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LOCAL CLASSICAL SOLUTION OF MUSKAT FREE BOUNDARY PROBLEM *

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Abstract In this paper we consider the two-dimensional Muskat free boundary problem: $\Delta u_i(x,t) = 0$ in space-time domain Q_i $(i = 1, 2)$, here t is a parameter. The unknown surface $\Gamma_{\rho T}$ (free boundary) is the common part of the boundaries of Q_1 and Q_2. The free boundary conditions are $u_1(x,t) = u_2(x,t)$ and $-k_1\dfrac{\partial u_1}{\partial n} = -k_2\dfrac{\partial u_2}{\partial n} = V_n$.
If the initial normal velocity of the free boundary is positive, we shall prove the existence of classical solution locally in time and uniqueness by making use of Newton's iteration method.
Key Words Classical solution; Muskat problem; Newton's iteration method
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1. Introduction and Main Result

Muskat problem is a very old open problem, it was proposed by Muskat in 1934 (see [1]). This problem describes the flows of two fluids in porous media, for example, oil and water. In 1987 Jiang had got a weak formulation for this problem ([2]). In 1989 Liang and Jiang researched an approximating Muskat problem (see [3]). But up to now there is not any mathematical result in the existence of weak or classical solutions.

In this paper, we shall prove the existence of classical solution locally in time and uniqueness by use of Newton's iteration method (see [4], Theorem 15.6). The solution is sought as the limit of the sequence

$$x_{n+1} = x_n - [D\mathcal{F}(x_n)]^{-1}\mathcal{F}(x_n)$$

The difficulty is to prove the invertibility of Frechet derivative operator. In order to state and prove our result, we introduce the following function spaces:

Let G be an open set in $\mathbb{R}^n$, $n = 1, 2$, define

$$C_T^{k+\alpha}(\overline{G}) = C^0([0,T]; C^{k+\alpha}(\overline{G})), \quad 0 < \alpha < 1,\ k = 1, 2, \cdots$$
$$\widehat{C}^{k+\alpha}(\overline{G}) = \{v \in C_T^{k+\alpha}(\overline{G}); \partial_t v \in C_T^{k-1+\alpha}(\overline{G})\} \text{ and}$$

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$$|v|_{\widehat{C}^{k+\alpha}(\overline{G})} = |v|_{C_T^{k+\alpha}(\overline{G})} + |\partial_t v|_{C_T^{k-1+\alpha}(\overline{G})}$$

Denote

$$\underset{\circ}{C}{}_T^{k+\alpha}(\overline{G}) = \{v \in C_T^{k+\alpha}(\overline{G}); v(\cdot,0) = 0\}$$

$$\underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\overline{G}) = \{v \in \widehat{C}_T^{2+\alpha}(\overline{G}); v(\cdot,0) = \partial_t v(\cdot,0) = 0\}$$

Let Ω be a bounded annular domain in $\mathbb{R}^2$ with $C^{4+\alpha}$ boundary $\partial\Omega = \Gamma^+ \cup \Gamma^-$, Γ^+ is the inside boundary and Γ^- is the outside boundary. $\Gamma_0 \in \Omega$ is the initial position of free boundary such that $\Gamma_0 \cap \Gamma^\pm = \emptyset$ and Γ^+ is inside of Γ_0 and Γ^- is outside of Γ_0. Define that $\Omega^\pm$ is the part between $\Gamma^\pm$ and Γ_0. For the points of the surface Γ_0 we introduce the coordinate ω; we also denote by $x(\omega) \in \Gamma_0$ and $\vec{n}(\omega)$ the unit normal to Γ_0 directed into Ω^+.

Let ν_0 be a given positive number such that the surface $\{x = x(\omega) \pm 2\vec{n}(\omega)\nu, 0 < \nu < \nu_0\}$ has no selfintersection and doesn't intersect $\Gamma^\pm$; let $\rho(\omega,t)$ be a function of class $\widehat{C}_T^{2+\alpha}(\Gamma_0)$ such that $\rho(\omega,0) = 0$ and $\max|\rho(\omega,t)| \le \nu_0/4$. We denote by $\Omega_{\rho T}^\pm$ the region bounded by the planes $t=0$, $t=T$, surface $\Gamma_T^\pm = \Gamma^\pm \times [0,T]$ and $\Gamma_{\rho T} = \{(x,t); x = x(\omega) + \vec{n}\rho(\omega,t), t \in [0,T]\}$.

The Muskat free boundary problem consists in finding the pressure $u^\pm(x,t)$ (of oil and water) and function $\rho(\omega,t)$ defining an *a priori* unknown surface $\Gamma_{\rho T}$ on the basis of the conditions

$$\Delta u^\pm(x,t) = 0 \quad \text{in } \Omega_{\rho T}^\pm \tag{1.1}$$

$$\partial_n u^+(x,t) = g^+(x,t) \quad \text{on } \Gamma_T^+ \tag{1.2}$$

$$u^-(x,t) = g^-(x,t) \quad \text{on } \Gamma_T^- \tag{1.3}$$

$$u^+(x,t) = u^-(x,t) \quad \text{on } \Gamma_{\rho T} \tag{1.4}$$

$$k^+\partial_n u^+ = k^-\partial_n u^- \quad \text{on } \Gamma_{\rho T} \tag{1.5}$$

$$V_n = -k^+\partial_n u^+ \quad \text{on } \Gamma_{\rho T} \tag{1.6}$$

Equation (1.1) is from Darcy's law neglecting gravity, (1.2) and (1.3) are boundary conditions on fixed boundaries in which n is the exterior unit normal to Γ^+, (1.3) represents supply of water. (1.4)–(1.6) are free boundary conditions, in which $k^\pm = \overline{k}^\pm/\mu^\pm$, $\overline{k}^\pm$ are permeabilities and $\mu^\pm$ are viscosity coefficients. (1.5) and (1.6) have the meaning of the law of energy conservation on the unknown boundary $\Gamma_{\rho T}$ and V_n is the velocity of the free boundary in the direction of $\vec{n}$.

