$

$$w_{1x}\cos\frac{\theta}{2}+w_{2x}\sin\frac{\theta}{2}+\frac{\theta_x}{2}\Big(w_1\sin\frac{\theta}{2}-w_2\cos\frac{\theta}{2}\Big)=0 \tag{2.29}$$

$$\lambda\Big(\frac{\dot\theta}{4}+w_3\Big)-\Big(w_{1x}\sin\frac{\theta}{2}-w_{2x}\cos\frac{\theta}{2}\Big)+\frac{\theta_x}{2}\Big(w_1\cos\frac{\theta}{2}+w_2\sin\frac{\theta}{2}\Big)=0 \tag{2.30}$$

$$-\frac{\dot\theta_y}{4}-w_{3y}-\frac{1}{\lambda}\Big(w_2\cos\frac{\theta}{2}+w_1\sin\frac{\theta}{2}\Big)=0 \tag{2.31}$$

$$w_{1y}\cos\frac{\theta}{2}-w_{2y}\sin\frac{\theta}{2}+\frac{\theta_y}{2}\Big(w_1\sin\frac{\theta}{2}+w_2\cos\frac{\theta}{2}\Big)=0 \tag{2.32}$$

$$\frac{1}{\lambda}\Big(-\frac{\dot\theta}{4}+w_3\Big)-\Big(w_{1y}\sin\frac{\theta}{2}+w_{2y}\cos\frac{\theta}{2}\Big)+\frac{\theta_y}{2}\Big(w_1\cos\frac{\theta}{2}-w_2\sin\frac{\theta}{2}\Big)=0 \tag{2.33}$$

Lemma 4 *From the equations* (2.30) *and* (2.33), *we can deduce the equations* (2.11), (2.12) *respectively.*

Proof From (2.25) and (2.26), we have

$$w_1\cos\frac{\theta}{2}+w_2\sin\frac{\theta}{2}=-\frac{\lambda u}{2}+\frac{1}{2\lambda}v\cos\theta+\frac{1}{2\lambda}w_x\cot\theta-\frac{\lambda}{2}w_y\csc\theta \tag{2.34}$$

$$w_1\cos\frac{\theta}{2}-w_2\sin\frac{\theta}{2}=-\frac{\lambda u}{2}\cos\theta+\frac{1}{2\lambda}v+\frac{1}{2\lambda}w_x\csc\theta-\frac{\lambda}{2}w_y\cot\theta \tag{2.35}$$

By using $\Big(w_2\cos\frac{\theta}{2}-w_1\sin\frac{\theta}{2}\Big)_x=w_{2x}\cos\frac{\theta}{2}-w_{1x}\sin\frac{\theta}{2}+\frac{\theta_x}{2}\Big(-w_2\sin\frac{\theta}{2}-w_1\cos\frac{\theta}{2}\Big)$, (2.30) can be written as

$$\lambda\Big(\frac{\dot\theta}{4}+w_3\Big)+\Big(w_2\cos\frac{\theta}{2}-w_1\sin\frac{\theta}{2}\Big)_x+\theta_x\Big(w_2\sin\frac{\theta}{2}+w_1\cos\frac{\theta}{2}\Big)=0$$

By using (2.22), (2.17) and (2.34), it becomes

$$\lambda\Big(\frac{1}{2}w-\frac{\theta_x}{2}u-\frac{1}{2\lambda^2}v_x\sin\theta\Big)-\frac{1}{2\lambda}w_{xx}-\frac{1}{2\lambda}v_x\sin\theta-\frac{1}{2\lambda}v\theta_x\cos\theta$$
$$+\theta_x\Big(-\frac{1}{2}\lambda u+\frac{1}{2\lambda}v\cos\theta+\frac{1}{2\lambda}w_x\cot\theta-\frac{\lambda}{2}w_y\csc\theta\Big)=0$$

it is the equation (2.11).

By using $-\Big(w_1\sin\frac{\theta}{2}+w_2\cos\frac{\theta}{2}\Big)_y=-w_{1y}\sin\frac{\theta}{2}-w_{2y}\cos\frac{\theta}{2}-\frac{\theta_y}{2}\Big(w_1\cos\frac{\theta}{2}-w_2\sin\frac{\theta}{2}\Big)$, (2.33) can be written as

$$\frac{1}{\lambda}\Big(w_3-\frac{\dot\theta}{4}\Big)+\Big(-w_1\sin\frac{\theta}{2}-w_2\cos\frac{\theta}{2}\Big)_y+\theta_y\Big(w_1\cos\frac{\theta}{2}-w_2\sin\frac{\theta}{2}\Big)=0$$

By using (2.34) and (2.35), it becomes

$$\frac{1}{\lambda}\Big(-\frac{1}{2}w-\frac{\theta_y}{2}v+\frac{1}{2}\lambda^2u_y\sin\theta\Big)+\frac{\lambda}{2}w_{yy}+\frac{\lambda}{2}u_y\sin\theta+\frac{\lambda}{2}u\theta_y\cos\theta$$
$$+\theta_y\Big(-\frac{1}{2}\lambda u\cos\theta+\frac{1}{2\lambda}v+\frac{1}{2\lambda}w_x\csc\theta-\frac{\lambda}{2}w_y\cot\theta\Big)=0$$

it is the equation (2.12).

