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COLORED BLACK HOLES

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract We describe some recent results on solutions of the static, stationary, spherically symmetric solutions of the $SU(z)$ Einstein-Yang/Mills equations. The main result is that any solution which is defined in the far field and has finite (ADM) mass, is defined for all $r > 0$.

Key Words Einstein-Yang-Mills equations; colored black holes; event horizon.
Classification 83C15, 83C57, 83C20, 58E15.

1. In this paper we describe some recent results on spherically symmetric black hole solutions of the $SU(2)$ Einstein-Yang/Mills (EYM) equations; such solutions are called "colored black holes." Our main result is that given any solution to the EYM equations which is defined in the far field ($r \gg 1$) and has finite (ADM) mass, is defined for all $r > 0$; see [1]. Since we know (see [2, 3, 4]) that given any event horizon, there are an infinite number of black-hole solutions having event horizon ρ, our result implies that all of these solutions can be continued back to zero. In particular this gives information as to the behavior of the gravitational field and the Yang-Mills field, inside a black hole, a subject of recent interest; [5, 6].

Our main result is surprising since, generally speaking, for nonlinear equations, existence theorems are only local, with perhaps global existence only for special parameter values. However for the EYM equations we prove global existence for any solution which is defined in a neighborhood of infinity. This result is not true in the "other direction"; namely if a solution is defined near $r = 0$, with particle-like boundary conditions (see [7]), a singularity can develop at some $\overline{r} > 0$, and the solution cannot be extended beyond $r = \overline{r}$ (see [8, Thm. 4.1]).

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2. The coupled EYM equations in $3+1$ space time with gauge group G, can be written in the form [1-20]

$$R_{ij} - \frac{1}{2} R g_{ij} = \sigma T_{ij}, \quad d^* F_{ij} = 0, \quad i,j = 0,1,2,3 \tag{1}$$

Here $R_{ij} - \frac{1}{2} R g_{ij}$ is the Einstein tensor computed with respect to the (unknown) metric g_{ij} and T_{ij} is the stress-energy tensor associated to the g-valued Yang/Mills curvature 2-form F_{ij} where g is the Lie algebra of G. If $G = SU(2)$, and we seek static, spherically symmetric solutions (solutions depending only on r), then we may write the metric as

$$ds^2 = -AC^2 dt^2 + A^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2}$$

and the YM curvature 2-form as [7],

$$F = w' \tau_1 dr \wedge d\theta + w' \tau_2 dr \wedge (\sin\theta d\phi) - (1-w^2)\tau_3 d\theta \wedge (\sin\theta d\phi) \tag{3}$$

Here (A, C), and w denote the unknown metric and connection coefficients respectively, and τ_1, τ_2 and τ_3 form a basis (the Paul matrices) for the Lie algebra $su(2)$. As has been discussed elsewhere, (cf. [1-20]), it follows from (1)–(3), that the spherically symmetric $SU(2)$ EYM equations are

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1-w^2)^2}{r^2} \tag{4}$$

$$r^2 A'' + \left[r(1-A) - \frac{(1-w^2)^2}{r^2} \right] w' + w(1-w^2) = 0 \tag{5}$$

and

$$\frac{C'}{C} = \frac{2w'^2}{r} \tag{6}$$

Notice that (4) and (5) don't involve C, so the major part of our effort is to solve (4) and (5).

A (colored) black-hole solution of (5)–(6) having event horizon $\rho > 0$ is a solution defined for all $r > \rho$ and satisfying

$$A(\rho) = 0, \quad A(r) > 0 \quad \text{if } r > \rho \tag{7}$$

$$\lim_{r\to\infty} \mu(r) \equiv \lim_{r\to\infty} r(1 - A(r)) = \overline{\mu} < \infty \tag{8}$$

and

$$\lim_{r\to\infty} (w^2(r), w'(r)) = (1, 0) \tag{9}$$

These conditions imply that the metric is asymptotically flat and has finite (ADM) mass $\overline{\mu}$, and that the YM field is well-behaved.

In considering black-hole solutions, we see that the equations are singular at $r = \rho$ in the sense that $A(\rho) = 0$. To overcome this difficulty, the data $(w(\rho), w'(\rho))$ must lie on a curve $C(\rho)$ in the $w - w'$ plane ([2]). The main result in [2] is that for every $\rho > 0$, there is a sequence of initial values $\{(\gamma_n, \beta_n) : n \in \mathbb{Z}_+\} \subset C(\rho)$, for which the corresponding solutions $(A_n(r), w_n(r_n), w_n'(r))$ of (4), (5) satisfy (7)–(9). If $\theta_n(r)$ is defined by

$$\theta_n(r) = \mathrm{Tan}^{-1}(w_n'(r)/w_n(r)), \quad \frac{-\pi}{2} < \theta_n(r) < 0$$

and the rotation number Ω_n is given by

$$\Omega_n = -\frac{1}{\pi} \lim_{r\to\infty} [\theta_n(r) - \theta_n(\rho)]$$

then for each n, $\Omega_n = n$. That is, for each value of the event horizon $\rho > 0$, there is a countable number of black hole solutions having event horizon ρ.