We shall assume

$$\Gamma^\pm, \Gamma_0 \in C^{4+\alpha} \text{ with } 0 < \alpha < 1 \tag{1.7}$$

$$k^\pm \text{ are constants with } k^+ > k^- > 0 \tag{1.8}$$

$$g^+(x,t) \in C_T^{3+\alpha}(\Gamma^+) \text{ and } \partial_t g^+(x,t) \in C_T^{1+\alpha}(\Gamma^+) \tag{1.9}$$

$$g^-(x,t) \in C_T^{4+\alpha}(\Gamma^-) \text{ and } \partial_t g^-(x,t) \in C_T^{2+\alpha}(\Gamma^-) \tag{1.10}$$

$$V_n\mid_{t=0}>0 \tag{1.11}$$

The condition (1.11) means the initial speed of free boundary is positive. Because $V_n\mid_{t=0}$ depends on $g^{\pm}(x,t)$, $\Gamma^{\pm}$ and Γ_0, in the last section, we shall discuss the conditions which guarantee the correctness of (1.11).

Now we can state our main result as follows:

Theorem 1.1 *Under the assumptions of* (1.7)–(1.11), *if* T *is small enough, the problem* (1.1)–(1.6) *has a unique classical solution* $\rho\in\widehat{C}_T^{2+\alpha}(\Gamma_0)$, $u^{\pm}\in C^0([0,T];C^{2+\alpha}(\Omega^{\pm}_{\rho T}\times\{t\}))$.

In next section, we make Hanzawa diffeomorphism to change the problem (1.1)–(1.6) into a cylindrical domain. In Section 3, we construct an initial approximation and describe the sketch of the proof of main theorem. In Section 4 we prove the invertibility of Frechet derivative operator and prove the main result. In the last section, we drive out some sufficient conditions which ensure the correctness of the condition (1.11).

2. Reduction of the Problem

To prove the solvability of the problem (1.1)–(1.6), it is convenient to reduce it to a problem in a fixed domain. To this end we use Hanzawa diffeomorphism presented in [5]. Suppose ν_0 introduced above is so small that mapping $x:\Gamma_0\times[-\nu_0,\nu_0]\to\mathbb{R}^2$ defined by the rule $x(\omega,\lambda)=x(\omega)+\lambda\vec{n}(\omega)$ is regular and is one to one. Let the range of this mapping be

$$N_0=\{x(\omega,\lambda);(\omega,\lambda)\in\Gamma_0\times[-\nu_0,\nu_0]\}$$

The inverse mapping from N_0 to $\Gamma_0\times[-\nu_0,\nu_0]$ is defined as follows: $x\to(\omega(x),\lambda(x))$. We set

$$\phi^{(1)}(\omega,\lambda)=\nabla_x\omega(x)\mid_{x=x(\omega,\lambda)},$$
$$\phi^{(2)}(\omega,\lambda)=\nabla_x\lambda(x)\mid_{x=x(\omega,\lambda)}$$

We shall show below that for sufficiently small T, the free boundary surface $\Gamma_{\rho T}$ can be described by the equation

$$h_\rho(x,t)\equiv\lambda(x)-\rho(\omega(x),t)=0$$

This makes it possible to compute unit normal to $\Gamma_{\rho T}\times\{t\}$:

$$n=\frac{\nabla_x h_\rho}{\mid\nabla_x h_\rho\mid}$$

Further, let $\chi(\lambda)\in C^\infty([-\nu_0,\nu_0])$ such that

$$\chi(\lambda)=0,\quad \text{if } |\lambda|>\frac{3}{4}\nu_0$$

$$\chi(\lambda) = 1, \quad \text{if } |\lambda| < \frac{1}{4}\nu_0$$
$$|\chi'(\lambda)| \le \frac{4}{3}\nu_0^{-1}$$

then $1 + \chi'(\lambda)\mu \ge 2/3$ if $|\mu| \le \nu_0/4$.

For any $\rho(\omega, t) \in C^1(\Gamma_{0T})$, where $\Gamma_{0T} = \Gamma_0 \times [0, T]$, with $\rho\,|_{t=0} = 0$ and $\max_{\Gamma_{0T}} |\rho| < \nu_0/4$, we define Hanzawa diffeomorphism

$$e_{\rho T} : \mathbb{R}_y^2 \times [0, T] \to \mathbb{R}_x^2 \times [0, T]$$

in the following manner:

$$(x, t) = e_{\rho T}(y, t) \text{ with}$$
$$\begin{cases} x = y & \text{if dist}\,(y, \Gamma_0) \ge \frac{3}{4}\nu_0 \\ x = x(\omega) + (\eta + \chi(\eta)\rho(\omega, t))\vec{n}(\omega) & \text{if dist}\,(y, \Gamma_0) \le \frac{3}{4}\nu_0 \end{cases}$$

where $y = y(\omega, \eta)$ is in N_0 and (ω, η) are local coordinates of y in $\Gamma_0 \times [-\nu_0, \nu_0]$. In local coordinates of $N_0 \times (0, T)$, we have

$$e_{\rho T}(\omega, \eta; t) = (\omega, \eta + \chi(\eta)\rho(\omega, t); t) \equiv (\omega, \lambda; t)$$

where $\lambda = \eta + \chi(\eta)\rho(\omega, t)$.