Lemma 5 *The equations* (2.29) *and* (2.32) *can be reduced to the same equation* (2.10).

Proof By using $\left(w_1\cos\frac{\theta}{2}+w_2\sin\frac{\theta}{2}\right)_x=w_{1x}\cos\frac{\theta}{2}+w_{2x}\sin\frac{\theta}{2}+\frac{\theta_x}{2}\left(-w_1\sin\frac{\theta}{2}+w_2\cos\frac{\theta}{2}\right)$, (2.34) and (2.17), we can write (2.29) as follows

$$\begin{aligned}0=&\left(w_1\cos\frac{\theta}{2}+w_2\sin\frac{\theta}{2}\right)_x+\theta_x2\left(w_1\sin\frac{\theta}{2}-w_2\cos\frac{\theta}{2}\right)\\=&\theta_x\left(\frac{1}{2\lambda}w_x+\frac{1}{2\lambda}v\sin\theta\right)-\frac{\lambda}{2}u_x+\frac{1}{2\lambda}v_x\cos\theta-\frac{1}{2\lambda}v\theta_x\sin\theta+\frac{1}{2\lambda}w_{xx}\cot\theta\\&-\frac{1}{2\lambda}w_x\theta_x\csc^2\theta-\frac{\lambda}{2}w_{xy}\csc\theta+\frac{\lambda}{2}w_y\theta_x\cot\theta\csc\theta\end{aligned}\tag{2.36}$$

From (2.11), $\frac{1}{2\lambda}w_{xx}\cot\theta=\frac{1}{2\lambda}(-2v_x\cos\theta+\lambda^2w\cot\theta+\theta_xw_x\cot^2\theta-\lambda^2\theta_xw_y\cot\theta\csc\theta)$ and from (2.21), $-\lambda u_x=\frac{1}{\lambda}v_x\cos\theta$, (2.36) is reduced to (2.10).

By using the same approach, we rewrite the equation (2.32) as $0=\left(w_1\cos\frac{\theta}{2}-w_2\sin\frac{\theta}{2}\right)_y+\theta_y\left(w_1\sin\frac{\theta}{2}+w_2\cos\frac{\theta}{2}\right)$, then we use (2.35), (2.20), (2.12) and (2.23), (2.32) is reduced to (2.10).

Up to now, we have deduced the five fundamental equations, the remained problems are to prove that the equations (2.28) and (2.31) are satisfied automatically and the equations (2.21) and (2.23) are compatible with the five fundamental equations. Now we prove the useful lemma.

Lemma 6

$$(v_x\sin\theta)_y=\lambda^2\theta_xu_y\tag{2.37}$$

$$(y_y\sin\theta)_x=\frac{1}{\lambda^2}\theta_yv_x\tag{2.38}$$

Proof From (2.11), and using (2.9), (2.10) and (2.12), we have

$$\begin{aligned}(v_x\sin\theta)_y=&\frac{1}{2}(-w_{xx}+\lambda^2w+\theta_xw_x\cot\theta-\lambda^2\theta_xw_y\csc\theta)_y\\=&\frac{1}{2}\Big[-w_x\cos\theta+\theta_xw\sin\theta+\lambda^2w_y+w_x\sin\theta+\theta_xw\cos\theta\cot\theta\\&-\theta_x\theta_yw_x\csc^2\theta-\lambda^2w_y+\lambda^2\theta_x\theta_yw_y\cot\theta\csc\theta\\&-\lambda^2\theta_x\left(-2u_y+\frac{1}{\lambda^2}w\csc\theta+\theta_yw_y\cot\theta\csc\theta-\frac{1}{\lambda^2}\theta_yw_x\csc^2\theta\right)\Big]\end{aligned}$$

The (2.37) is deduced. By the same approach, from (2.12) and by using (2.9), (2.10) and (2.11), (2.38) is yielded.