3. Defining $\mu(r) = r(1 - A(r))$, we have $A(r) = 1 - \dfrac{\mu(r)}{r}$ and $\mu(r)$ is called the mass function. The total mass $\overline{\mu}$ defined in (8), (that is, the ADM mass), is an invariant quantity, and thus has physical significance. We now consider solutions of the EYM equations satisfying the following minimal physical hypotheses:

$$1 > A(r) > 0 \quad \text{if } r \gg 1 \tag{10}$$

and

$$\lim_{r\to\infty} \mu(r) = \overline{\mu} < \infty \tag{11}$$

Note that (10) implies $\mu(r) > 0$ in the far field ($r \gg 1$), and (11) implies that the solution has finite total mass.

For such solutions defined in the far field, we let r decrease, and ask what can the solution do? There are two possibilities:

(i) $A(r) < 1$ for all r, or

(ii) $A(\sigma) = 1$ for some $\sigma > 0$.

In case (i), if we come back to $r = 0$, with $A(r) > 0$ for all $r > 0$, then ([9]), it must be that $A(0) = 1$, $w^2(0) = 1$, and $w'(0) = 0$; such solutions are called particle-like solutions, see [9]. If on the other hand we come back to some (first) $\rho > 0$ where $A(\rho) = 0$, it was shown in [2], that this implies that the solution is a black-hole solution; i.e. $(w(\rho), w'(\rho)) \in C(\rho)$.

In case (ii), we call such a solution Reissner-Nordström-Like (RNL). In this case the solution is defined for all $r > 0$ and satisfies

$$\lim_{r\searrow 0} (A(r), w(r), w'(r)) = (\infty, \overline{w}, 0) \tag{12}$$

for some $\overline{w}$, and $r^2A(r)$ is analytic at $r=0$. In [10] we prove the existence of infinitely-many RNL solutions satisfying $A(\sigma)=1$, $(w(\sigma),w'(\sigma))$ arbitrary, if $\sigma \geq \frac{1}{2}$. These solutions satisfy (8) and (9), and have finite rotation numbers. If $w(0)=0$, then near $r=0$, we can write

$$w(r) = Cr^3 + O(r^4)$$

so $\theta(r) = O\left(\frac{1}{r}\right)$, and hence $\lim_{r\searrow 0}\theta(r) = \pm\frac{\pi}{2} \pmod{2\pi}$. These solutions thus have half-integral rotation numbers.

For our RNL solutions, we prove that $A(r)>0$ if $r>\sigma$, $A'(r)<0$ if $r\leq\sigma$, so that A is singular only at $r=0$. We prove also that $r=0$ is a non-removable singularity since the curvature invariant $\mathcal{R} = R_{ijk\ell}\, R^{ijkl}$ satisfies $\mathcal{R} \geq \text{const.}/r^6$. But this singularity is not inside a black hole; i.e. it is not "hidden", but rather, $r=0$ is a "naked" singularity.

4. Physically relevant solutions should satisfy this minimal hypothesis:

(H) For $r \gg 1$, $A(r)>0$ and $\lim_{r\to\infty}\mu(r)<\infty$

For such solution, set

$$\rho = \inf\{r : A(s) > 0 \quad \text{if } s > r \geq 0\}$$

then $\rho \geq 0$, and if $\rho = 0$, the solution is a particle-like solution, while if $\rho > 0$, it is a black hole solution. We call a solution regular if it satisfies (H) and $0 < A(r) < 1$ for all $r > \rho$. We then have the following theorem; see [10]:

Theorem A *All regular solutions of the EYM equations are black-hole or particle-like solutions, whose existence was demonstrated in the papers* [2, 4, 8, 11].

Thus, there are no other physically relevant particle-like, or black hole solutions different from the ones we previously obtained. Moreover, in [10], we showed the following theorem which characterizes solutions satisfying (H):

Theorem B *Any spherically symmetric $SU(2)$ solution of the EYM equations, which satisfies hypothesis (H) lies in one of the following sets:*

(i) *Regular solutions, $0 \leq A(r) < 1$ for all $r > \rho$;*

(a) *if $\rho > 0$, solution is a black-hole solution,*

(b) *if $\rho = 0$, solution is particle-like.*

(ii) *Reissner-Nordström-Like: $A(\sigma) = 1$ for some $\sigma > 0$. All such solutions have finite mass, and are asymptotically flat at infinity: $(A(r), C(r)) \to (1,1)$ as $r \to \infty$.*

This theorem characterizes all physically relevant solutions outside of a black hole. Now we ask: what happens inside of a black-hole? To answer this, we first state the following theorem:

Theorem C (Smoller-Wasserman [1]): *Let (A, w) be any solution of the $SU(2)$ radially symmetric EYM equations (4), (5), which is defined in the far field. Suppose too that one of the following two conditions hold:*

(i) $\lim_{r\to\infty} \mu(r) < \infty$ (finite total mass), *or equivalently,*

(ii) $A(r_1) > 0$ for some $r_1 > 1$.