The transformation $e_{\rho T}^{-1}$ makes it possible to take the noncylindrical domain $\Omega_{\rho T}^{\pm}$ into the cylindrical domain $\Omega_T^{\pm}$. We can make the change of variables $(x, t) = e_{\rho T}(y, t)$ and get

$$v^{\pm}(y, t) = u^{\pm}(e_{\rho T}(y, t))$$

then the problem (1.1)–(1.6) becomes

$$\mathcal{L}_\rho v^{\pm}(y, t) = 0 \quad \text{in } \Omega_T^{\pm} \tag{2.1}$$
$$\partial_n v^{+}(y, t) = g^{+}(y, t) \quad \text{on } \Gamma_T^{+} \tag{2.2}$$
$$v^{-}(y, t) = g^{-}(y, t) \quad \text{on } \Gamma_T^{-} \tag{2.3}$$
$$v^{+}(y, t) = v^{-}(y, t) \quad \text{on } \Gamma_{0T} \tag{2.4}$$
$$k^+ S_\rho \partial_n v^+ - k^+ K_{\rho\omega}\partial_\omega v^+ = k^- S_\rho \partial_n v^- - k^- K_{\rho\omega}\partial_\omega v^- \quad \text{on } \Gamma_{0T} \tag{2.5}$$
$$\partial_t \rho = -k^+ S_\rho \partial_n v^+ + k^+ K_{\rho\omega}\partial_\omega v^+ \quad \text{on } \Gamma_{0T} \tag{2.6}$$

where

$$\mathcal{L}_\rho = \sum_{i,j=1}^{2} a_\rho^{ij}\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{2} a_\rho^{i}\frac{\partial}{\partial y_i}$$
$$a_\rho^{ij} = a^{ij}(\rho, \partial_\omega \rho), \quad 1 \le i, j \le 2,$$
$$a_\rho^{i} = a^{i}(\rho, \partial_\omega \rho, \partial_\omega^2 \rho), \quad 1 \le i \le 2$$

$$\sigma_0|\xi|^2 \le a_\rho^{ij}\xi_i\xi_j \le \sigma_1|\xi|^2$$

$\sigma_0, \sigma_1 > 0$, depend on ρ, T, and (see[6])

$$S_\rho = 1 + (\partial_\omega \rho)^2 |\phi^{(1)}|^2,$$

$$K_{\rho\omega} = \partial_\omega \rho |\phi^{(1)}|^2$$

We now consider a nonlinear operator:

$$\mathcal{F}(\rho) \equiv \partial_t \rho + k^+ S_\rho \partial_n v_\rho^+ - k^+ K_{\rho\omega} \partial_\omega v_\rho^+ \tag{2.7}$$

on the function $\rho \in \widehat{C}_T^{2+\alpha}(\Gamma_0)$ with $\max|\rho(\omega,t)| \le \nu_0/4$ and $v_\rho^\pm$ is a solution of (2.1)–(2.5) with given $\rho \in \widehat{C}_T^{2+\alpha}(\Gamma_0)$ where t is a parameter. Obviously (1.1)–(1.6) has the solution $(u^\pm, \rho)$ which is equivalent to the existence of a solution of the equation $\mathcal{F}(\rho) = 0$.

So Theorem 1.1 of Section 1 can be reformulated as follows:

Theorem 2.1 (Reformulated form) *Under the same assumptions as in Theorem 1.1, for a sufficiently small T, there exists a unique $\rho \in \widehat{C}_T^{2+\alpha}(\Gamma_0)$, with $\mathcal{F}(\rho) = 0$.*

Here we suppose ρ is the only unknown function of the problem because $v^\pm$ is obtained once ρ is determined.

3. Construction of an Initial Approximation

It is well known that the first approximation is important in Newton iteration method, so at first, let us analyse the value of $\partial_t^j \rho\mid_{t=0}$, $j = 0, 1$. Of course,

$$\rho\mid_{t=0} = 0 \equiv R_0(\omega) \in C^{4+\alpha}(\Gamma_0)$$

and from (1.1)–(1.6) we know $u^\pm(x,0) \in C^{4+\alpha}(\overline{\Omega^\pm})$, so

$$V_n\mid_{t=0} = -k^+ \frac{\nabla h_\rho}{|\nabla h_\rho|} \nabla_x u^+(x,0)$$

Since $V_n = \partial_t \rho / |\nabla h_\rho|$,

$$\partial_t \rho\mid_{t=0} = -k^+ (\nabla h_\rho)\mid_{t=0} \nabla_x u^+(x,0) = k^+ \nabla_x \lambda \cdot \nabla_x u^+(x,0) \equiv R_1(\omega) \in C^{2+\alpha}(\Gamma_0)$$

Suppose that $\rho_0(\omega,t) \in C^{4+\alpha, 2+\alpha/2}(\Gamma_{0T})$ satisfies $\partial_t^j \rho_0\mid_{t=0} = R_j$, $j = 0, 1$, (see [7], p.298, Th.4.3) and

$$|\rho_0|_{C^{4+\alpha,2+\alpha/2}(\Gamma_{0T})} \le C \tag{3.1}$$

here C depends only on $|g^+(x,0)|_{C_T^{1+\alpha}(\Gamma^+)}$ and $|g^-(x,0)|_{C_T^{2+\alpha}(\Gamma^-)}$.

Let $v_{\rho_0}^\pm$ admit the following problem

$$\mathcal{L}_{\rho_0} v_{\rho_0}^\pm(y,t) = 0 \quad \text{in } \Omega_T^\pm \tag{3.2}$$

$$\partial_n v_{\rho_0}^+(y,t) = g^+(y,t) \quad \text{on } \Gamma_T^+ \tag{3.3}$$

$$v_{\rho_0}^-(y,t) = g^-(y,t) \quad \text{on } \Gamma_T^- \tag{3.4}$$

$$v_{\rho_0}^+(y,t) = v_{\rho_0}^-(y,t) \quad \text{on } \Gamma_{0T} \tag{3.5}$$

$$k^+ S_{\rho_0}\partial_n v_{\rho_0}^+ - k^+ K_{\rho_0\omega}\partial_\omega v_{\rho_0}^+ = k^- S_{\rho_0}\partial_n v_{\rho_0} - k^- K_{\rho_0\omega}\partial_\omega v_{\rho_0}^- \quad \text{on } \Gamma_{0T} \tag{3.6}$$