Lemma 7 *If u,v,w and θ satisfy the equations* (2.9)-(2.13), *then the equations* (2.38) *and* (2.31) *are satisfied automatically.*

Proof By using (2.24), (2.17), (2.9) and (2.38), (2.28) becomes

$$\left(\frac{1}{2}w+\theta_y\frac{v}{2}-\frac{1}{2}\lambda^2u_y\sin\theta\right)+\lambda\left(-\frac{1}{2\lambda}w_x-\frac{1}{2\lambda}v\sin\theta\right)$$

$$=\frac{1}{2}w_x+\frac{v}{2}\sin\theta+\theta_y\frac{1}{2}v_x-\frac{1}{2}v_x\theta_y-\frac{1}{2}w_x-\frac{1}{2}v\sin\theta=0$$

By using (2.22), (2.20), (2.9) and (2.37), (2.31) becomes

$$-\Big(\frac{1}{2}w+\theta_x\frac{u}{2}-\frac{1}{2\lambda^2}v_x\sin\theta\Big)_y-\frac{1}{\lambda}\Big(-\frac{\lambda}{2}w_y-\frac{\lambda}{2}u\sin\theta\Big)$$
$$=-\frac{1}{2}w_y-\frac{u}{2}\sin\theta-\theta_x\frac{1}{2}u_y+\frac{1}{2}u_y\theta_x+\frac{1}{2}w_y+\frac{1}{2}u\sin\theta=0$$

Lemma 8 *If u, v, w and θ satisfy the equations (2.9)–(2.13), then the equations (2.21) and (2.23) are compatible with them.*

Proof Derivating (2.21) with y and derivating (2.23) with x, then adding and subtracting them, we get

$$\Big(\lambda u_{xy}+\frac{1}{\lambda}v_{xy}\Big)=\frac{\sin\theta}{1+\cos\theta}\Big(\lambda u_y\theta_x+\frac{1}{\lambda}u_x\theta_y\Big) \tag{2.39}$$

$$\Big(\lambda u_{xy}-\frac{1}{\lambda}v_{xy}\Big)=\frac{\sin\theta}{1-\cos\theta}\Big(\frac{1}{\lambda}u_x\theta_y-\lambda u_y\theta_x\Big) \tag{2.40}$$

To add and subtract the equations (2.37) and (2.38), we get

$$\Big(\lambda u_{xy}+\frac{1}{\lambda}v_{xy}\Big)=\frac{1-\cos\theta}{\sin\theta}\Big(\lambda u_y\theta_x+\frac{1}{\lambda}v_x\theta_y\Big) \tag{2.41}$$

$$\Big(\lambda u_{xy}-\frac{1}{\lambda}v_{xy}\Big)=\frac{1+\cos\theta}{\sin\theta}\Big(\frac{1}{\lambda}v_x\theta_y-\lambda u_y\theta_x\Big) \tag{2.42}$$

Since $\dfrac{1+\cos\theta}{\sin\theta}=\dfrac{\sin\theta}{1-\cos\theta}$, we have (2.39) and (2.40) equal to (2.41), (2.42) respectively.

From Lemmas 3–5, 7 and Lemma 8, Theorem 1 is yielded.

3. One Example

Note that the equation (2.10) is just the linearization of the Sine-Gordon equation (2.9), if θ is a solution of (2.9), then θ_x, θ_y are the solution of the equation (2.10). We take the normal velocity component w as follows

$$w=\alpha\theta_x+\beta\theta_y \tag{3.1}$$

in which α and β are constants. From (2.11) and (2.12) we find that

$$u=\frac{\alpha}{4\lambda^2}(\theta_x^2+2\theta_{xx}\cot\theta)-\frac{\beta}{2}\theta_{yy}\csc\theta,\ v=-\frac{\alpha}{2}\theta_{xx}\csc\theta+\frac{\beta\lambda^2}{4}(\theta_y^2+2\theta_{yy}\cot\theta) \tag{3.2}$$

From (2.13), the time evolution of θ reads

$$\dot\theta=\alpha\Big[\frac{3}{2}\theta_x+\frac{1}{4\lambda^2}(\theta_x^3+2\theta_{xxx})\Big]+\beta\Big[\frac{3}{2}\theta_y+\frac{\lambda^2}{4}(\theta_y^3+2\theta_{yyy})\Big] \tag{3.3}$$

Theorem 2

$$\theta = 4\tan^{-1} e^{\xi}$$

$$\xi = cx + \frac{y}{c} + \frac{\alpha}{2}\Big(3c + \frac{c^3}{\lambda^2}\Big)t + \frac{\beta}{2}\Big(\frac{3}{c} + \frac{\lambda^2}{c^3}\Big)t \tag{3.4}$$

is one soliton solution of the equations (2.9) *and* (3.3).