Then the solution is defined for all $r > 0$.

This result displays the rather amazing structure of the EYM equations; indeed, these very complicated nonlinear equations have globally defined solutions provided that the solution satisfies the minimal physical assumption of having finite total mass.

As a corollary of the last theorem, it follows that the black-hole solutions whose existence was first proved in [2], are actually defined inside the event horizon all the way up to $r = 0$; namely, we have

Theorem D *Let (A, w) be any black hole solution of equations (4), (5) having event horizon $\rho > 0$, defined for all $r > \rho$, and satisfying (7)–(9). Then the solution is defined for all $r > 0$.*

5. We have also proved in [1] that for solutions having finite total mass, the zeros of the metric function A are discrete, except possibly at $r = 0$. We conjecture that $r = 0$ cannot be a limit point of zeros of A, so that A has a finite number of zeros. In [6], the authors numerically found a colored black hole having two zeros. It would be interesting to give a rigorous proof of this result.

Using the methods in [2, 8, 11], we have proved the following theorem; see [1]

Theorem E *There is a continuous two parameter family of solutions $(A_{\alpha,\beta}(r), w_{\alpha,\beta}(r))$ to the EYM equations (4), (5), defined in the far field, which are analytic functions of $s = \frac{1}{r}$. That is, if $((A(r), w(r))$ is a solution to the EYM equations (4), (5), which is asymptotically flat, and is analytic in $s = \frac{1}{r}$, then $(A(r), w(r)) = (A_{\alpha,\beta}(r), w_{\alpha,\beta}(r))$ for some pair of parameter values (α, β).*

In this theorem, one parameter is the (ADM) mass β and in fact $A(s = 0) = 1$, and $\left.\frac{dA}{ds}\right|_{s=0} = -\beta$. The other parameter is $\alpha = \left.\frac{dw}{ds}\right|_{s=0}$ and $w^2(0) = 1$.

It follows from the results in [8 or 2], that the (ADM) mass β is finite for any solution which is defined in the far-field. Moreover, for such solutions, $rw'(r) \to 0$ as $r \to \infty$; cf. [2]. We do not know if $\lim_{r\to\infty} = \lim_{s\to 0} \frac{dw(s)}{ds}$ exists. Thus we pose the question as to whether every asymptotically flat solution to the EYM equations (4), (5) is analytic in $s = \frac{1}{r}$ at $s = 0$.

If the answer is affirmative, as we suspect, then (as discussed in [1]), we may consider the (α, β)-plane as representing those solutions having the following asymptotic form near $s = 0$:

$$A(s) = 1 - \beta s + h.o.t.$$
$$w(s) = 1 - \alpha s + h.o.t.$$

and all such solutions are described by a point in the (α, β)-plane (or in the plane corresponding to $w(s = 0) = -1$), or they correspond to the one parameter family of classical Reissner-Nordström solutions: $A(r) = 1 - \frac{c}{r} + \frac{1}{r^2}$, $w(r) \equiv 0$.

In the (α, β) plane, certain regions are easily identifiable. Thus, if $\alpha < 0$, or if $\alpha > 0$ and $\beta < 0$, these correspond to RNL-solutions. The line $\alpha = 0$ corresponds to Schwarzschild solutions of mass β. Particle-like, and black-hole solutions must lie in the first quadrant, $\alpha > 0$, $\beta > 0$. Presumably there are a countable number of distinct curves lying in the first quadrant which are distinguished by the number of zeros of w, parametrized by the event horizon $\rho = 0$; ($\rho = 0$ corresponds to particle-like solutions). All other points in the first quadrant correspond to RNL solutions, as follows from our above results, because any point in this plane represents a solution which is defined for all $r > 0$.

Thus near nay given black-hole solution, there are global solutions which are neither black-hole solutions, or particle like solutions; they must therefore be RNL solutions. It follows that for any such global solution (A, w), either A has a zero, in which case the corresponding point (α, β) lies on one of the above-mentioned countable number of curves, or it is one of the countable number of particle-like solutions, or it is an RNL-solution, [9, 10].

It follows that in any neighborhood of a black-hole solution $(A_0(r), w_0(r))$, there are RNL solutions. In particular, if $A_0(r_1) < 0$, then arbitrarily close to this solution there are solutions $(A(r), w(r))$ having $A(r_1) > 0$. This is a spectacular example of non-continuous dependence on initial conditions.

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