It is clear that $a_{\rho_0}^{ij} \in C^{3+\alpha,(3+\alpha)/2}(\overline{\Omega}_T^\pm)$, $a_{\rho_0}^i \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T^\pm)$, so $a_{\rho_0}^{ij}$, $a_{\rho_0}^i \in C_T^{2+\alpha}(\overline{\Omega}_T^\pm)$, it follows that $v_{\rho_0}^\pm \in C_T^{4+\alpha}(\overline{\Omega}^\pm)$ by $\Gamma^\pm, \Gamma_0 \in C^{4+\alpha}$, $g^+(x,t) \in C_T^{3+\alpha}(\Gamma^+)$, $g^-(x,t) \in C_T^{4+\alpha}(\Gamma^-)$, and the theory of elliptic equations. Considering $\partial_t g^+(y,t) \in C_T^{1+\alpha}(\Gamma^+)$, $\partial_t g^-(y,t) \in C_T^{2+\alpha}(\Gamma^-)$, we can differentiate (3.2)–(3.6) formally with respect to t, then we have $\partial_t v_{\rho_0}^\pm \in C_T^{2+\alpha}(\overline{\Omega}^\pm)$, and

$$\begin{aligned}|v_{\rho_0}^\pm|_{C_T^{4+\alpha}(\overline{\Omega}^\pm)} + |\partial_t v_{\rho_0}^\pm|_{C_T^{2+\alpha}(\overline{\Omega}^\pm)} &\le C(|\rho_0|_{C^{4+\alpha,2+\alpha/2}(\overline{\Gamma}_{0T})} \\ &+ |g^+|_{C_T^{3+\alpha}(\Gamma^+)} + |\partial_t g^+|_{C_T^{1+\alpha}(\Gamma^+)} + |g^-|_{C_T^{4+\alpha}(\Gamma^+)} + |\partial_t g^-|_{C_T^{2+\alpha}(\Gamma^-)})\end{aligned}$$

We call $(\rho_0, v_{\rho_0}^\pm)$ the initial approximation, and we will find $\delta\rho \in \underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\Gamma_0)$, such that $\rho_0 + \delta\rho$ is the solution of (2.1)–(2.6).

The proof of Theorem 2.1 is based on the theorem of the following with respect to Newton iteration method ([4], Theorem 15.6):

Let X, Y be Banach spaces and $F : B_r(x_0) \subset X \to Y$ a C^1-map such that

(a) $[F'(x_0)]^{-1} \in L(Y,X)$, $|[F'(x_0)]^{-1}F(x_0)| \le \alpha$ *and* $|[F'(x_0)]^{-1}| \le \beta$;

(b) $|[F'(x)] - [F'(\overline{x})]| \le k|x - \overline{x}|$ *for all* $x, \overline{x} \in B_r(x_0)$;

(c) $2k\alpha\beta < 1$ *and* $2\alpha < r$

are satisfied. Then F has a unique zero Z in $\overline{B}_{2\alpha}(x_0)$.

We define $F : \underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\Gamma_0) \to \underset{\circ}{C}{}_T^{1+\alpha}(\Gamma_0)$ by $F(\delta\rho) = \mathcal{F}(\rho_0 + \delta\rho)$, here $\mathcal{F}(\rho_0 + \delta\rho)$ is defined in (2.7).

The main difficulty is to prove $[\mathcal{F}'(\rho_0)]^{-1} \in L(Y,X)$, others are easy to be proved because $\mathcal{F}(\rho_0)\,|_{t=0} = 0$, and $\partial_t(\mathcal{F}(\rho_0))|_{t=0} = 0$, so if T is small enough then $|\mathcal{F}(\rho_0)|_{C_T^{1+\alpha}(\Gamma_0)}$ is small enough too. The proof of (b) is standard but needs complicated calculations, we omit the detail. In the next section we prove $[\mathcal{F}'(\rho_0)]^{-1} \in L(\underset{\circ}{C}{}_T^{1+\alpha}(\Gamma_0), \underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\Gamma_0))$ and estimate the norm of $[\mathcal{F}'(\rho_0)]^{-1}$. In this way, we complete the proof of theorem 2.1.

4. Invertibility of $\mathcal{F}'(\rho_0)$

At first we derive out the formula of $\mathcal{F}'(\rho_0)\delta\rho$, here $\delta\rho \in \underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\Gamma_0)$:

$$\begin{aligned}\mathcal{F}'(\rho_0)\delta\rho &= D_\tau[\mathcal{F}(\rho_0 + \tau\delta\rho)]\,|_{\tau=0} \\ &= \partial_t\delta\rho + k^+ S_{\rho_0}\partial_n(\delta v_{\rho_0}^+) + k^+(\delta S_{\rho_0})\partial_n v_{\rho_0}^+ - k^+(\delta K_{\rho_0\omega})\partial_\omega v_{\rho_0}^+ - k^+ K_{\rho_0\omega}\partial_\omega(\delta v_{\rho_0}^+) \\ &\equiv \partial_t\delta\rho + f^+(\delta\rho, \delta v_{\rho_0}^+)\end{aligned}$$

where $\delta S_{\rho_0} = (\partial S_{\rho_0}/\partial\rho_0)\delta\rho + [\partial S_{\rho_0}/\partial(\partial_\omega\rho_0)]\partial_\omega\delta\rho$, $\delta K_{\rho_0\omega} = (\partial K_{\rho_0\omega}/\partial\rho_0)\delta\rho + [\partial K_{\rho_0\omega}/\partial(\partial_\omega\rho_0)]\partial_\omega\delta\rho$ and $\delta v_{\rho_0}^\pm$ satisfy

$$\mathcal{L}_{\rho_0}(\delta v_{\rho_0}^\pm) = -(\delta\mathcal{L}_{\rho_0})v_{\rho_0}^\pm \quad \text{in } \Omega_T^\pm \tag{4.1}$$