Proof In this case, we have

$$\cos\theta = \tan h^2\xi - \operatorname{sech}^2\xi, \quad \sin\theta = -\operatorname{sech}\xi\tanh\xi \tag{3.5}$$

$$\theta_x = 2c\operatorname{sech}\xi, \theta_{xx} = -2c^2\operatorname{sech}\xi\tanh\xi, \ \theta_{xxx} = 2c^3(-\operatorname{sech}^3\xi + \operatorname{sech}\xi\tanh^2\xi)$$

$$\theta_y = \frac{2}{c}\operatorname{sech}\xi, \theta_{yy} = -\frac{2}{c^2}\operatorname{sech}\xi\tanh\xi, \theta_{yyy} = \frac{2}{c^3}(-\operatorname{sech}^3\xi + \operatorname{sech}\xi\tanh^2\xi)$$

$$\theta_{xy} = -2\operatorname{sech}\xi\tanh\xi = \sin\theta$$

$$\begin{aligned}\theta_t &= 2\operatorname{sech}\xi\Big[\frac{\alpha}{2}\Big(3c + \frac{c^3}{\lambda^2}\Big) + \frac{\beta}{2}\Big(\frac{3}{c} + \frac{\lambda^2}{c^3}\Big)\Big]\\
&= \alpha\Big[3c\operatorname{sech}\xi + \frac{1}{4\lambda^2}(8c^3\operatorname{sech}^3\xi - 4c^3\operatorname{sech}^3\xi + 4c^3\operatorname{sech}\xi\tanh^2\xi)\Big]\\
&\quad + \beta\Big[\frac{3}{c}\operatorname{sech}\xi + \frac{\lambda^2}{4}\Big(\frac{8}{c^3}\operatorname{sech}^3\xi - \frac{4}{c^3}\operatorname{sech}^3\xi + \frac{4}{c^3}\operatorname{sech}\xi\tanh^2\xi\Big)\Big]\\
&= 2\operatorname{sech}\xi\Big[\frac{\alpha}{2}\Big(3c + \frac{c^3}{\lambda^2}\Big) + \frac{\beta}{2}\Big(\frac{3}{c} + \frac{\lambda^2}{c^3}\Big)\Big]\end{aligned}$$

Theorem 2 is proved.

By direct calculation, we have the lemma.

Lemma 9 *We define*

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i}{4}\theta} & e^{-\frac{i}{4}\theta} \\ -e^{\frac{i}{4}\theta} & e^{\frac{i}{4}\theta}\end{pmatrix}; T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i}{4}\theta} & -e^{-\frac{i}{4}\theta} \\ e^{\frac{i}{4}\theta} & e^{-\frac{i}{4}\theta}\end{pmatrix} \tag{3.6}$$

$$\begin{aligned}&T^{-1}\begin{pmatrix} f_3 & f_1 - if_2 \\ f_1 + if_2 & -f_3\end{pmatrix}T\\
&= \begin{pmatrix} -f_1\cos\frac{\theta}{2} - f_2\sin\frac{\theta}{2} & f_3 - i\Big(f_2\cos\frac{\theta}{2} - f_1\frac{\sin\theta}{2}\Big) \\ f_3 + i\Big(f_2\cos\frac{\theta}{2} - f_1\frac{\sin\theta}{2}\Big) & f_1\cos\frac{\theta}{2} + f_2\sin\frac{\theta}{2}\end{pmatrix}\end{aligned} \tag{3.7}$$

Lemma 10 *By the gauge transformation*

$$\phi = T\psi \tag{3.8}$$

The equations (2.6), (2.7) *and* (2.10) *are mapped to*

$$\psi_x = \bar{U}\psi, \ \bar{U} = T^{-1}UT - T^{-1}T_x = \frac{i}{2}\begin{pmatrix}\lambda & \theta_x \\ \theta_x & -\lambda\end{pmatrix} \tag{3.9}$$

$$\psi_y = \bar{V}\psi, \ \bar{V} = T^{-1}VT - T^{-1}T_y = \frac{1}{2\lambda}\begin{pmatrix} -i\cos\theta & -\sin\theta \\ \sin\theta & i\cos\theta\end{pmatrix} \tag{3.10}$$

$$\psi_t = \bar{W}\psi,\ \bar{W} = T^{-1}WT - T^{-1}T_t$$
$$= i\begin{pmatrix} -w_1\cos\frac{\theta}{2} - w_2\sin\frac{\theta}{2} & \left(w_3+\frac{\dot\theta}{4}\right) - i\left(w_2\cos\frac{\theta}{2} - w_1\sin\frac{\theta}{2}\right) \\ \left(w_3+\frac{\dot\theta}{4}\right) + i\left(w_2\cos\frac{\theta}{2} - w_1\sin\frac{\theta}{2}\right) & w_1\cos\frac{\theta}{2} + w_2\sin\frac{\theta}{2} \end{pmatrix} \tag{3.11}$$

Proof We note $T^{-1}T_x = \frac{i\theta_x}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$; $T^{-1}T_y = \frac{i\theta_y}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$; $T^{-1}T_t = \frac{\dot\theta}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and from Lemma 9, (3.9), (3.10), (3.11) are obtained immediately.

Theorem 3

$$\psi_1 = (\lambda + ic\tanh\xi)e^{i\eta}, \quad \psi_2 = c\,\mathrm{sech}\,\xi e^{i\eta}$$
$$\eta = \frac{1}{2}\left(\lambda x - \frac{y}{\lambda} + \alpha\lambda t - \frac{\beta}{\lambda}t\right) \tag{3.12}$$

is a solution of the equations (3.9), (3.10) *and* (3.11).

Proof

$$\psi_{1x} = \frac{i\lambda}{2}\psi_1 + ic^2\mathrm{sech}^2\xi e^{i\eta} = \frac{i\lambda}{2}\psi_1 + ic^2\mathrm{sech}^2\xi e^{i\eta}$$
$$\psi_{2x} = \frac{i\lambda}{2}c\,\mathrm{sech}\,\xi e^{i\eta} - c^2\mathrm{sech}\,\xi\tanh\xi e^{i\eta}$$
$$= ic\,\mathrm{sech}\,\xi(\lambda + ic\tanh\xi)e^{i\eta} - \frac{i\lambda}{2}c\,\mathrm{sech}\,\xi e^{i\eta}$$
$$\psi_{1y} = \frac{-i}{2\lambda}(\lambda + ic\tanh\xi)e^{i\eta} + i\,\mathrm{sech}^2\xi e^{i\eta}$$
$$= \frac{-i}{2\lambda}(\tanh^2\xi - \mathrm{sech}^2\xi)(\lambda + ic\tanh\xi)e^{i\eta} - \frac{1}{2\lambda}(-2\mathrm{sech}\,\xi\tanh\xi)c\,\mathrm{sech}\,\xi e^{i\eta}$$
$$\psi_{2y} = \frac{-i}{2\lambda}c\,\mathrm{sech}\,\xi e^{i\eta} - \mathrm{sech}\,\xi\tanh\xi e^{i\eta}$$
$$= \frac{1}{2\lambda}(-2\mathrm{sech}\,\xi\tanh\xi)(\lambda + ic\tanh\xi)e^{i\eta} + \frac{i}{2\lambda}(\tanh^2\xi - \mathrm{sech}^2\xi)c\,\mathrm{sech}\,\xi e^{i\eta}$$

To prove ψ_1, ψ_2 satisfy (3.11), we note that when θ defining by (3.4), we have

$$u = \frac{\alpha}{4\lambda^2}(\theta_x^2 + 2\theta_{xx}\cot\theta) - \frac{\beta}{2}\theta_{yy}\csc\theta = \frac{\alpha}{2\lambda^2}c^2 - \frac{\beta}{2c^2} \tag{3.13}$$
$$v = -\frac{\alpha}{2}\theta_{xx}\csc\theta + \frac{\beta\lambda^2}{4}(\theta_y^2 + 2\theta_{xx}\cot\theta) = -\frac{\alpha}{2}c^2 + \frac{\beta\lambda^2}{2c^2} \tag{3.14}$$
$$w_3 + \frac{\dot\theta}{4} = \frac{1}{2}w + \frac{\theta_x}{2}u - \frac{1}{2}\frac{v_x}{\lambda}\sin\theta = \left[\alpha\left(c + \frac{c^3}{2\lambda^2}\right) + \frac{\beta}{2c}\right]\mathrm{sech}\,\xi \tag{3.15}$$
$$w_1\sin\frac{\theta}{2} - w_2\cos\frac{\theta}{2} = \frac{1}{2\lambda}w_x + \frac{1}{2\lambda}v\sin\theta = \left[\frac{-\alpha c^2}{2\lambda} - \beta\left(\frac{1}{\lambda} + \frac{\lambda}{2c^2}\right)\right]\mathrm{sech}\,\xi\tanh\xi \tag{3.16}$$

$$w_1\cos\frac{\theta}{2}+w_2\sin\frac{\theta}{2}=-\frac{\lambda}{2}u+\frac{1}{2\lambda}v\cos\theta+\frac{1}{2\lambda}w_x\cot\theta-\frac{\lambda}{2}w_y\csc\theta$$

$$=\alpha\Big[\frac{-c^2}{2\lambda}\operatorname{sech}^2\xi-\frac{\lambda}{2}\Big]+\beta\Big[\frac{-\lambda}{2c^2}\operatorname{sech}^2\xi+\frac{1}{2\lambda}\Big(\tanh^2\xi-\operatorname{sech}^2\xi\Big)\Big] \tag{3.17}$$