$$\partial_n(\delta v^+_{\rho_0})(y,t) = 0 \quad \text{on } \Gamma^+_T \tag{4.2}$$
$$(\delta v^-_{\rho_0})(y,t) = 0 \quad \text{on } \Gamma^-_T \tag{4.3}$$
$$(\delta v^+_{\rho_0}) = (\delta v^-_{\rho_0}) \quad \text{on } \Gamma_{0T} \tag{4.4}$$
$$f^+(\delta\rho, \delta v^+_{\rho_0}) = f^-(\delta\rho, \delta v^-_{\rho_0}) \quad \text{on } \Gamma_{0T} \tag{4.5}$$
$$\partial_t \delta\rho + f^+(\delta\rho, \delta v^+_{\rho_0}) = G \quad \text{on } \Gamma_{0T} \tag{4.6}$$

in which $\delta\rho$ and $\delta v^\pm_{\rho_0}$ are unknown, and $\delta\mathcal{L}_{\rho_0}$ is the variation of the operator $\mathcal{L}_\rho$ at $\rho = \rho_0$. The invertibility of $\mathcal{F}'(\rho_0)$ is that for any $G \in \underset{\circ}{C}{}^{1+\alpha}_T(\Gamma_0)$, we must find $\delta\rho$ and $\delta v^\pm_{\rho_0}$ satisfying (4.1)–(4.6). In fact we can prove the following theorem:

Theorem 4.1 *For any $G \in \underset{\circ}{C}{}^{1+\alpha}_T(\Gamma_0)$, the proboem (4.1)–(4.6) has a unique solution $(\delta\rho, \delta v^\pm_{\rho_0})$, $\delta\rho \in \underset{\circ}{\widehat{C}}{}^{2+\alpha}_T(\Gamma_0)$ and*

$$|\delta\rho|_{\widehat{C}^{2+\alpha}_T(\Gamma_0)} \le C|G|_{C^{1+\alpha}_T(\Gamma_0)}$$

here C is the norm of $[\mathcal{F}'(\rho_0)]^{-1}$ only depending on $|v^\pm_{\rho_0}|_{C^{1+\alpha}_T}, |\partial_t v^\pm_{\rho_0}|_{C^{2+\alpha}_T}$ and T.

Proof In order to solve the problem (4.1)–(4.6), as usual, we use the Hanzawa change of function:

$$W^\pm \equiv \delta v^\pm_{\rho_0} - \langle (\nabla_x u^\pm_{\rho_0}) \circ e_{\rho_0 T}, \delta e_{\rho_0 T}\rangle \tag{4.7}$$

here

$$\delta e_{\rho_0 T}(y(\omega,\eta),t) = \Big(\frac{\partial}{\partial\rho_0}x(\omega,\lambda), 0\Big)\delta\rho, \quad \text{for } (y,t) \in N_0 \times [0,T]$$
$$\delta e_{\rho_0 T}(y(\omega,\eta),t) = (0,0) \quad \text{for } (y,t) \notin N_0 \times [0,T]$$

and $u_{\rho_0} = v_{\rho_0} \circ e^{-1}_{\rho_0 T}$. So in fact the mapping $\delta e_{\rho_0 T}$ is the variation of mapping $e_{\rho_0 T}$. It is clear that $W^\pm = \delta v^\pm_{\rho_0} - \partial_n v^\pm_{\rho_0}\delta\rho$ on Γ_{0T}. It is not difficult to calculate (see [5], page 327), under the change of (4.7), that the problem (4.1)–(4.6) becomes:

$$\mathcal{L}_{\rho_0} W^\pm = 0 \quad \text{in } \Omega^\pm_T \tag{4.8}$$
$$\partial_n W^+ = 0 \quad \text{on } \Gamma^+_T \tag{4.9}$$
$$W^- = 0 \quad \text{on } \Gamma^-_T \tag{4.10}$$
$$W^+ + \partial_n v^+_{\rho_0}\delta\rho = W^- + \partial_n v^-_{\rho_0}\delta\rho \quad \text{on } \Gamma_{0T} \tag{4.11}$$
$$f^+(\delta\rho, W^+ + \partial_n v^+_{\rho_0}\delta\rho) = f^-(\delta\rho, W^- + \partial_n v^-_{\rho_0}\delta\rho) \quad \text{on } \Gamma_{0T} \tag{4.12}$$
$$\partial_t\delta\rho + f^+(\delta\rho, W^+ + \partial_n v^+_{\rho_0}\delta\rho) = G \quad \text{on } \Gamma_{0T} \tag{4.13}$$

Thus, the existence and the boundedness of the norm of $[\mathcal{F}'(\rho_0)]^{-1}$ and, consequently, the existence and uniqueness of a classical solution of the problem (2.1)–(2.6) are a consequence of the well-posedness of the problem (4.8)–(4.13) in the respective classes:

$$\delta\rho \in \underset{\circ}{\widehat{C}}{}^{2+\alpha}_T(\Gamma_0) \quad \text{and } W^\pm \in \underset{\circ}{C}{}^{2+\alpha}_T(\overline{\Omega^\pm})$$

In order to eliminate $\delta\rho$ in (4.8)–(4.13), we utilize the condition that the initial speed of the motion of the free surface is different from zero:

$$V_n\mid_{t=0}=-k^+\partial_n v_{\rho_0}^+\mid_{t=0}=-k^-\partial_n v_{\rho_0}^-\mid_{t=0}\ \text{ on } \Gamma_0$$

So $\partial_n v_{\rho_0}^+\mid_{t=0}<0$ and $\partial_n v_{\rho_0}^-\mid_{t=0}<0$ on Γ_0. Considering $k^+\partial_n v_{\rho_0}^+\mid_{t=0}=k^-\partial_n v_{\rho_0}^-|_{t=0}$ on Γ_0 and $k^+>k^->0$, we have $\partial_n v_{\rho_0}^-|_{t=0}<\partial_n v_{\rho_0}^+|_{t=0}<0$ on Γ_0, therefore $(\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-)|_{t=0}>0$ on Γ_0. Since $\partial_n v_{\rho_0}^\pm\in C_T^{2+\alpha}(\overline{\Omega}^\pm)$, if T is small enough, then

$$\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^->0 \text{ on } \Gamma_{0T}$$