$$\psi_{1t}=\Big(\frac{i\alpha\lambda}{2}-\frac{i\beta}{2\lambda}\Big)(\lambda+ic\tanh\xi)e^{i\eta}-ic\operatorname{sech}^2\xi e^{i\eta}\Big[\frac{\alpha}{2}\Big(3c+\frac{c^3}{\lambda^2}\Big)+\frac{\beta}{2}\Big(\frac{3}{c}+\frac{\lambda^2}{c^3}\Big)\Big]$$

$$\begin{aligned}&=\alpha\Big\{\Big(\frac{i}{2}\lambda^2+\frac{i}{2}c^2\operatorname{sech}^2\xi\Big)-\frac{1}{2\lambda}c^3\operatorname{sech}^2\xi\tanh\xi-\frac{\lambda c}{2}\tanh\xi\\&\quad+i\Big(c^2+\frac{c^4}{2\lambda}\Big)\operatorname{sech}^2\xi+\frac{c^3}{2\lambda}\operatorname{sech}^2\xi\tanh\xi\Big\}e^{i\eta}\\&\quad+\beta\Big\{\frac{i\lambda^2}{2c^2}\operatorname{sech}^2\xi-\frac{i}{2}\Big(\tanh^2\xi-\operatorname{sech}^2\xi\Big)-\frac{\lambda}{2c}\operatorname{sech}^2\xi\tanh\xi\\&\quad+\frac{c}{2\lambda}(\tanh^2\xi-\operatorname{sech}^2\xi)\tanh\xi\\&\quad+\frac{i}{2}\operatorname{sech}^2\xi+\Big(\frac{c}{\lambda}+\frac{\lambda}{2c}\Big)\operatorname{sech}^2\xi\tanh\xi\Big\}e^{i\eta}\end{aligned}$$

We find that the left handside is equal to right handside

$$\psi_{2t}=\Big(\frac{i\alpha\lambda}{2}-\frac{i\beta}{2\lambda}\Big)c\operatorname{sech}\xi e^{i\eta}-c\operatorname{sech}\xi\tanh\xi e^{i\eta}\Big[\frac{\alpha}{2}\Big(3c+\frac{c^3}{\lambda^2}\Big)+\frac{\beta}{2}\Big(\frac{3}{c}+\frac{\lambda^2}{c^3}\Big)\Big]$$

$$\begin{aligned}&=\alpha\Big\{i\lambda\Big(c+\frac{c^3}{2\lambda^2}\Big)\operatorname{sech}\xi-c\tanh\xi\Big(c+\frac{c^3}{2\lambda^2}\Big)\operatorname{sech}\xi\\&\quad-\frac{1}{2}c^2\operatorname{sech}\xi\tanh\xi-\frac{i}{2\lambda}c^3\operatorname{sech}\xi\tanh^2\xi\\&\quad-\frac{i\lambda}{2}c\operatorname{sech}\xi-\frac{i}{2\lambda}c^3\operatorname{sech}\xi+\frac{i}{2\lambda}c^3\tanh^2\xi\operatorname{sech}\xi\Big\}e^{i\eta}\\&\quad+\beta\Big\{\frac{i\lambda}{2c}\operatorname{sech}\xi-c\tanh\xi\frac{1}{2c}\operatorname{sech}\xi-\operatorname{sech}\xi\tanh\xi\Big(1+\frac{\lambda^2}{2c^2}\Big)\\&\quad-ic\tanh^2\xi\operatorname{sech}\xi\Big(\frac{1}{\lambda}+\frac{\lambda}{2c^2}\Big)+ic\operatorname{sech}\xi\Big(\frac{-\lambda}{2c^2}\operatorname{sech}^2\xi+\frac{1}{2\lambda}(\tanh^2\xi-\operatorname{sech}^2\xi)\Big)\Big\}e^{i\eta}\end{aligned}$$

we find that the left handside is equal to right handside.

Lemma 11 *If* $\psi=\begin{pmatrix}\psi_1 & -\psi_2^*\\ \psi_2 & \psi_1^*\end{pmatrix}$, *then we have*

$$\psi^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\psi=\frac{1}{\triangle}\begin{pmatrix}-\psi_1\psi_1^*+\psi_2\psi_2^* & 2\psi_1^*\psi_2^*\\ 2\psi_1\psi_2 & \psi_1\psi_1^*-\psi_2\psi_2^*\end{pmatrix},\quad \triangle=\lambda^2+c^2$$

$$\begin{cases}I_1=-\psi_1\psi_1^*+\psi_2\psi_2^*=-\lambda^2-c^2+2c^2\operatorname{sech}^2\xi\\ I_2=2\psi_1\psi_2=2(\lambda c\operatorname{sech}\xi+ic^2\operatorname{sech}\xi\tanh\xi)e^{i2\eta}\end{cases} \tag{3.18}$$

$$\psi^{-1}\begin{pmatrix}-\cos\theta & i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}\psi$$

$$=\frac{1}{\triangle}\begin{pmatrix}\cos\theta I_1+i\sin\theta(\psi_1^*\psi_2-\psi_1\psi_2^*) & \cos\theta I_2^*+i\sin\theta(\psi_2^{*2}+\psi_1^{*2})\\ \cos\theta I_2-i\sin\theta(\psi_2^2+\psi_1^2) & -\cos\theta I_1-i\sin\theta(\psi_1^*\psi_2-\psi_1\psi_2^*)\end{pmatrix}$$

$$\begin{cases}J_1=\cos\theta I_1+i\sin\theta(\psi_1^*\psi_2-\psi_1\psi_2^*)=-\lambda^2-c^2+2\lambda^2\operatorname{sech}^2\xi\\ J_2=\cos\theta I_2-i\sin\theta(\psi_2^2+\psi_1^2)=2(-\lambda c\operatorname{sech}\xi+i\lambda^2\operatorname{sech}\xi\tanh\xi)e^{i2\eta}\end{cases} \tag{3.19}$$

$$\psi^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\psi = \frac{1}{\triangle}\begin{pmatrix}\psi_1^*\psi_2+\psi_1\psi_2^* & \psi_1^{*2}-\psi_2^{*2}\\ \psi_1^2-\psi_2^2 & -\psi_1^*\psi_2+\psi_1\psi_2^*\end{pmatrix}$$

$$L_1=\psi_1^*\psi_2+\psi_1\psi_2^*=2\lambda c\,\mathrm{sech}\,\xi,\ \ L_2=\psi_1^2-\psi_2^2=(\lambda^2-c^2+2i\lambda c\tanh\xi)e^{i2\eta} \tag{3.20}$$

Proof It can be proved by direct calculation.

Since θ contains parameter $\lambda(\lambda=e^{\varepsilon t})$, the formula $F=2\phi^{-1}\dfrac{\partial}{\partial\varepsilon}\phi$ in [4] can not be used. We need the following lemma.

Lemma 12 When w and θ are defined by (3.1) and (3.4) respectively, then

$$\begin{cases} F_{1x}=\dfrac{2\lambda}{\triangle}(\lambda c\,\mathrm{sech}\,\xi\cos2\eta-c^2\mathrm{sech}\,\xi\tanh\xi\sin2\eta)\\ F_{2x}=\dfrac{2\lambda}{\triangle}(\lambda c\,\mathrm{sech}\,\xi\sin2\eta+c^2\mathrm{sech}\,\xi\tanh\xi\cos2\eta)\\ F_{3x}=-\lambda+\dfrac{2\lambda c^2\mathrm{sech}^2\xi}{\triangle}\end{cases} \tag{3.21}$$

$$\begin{cases} F_{1y}=\dfrac{2}{\triangle}(-c\,\mathrm{sech}\,\xi\cos2\eta-\lambda\,\mathrm{sech}\,\xi\tanh\xi\sin2\eta)\\ F_{2y}=\dfrac{2}{\triangle}(c\,\mathrm{sech}\,\xi\sin2\eta+\lambda\,\mathrm{sech}\,\xi\tanh\xi\sin2\eta)\\ F_{3y}=\dfrac{-1}{\lambda}+\dfrac{2\lambda\,\mathrm{sech}^2\xi}{\triangle}\end{cases} \tag{3.22}$$

$$\begin{cases} F_{1t}=\dfrac{1}{\triangle}\Big\{2(\alpha\lambda^2c-\beta c)\mathrm{sech}\,\xi\cos2\eta-\Big[\alpha\Big(3\lambda c^2+\dfrac{c^4}{\lambda}\Big)+\beta\Big(3\lambda+\dfrac{\lambda^3}{c^2}\Big)\Big]\\ \qquad\cdot\mathrm{sech}\,\xi\tanh\xi\sin2\eta\Big\}\\ F_{2t}=\dfrac{1}{\triangle}\Big\{2(\alpha\lambda^2c-\beta c)\mathrm{sech}\,\xi\sin2\eta+\Big[\alpha\Big(3\lambda c^2+\dfrac{c^4}{\lambda}\Big)+\beta\Big(3\lambda+\dfrac{\lambda^3}{c^2}\Big)\Big]\\ \qquad\cdot\mathrm{sech}\,\xi\tanh\xi\sin2\eta\Big\}\\ F_{3t}=\Big\{\alpha\Big[\Big(3\lambda c^2+\dfrac{c^4}{\lambda}\Big)+\beta\Big(3\lambda+\dfrac{\lambda^3}{c^2}\Big)\Big]\dfrac{\mathrm{sech}^2\xi}{\triangle}\end{cases} \tag{3.23}$$