From (4.11), we get

$$\delta\rho=(W^--W^+)/(\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-) \tag{4.14}$$

Substituting (4.14) into boundary conditions (4.12) and (4.13), we have the linear quasi-stationary diffraction problem for elliptic equations:

$$\mathcal{L}_{\rho_0}W^\pm=0 \quad \text{in } \Omega_T^\pm \tag{4.15}$$

$$\partial_n W^+=0 \quad \text{on } \Gamma_T^+ \tag{4.16}$$

$$W^-=0 \quad \text{on } \Gamma_T^- \tag{4.17}$$

$$\begin{aligned}&f^+\Big(\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-},W^++\partial_n v_{\rho_0}^+\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-}\Big)\\&=f^-\Big(\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-},W^-+\partial_n v_{\rho_0}^-\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-}\Big)\quad \text{on } \Gamma_{0T}\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\partial_t\Big(\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-}\Big)+f^+\Big(\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-},W^++\partial_n v_{\rho_0}^+\frac{W^--W^+}{\partial_n v_{\rho_0}^+-\partial_n v_{\rho_0}^-}\Big)\\&=G\quad \text{on } \Gamma_0\end{aligned} \tag{4.19}$$

It is not hard to show by the traditional method of "freezing" of coefficients, the existence and uniqueness of the solution $W^\pm\in \overset{\circ}{C}{}_T^{2+\alpha}(\overline{\Omega}^\pm)$ of the problem (4.15)–(4.19) reduce to the problem of well-posedness of the following model problem:

$$\Delta W^\pm=0 \quad \text{in } \Omega_T^\pm \tag{4.20}$$

$$\partial_n W^+=0 \quad \text{in } \Gamma_T^+ \tag{4.21}$$

$$W^-=0 \quad \text{on } \Gamma_T^- \tag{4.22}$$

$$k^+\partial_n W^+=k^-\partial_n W^- \quad \text{on } \Gamma_{0T} \tag{4.23}$$

$$\partial_t(W^--W^+)+C_0k^-\partial_n W^-=G \quad \text{on } \Gamma_{0T} \tag{4.24}$$

$$W^+(y,0)=W^-(y,0) \quad \text{on } \Gamma_0. \tag{4.25}$$

here $C_0>0$ is a constant.

Next we use the method of parameter extension to solve the problem (4.20)–(4.25), that is replacing (4.23) and (4.25) by

$$k^+\partial_n W^+=\tau k^-\partial_n W^- \quad \text{on } \Gamma_{0T}, \tag*{(4.23)$_\tau$}$$

$$W^+(y,0) = \tau W^-(y,0) \quad \text{on } \Gamma_0 \tag{4.25$_\tau$}$$

We consider the proboem (4.20)–(4.22), $(4.23)_\tau$, (4.24) and $(4.25)_\tau$. When $\tau = 1$, this problem is just the problem (4.20)–(4.25). When $\tau = 0$, this problem splits into two problems:

$$\begin{cases} \Delta W^+ = 0 & \text{in } \Omega_T^+ \\ \partial_n W^+ = 0 & \text{on } \Gamma_T^+ \\ \partial_n W^+ = 0 & \text{on } \Gamma_{0T} \\ W^+(y,0) = 0 & \text{on } \Gamma_0 \end{cases} \tag{I}$$

so $W^+ \equiv 0$ in Ω_T^+, and

$$\begin{cases} \Delta W^- = 0 & \text{in } \Omega_T^- \\ W^- = 0 & \text{on } \Gamma_T^- \\ \partial_t W^- + C_0 k^- \partial_n W^- = G & \text{on } \Gamma_{0T} \end{cases} \tag{II}$$

It is not hard to prove that the problem (II) has the unique (global) solution $W^- \in \overset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\overline{\Omega}^-)$ and (see [8], Th. 5.1)

$$|W^-|_{\widehat{C}_T^{2+\alpha}(\overline{\Omega}^-)} \le C_1 |G|_{C_T^{1+\alpha}(\Gamma_0)}$$

here C_1 depends only on C_0, k^- and Γ_0, Γ^-.

In order to get the well-posedness of the problem (4.20)–(4.25), we must have a uniform *a priori* estimate with respect to the solutions $W^\pm \in \overset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\overline{\Omega^\pm})$ of the problem (4.20)–(4.22), $(4.23)_\tau$, (4.24) and $(4.25)_\tau$.

It is clear that we only need to prove the estimate in the neighborhood $N_0 \times [0,T]$ of Γ_{0T}. To do this end, we use the local coordinates (ω, n) in the neighborhood N_0 of Γ_0. After localization, we suppose $W_\tau^\pm$ have compact supports with respect to ω in $\mathbb{R}^1$ and n in $[0, \pm\infty)$. Neglecting the term of lower order, we have

$$\Delta_{\omega,n} W_\tau^\pm(\omega, n, t) = 0 \quad \text{in } \mathbb{R}^1 \times \mathbb{R}^\pm \times [0,T] \tag{4.26}$$

$$k^+ \partial_n W_\tau^+ = \tau k^- \partial_n W_\tau^- \quad \text{on } \mathbb{R}^1 \times [0,T] \tag{4.27}$$

$$\partial_t W_\tau^- - \partial_t W_\tau^+ + C_0 k^- \partial_n W_\tau^- = G \quad \text{on } \mathbb{R}^1 \times [0,T] \tag{4.28}$$

$$W_\tau^+(\omega, 0, 0) = \tau W_\tau^-(\omega, 0, 0) \tag{4.29}$$

Carrying out the Fourier transformation with respect to ω to (4.26)–(4.29), we get

$$\left(-|\xi|^2 + \frac{d^2}{dn^2}\right)\widetilde{W}_\tau^\pm = 0 \quad \text{in } \mathbb{R}_\xi^1 \times \mathbb{R}^\pm \times [0,T] \tag{4.30}$$

$$k^+ \partial_n \widetilde{W}_\tau^+ = \tau k^- \partial_n \widetilde{W}_\tau^- \quad \text{on } \mathbb{R}_\xi^1 \times [0,T] \tag{4.31}$$

$$\partial_t \widetilde{W}_\tau^- - \partial_t \widetilde{W}_\tau^+ + C_0 k^- \partial_n \widetilde{W}_\tau^- = \widetilde{G} \quad \text{on } \mathbb{R}_\xi^1 \times [0,T] \tag{4.32}$$

$$\widetilde{W}_\tau^+(\xi, 0, 0) = \tau \widetilde{W}_\tau^-(\xi, 0, 0) \quad \text{on } \mathbb{R}_\xi^1 \tag{4.33}$$

here $\widetilde{W}_\tau^\pm(\xi,n,t)$ is the Fourier transformation of $W_\tau^\pm(\omega,n,t)$ with respect to ω.

From (4.30), considering $\lim\limits_{|\xi|\to\infty} \widetilde{W}_\tau^\pm = 0$ we know that

$$\widetilde{W}_\tau^\pm(\xi,n,t) = p^\pm(\xi,t)e^{\mp|\xi|n}$$

$p^\pm(\xi,t)$ are to be determined later on. Substituting it into (4.31)-(4.33), we get

$$-k^+|\xi|p^+(\xi,t) = \tau k^-|\xi|p^-(\xi,t) \tag{4.34}$$

$$\partial_t p^- - \partial_t p^+ + C_0 k^-|\xi|p^- = \widetilde{G} \tag{4.35}$$

$$p^+(\xi,0) = \tau p^-(\xi,0) \tag{4.36}$$

From (4.34), we have

$$p^+(\xi,t) = -\frac{\tau k^-}{k^+}p^-(\xi,t) \tag{4.37}$$

Substituting (4.37) into (4.35) and (4.36), we get

$$\frac{k^+ + \tau k^-}{k^+}\partial_t p^- + C_0 k^-|\xi|p^- = \widetilde{G}$$

$$p^-(\xi,0) = 0$$

The solution of this ordinary differential equation is

$$p^-(\xi,t) = \frac{k^+}{k^+ + \tau k^-}\int_0^t e^{-\overline{C}_0|\xi|(t-\mu)}\widetilde{G}(\xi,\mu)d\mu$$

here $\overline{C}_0 = \dfrac{C_0 k^+ k^-}{k^+ + \tau k^-} > 0$. So

$$W_\tau^-(\omega,0,t) = \frac{C_0 k^+ k^-}{k^+ + \tau k^-}\int_0^t d\mu \int_{\mathbb{R}^1} k(\omega - z, t-\mu)G(z,\mu)dz$$

in which

$$k(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^1} e^{-\overline{C}_0|\xi|t}e^{i\omega\xi}d\xi = \frac{\overline{C}_0 t}{\overline{C}_0^2 t^2 + \omega^2}$$

It is easy to find that $K(\omega,t)$ is even in ω and

$$|\partial_t^r \partial_\omega^s k(\omega,t)| \le C(\omega^2 + t^2)^{-\frac{1}{2}(1+r+s)}$$

from this we know that (see the proof of Lemma 1 of [10]) $W_\tau^-(\omega,0,t) \in \underset{\circ}{C}{}_T^{2+\alpha}(\mathbb{R}^1)$ and

$$|W_\tau^-|_{C_T^{2+\alpha}(\mathbb{R}^1)} \le C|G|_{C_T^{1+\alpha}(\mathbb{R}^1)} \tag{4.38}$$

From (4.37), we have

$$|W_\tau^+|_{C_T^{2+\alpha}(\mathbb{R}^1)} \le C|G|_{C_T^{1+\alpha}(\mathbb{R}^1)} \tag{4.39}$$

Differentiating with respect to t in the formula of $w_\tau^-(\omega,0,t)$, and considering

$$\partial_t k(\omega,t) = -\overline{C}_0 \frac{\overline{C}_0^2 t^2 - \omega^2}{(\overline{C}_0^2 t^2 + \omega^2)^2}$$

we have that $\partial_t W_\tau^-(\omega,0,t) \in \underset{\circ}{C}{}_T^{1+\alpha}(\mathbb{R}^1)$ and

$$|\partial_t W_\tau^-|_{C_T^{1+\alpha}(\mathbb{R}^1)} \le C|G|_{C_T^{1+\alpha}(\mathbb{R}^1)} \tag{4.40}$$

From (4.37), we have again

$$|\partial_t W_\tau^+|_{C_T^{1+\alpha}(\mathbb{R}^1)} \le C|G|_{C_T^{1+\alpha}(\mathbb{R}^1)} \tag{4.41}$$

(4.38)–(4.41) point out

$$|W_\tau^\pm|_{\widehat{C}_T^{2+\alpha}(\Gamma_0)} \le C|G|_{C_T^{1+\alpha}(\Gamma_0)}$$

here C is independent of τ.

Comebacking to the problem (4.20)–(4.23)$_\tau$, (4.24) and (4.25)$_\tau$, we have uniform estimate

$$|W_\tau^\pm|_{\widehat{C}_T^{2+\alpha}(\overline{\Omega}^\pm)} \le C|G|_{C_T^{1+\alpha}(\Gamma_0)}$$

of course, C is independent of τ.

In this way, we have proved the well-posedness of the problem (4.20)–(4.25). It means the problem (4.15)–(4.19) has the unique solution $W^\pm \in \underset{\circ}{C}{}_T^{2+\alpha}(\overline{\Omega}^\pm)$ and

$$|W^\pm|_{C_T^{2+\alpha}(\overline{\Omega}^\pm)} \le C|G|_{C_T^{1+\alpha}(\Gamma_0)}$$

From (4.14) we know $\delta\rho \in C_T^{2+\alpha}(\Gamma_0)$ and $\delta\rho\,|_{t=0}=0$, from (4.13) we have that $\partial_t\delta\rho \in C_T^{1+\alpha}(\Gamma_0)$ and $\partial_t\delta\rho\,|_{t=0}=0$, therefore $\delta\rho \in \underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\Gamma_0)$, and

$$|\delta\rho|_{\widehat{C}_T^{2+\alpha}(\Gamma_0)} \le C|G|_{C_T^{1+\alpha}(\Gamma_0)}$$

This completes the proof of Theorem 4.1.