Proof Since $F_x=-i\lambda\phi^{-1}\begin{pmatrix}0 & e^{-i\frac{\theta}{2}}\\ e^{i\frac{\theta}{2}} & 0\end{pmatrix}\phi=-i\lambda\psi^{-1}T^{-1}\begin{pmatrix}0 & e^{-i\frac{\theta}{2}}\\ e^{i\frac{\theta}{2}} & 0\end{pmatrix}T\psi$ $=-i\lambda\psi^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\psi$, by Lemma 9, from (3.18), we have (3.21)

$$F_y=\frac{-i}{\lambda}\phi^{-1}\begin{pmatrix}0 & e^{i\frac{\theta}{2}}\\ e^{-i\frac{\theta}{2}} & 0\end{pmatrix}\phi=-i\frac{1}{\lambda}\psi^{-1}\begin{pmatrix}-\cos\theta & i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}\psi$$

from Lemma 9, and (3.19), we have (3.22).

From (2.8), Lemma 9 and (2.18)–(2.20), we have

$$F_{3t}=\lambda u\Big(-1+\frac{2c^2\mathrm{sech}^2\xi}{\triangle}\Big)+\frac{v}{\lambda}\Big(-1+\frac{2\lambda^2\mathrm{sech}^2\xi}{\triangle}\Big)+\frac{w2\lambda c\,\mathrm{sech}\,\xi}{\triangle}$$

$$=\Big\{2c^2\Big(\frac{\alpha c^2}{2\lambda}-\frac{\beta\lambda}{2c^2}\Big)+2\lambda^2\Big(-\frac{\alpha c^2}{2\lambda}+\frac{\beta\lambda}{2c^2}\Big)+2\lambda c\Big(2\alpha c+\frac{2\beta}{c}\Big)\Big\}\frac{\operatorname{sech}^2\xi}{\triangle}$$

$$=\Big\{\alpha\Big(\frac{c^4}{\lambda}+3c^2\lambda\Big)+\beta\Big(\frac{\lambda^3}{c^2}+3\lambda\Big)\Big\}\frac{\operatorname{sech}^2\xi}{\triangle}$$

$$F_{1t}=\Big\{2\lambda c\Big(\frac{\alpha c^2}{2\lambda}-\frac{\beta\lambda}{2c^2}\Big)-2\lambda c\Big(-\frac{\alpha c^2}{2\lambda}+\frac{\beta\lambda}{2c^2}\Big)+\Big(2\alpha c+\frac{2\beta}{c}\Big)(\lambda^2-c^2)\Big\}\frac{\operatorname{sech}\xi\cos 2\eta}{\triangle}$$
$$-\Big\{2c^2\Big(\frac{\alpha c^2}{2\lambda}-\frac{\beta\lambda}{2c^2}\Big)+2\lambda^2\Big(-\frac{\alpha c^2}{2\lambda}+\frac{\beta\lambda}{2c^2}\Big)+2\lambda c\Big(2\alpha c+\frac{2\beta}{c}\Big)\Big\}\frac{\operatorname{sech}\xi\tanh\xi\sin 2\eta}{\triangle}$$

$$F_{2t}=\Big\{2\lambda c\Big(\frac{\alpha c^2}{2\lambda}-\frac{\beta\lambda}{2c^2}\Big)-2\lambda c\Big(-\frac{\alpha c^2}{2\lambda}+\frac{\beta\lambda}{2c^2}\Big)+\Big(2\alpha c+\frac{2\beta}{c}\Big)(\lambda^2-c^2)\Big\}\frac{\operatorname{sech}\xi\sin 2\eta}{\triangle}$$
$$+\Big\{2c^2\Big(\frac{\alpha c^2}{2\lambda}-\frac{\beta\lambda}{2c^2}\Big)+2\lambda^2\Big(-\frac{\alpha c^2}{2\lambda}+\frac{\beta\lambda}{2c^2}\Big)+2\lambda c\Big(2\alpha c+\frac{2\beta}{c}\Big)\Big\}\frac{\operatorname{sech}\xi\tanh\xi\cos 2\eta}{\triangle},$$

it is nothing but the last expression of (3.23), Lemma 12 is proved.

We can integrate (3.21), (3.22), (3.23) to get F difference a constant, one has the following theorem.

Theorem 4 *If $w=\alpha\theta_x+\beta\theta_y$, and θ is defined by (3.4), then the motion of the constant Gaussian curvature $k=-1$ can be expressed as follows,*

$$F_1=\frac{2\lambda c\operatorname{sech}\xi\sin 2\eta}{\triangle},\quad F_2=-\frac{2\lambda c\operatorname{sech}\xi\cos\eta}{\triangle},\quad F_3=-\Big(\lambda x+\frac{1}{\lambda}y\Big)+\frac{2\lambda c}{\triangle}\tanh\xi \quad (3.24)$$

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