Theorem 2.1 has been proved.

5. Analysis of Condition (1.11)

In this last section we give two sufficient conditions which guarantee the correctness of the condition (1.11).

As we know, at initial time $V_n = -k^+\partial_n u^+ = -k^-\partial_n u^-$, so $V_n\,|_{t=0} > 0$ is equivalent to $\partial_n u^+\,|_{t=0} < 0$ or $\partial_n u^-\,|_{t=0} < 0$.

Lemma 5.1 (First sufficient condition) *Assume that Γ^+ and $\widetilde{\Gamma}_0$ are circles with the same center, Γ^- has polar coordinates representation $r = f(\theta)$, $\widetilde{\Gamma}_0$ is between*

Γ^+ and Γ^-. If Γ_0 is closer to $\widetilde{\Gamma}_0$, $g^+(x,0) < 0$, $g^-(x,0) = cosnt$, then, at $t = 0$, $\partial_n u^\pm |_{\Gamma_0} < 0$.

Proof We use polar coordinates on $t = 0$, suppose that $\widetilde{u}^\pm$ satisfy

$$\begin{aligned} &\frac{1}{r}\partial_r(r\partial_r\widetilde{u}^\pm) + \frac{1}{r^2}\partial_\theta^2\widetilde{u}^\pm = 0 \quad \text{in } \Omega^\pm \\ &\partial_r\widetilde{u}^+ = -g^+(>0) \quad \text{on } \Gamma^+ \\ &\widetilde{u}^- = g^-(=\text{const}) \quad \text{on } \Gamma^- \\ &\widetilde{u}^+ = \widetilde{u}^- \quad \text{on } \Gamma_0 \\ &k^+\partial_r\widetilde{u}^+ = k^-\partial_r\widetilde{u}^- \quad \text{on } \Gamma_0 \end{aligned}$$

From the maximum principle we get $\max\widetilde{u}^- = g^-$. Since Γ^- has a representation $r = f(\theta)$, it follows that $\partial_r\widetilde{u}^- > 0$ on Γ^-. Set

$$U^\pm = k^\pm r\partial_r\widetilde{u}^\pm \text{ in } \quad \Omega^\pm$$

then $U^\pm$ satisfy

$$\begin{aligned} &\partial_r^2U^\pm + \frac{1}{r}\partial_rU^\pm + \frac{1}{r^2}\partial_\theta^2U^\pm = 0 \quad \text{in } \Omega^\pm \\ &U^+ > 0 \quad \text{on } \Gamma^+ \\ &U^- > 0 \quad \text{on } \Gamma^- \\ &U^+ = U^- \quad \text{on } \Gamma_0 \\ &\frac{1}{k^+}\partial_rU^+ = \frac{1}{k^-}\partial_rU^- \quad \text{on } \Gamma_0 \end{aligned}$$

From the maximum principle we get $U^\pm > 0$ in $\Omega^\pm$, so $U^\pm > 0$ on Γ_0, it means $\partial_r\widetilde{u}^\pm > 0$ on Γ_0 ($\partial_n\widetilde{u}^\pm |_{\Gamma_0} < 0$).

Using the perturbation of the interface of the diffraction problem (see [3], Theorem 1.1) we know that

$$\Big|\partial_n u^\pm |_{\Gamma_0} - \partial_n\widetilde{u}^\pm |_{\Gamma_0}\Big|_{C^{1+\alpha}} \le |\Gamma_0 - \widetilde{\Gamma}_0|_{C^{2+\alpha}}$$

so if Γ_0 is closer to $\widetilde{\Gamma}_0$, we have

$$\partial_n u^\pm|_{\Gamma_0} < 0$$

Lemma 5.2 (Second sufficient condition) *Suppose that $g^+(x,0) < 0$, $g^-(x,0) > 0$, if Γ_0 is the solution of steady two-phase Stefan problem, then we have*

$$\partial_n u^\pm(x,0)|_{\Gamma_0} < 0$$

Proof From assumptions, we know that Γ_0 and $u^\pm(x,0)$ satisfy

$$\Delta u^\pm = 0 \quad \text{in } \Omega^\pm \tag{5.1}$$

$$\partial_n u^+ = g^+(<0) \quad \text{on } \Gamma^+ \tag{5.2}$$

$$u^- = g^-(>0) \quad \text{on } \Gamma^- \tag{5.3}$$

$$u^+ = u^- = 0 \quad \text{on } \Gamma_0 \tag{5.4}$$

$$k^+ \frac{\partial u^+}{\partial n} = k^- \frac{\partial u^-}{\partial n} \quad \text{on } \Gamma_0 \tag{5.5}$$

If we set

$$U = \begin{cases} k^+ u^+ & \text{in } \Omega^+ \\ k^- u^- & \text{in } \Omega^- \end{cases}$$

then, from (5.1)–(5.5), U satisfies

$$\Delta U = 0 \quad \text{in } \Omega^+ \cup \Gamma_0 \cup \Omega^-$$
$$\partial_n U = k^+ g^+ \quad \text{on } \Gamma^+,$$
$$U = k^- g^- \quad \text{on } \Gamma^-$$

By the maximum principle we have

$$U < 0 \quad \text{in } \Omega^+,\ U > 0 \quad \text{in } \Omega^-, \text{ and } \partial_n U < 0 \quad \text{on } \Gamma_0$$

so we have $\partial_n u^\pm|_{\Gamma_0} < 0$.

On the other hand, Γ_0 is the zero-level curve of harmonic function U and it is therefore an analytic curve (see [9]).

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