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DECAY PROPERTIES OF GLOBAL SOLUTIONS FOR BENJAMIN-ONO EQUATION OF HIGH ORDER

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Abstract The decay properties of global solutions for the Benjamin-Ono equation of high order are obtained as $|x| \to \infty$. An Iorio's type result is derived for this equation.
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1. Introduction

In this paper, we study the decay properties of global solutions for Benjamin-Ono (BO) equation of high order. In [1–3], the authors establish the existence and uniqueness of global solutions for the BO equation and its high order approximate form. The decay properties of the solution to the BO equation in Sobolev spaces $J_r = H^r(R) \cap L_r^2(R)$ are also obtained in [1]. Our aim in the present paper is to discuss the decay properties of the solution for the BO equation of high order and an Iorio's type result is derived. Before starting our work we introduce the definition of a weighted Sobolev space.

Definition 1.1 *Define $J_{r,s} = H^s(R) \cap L_r^2(R)$ with the following norm*

$$\|f\|_{J_{r,s}}^2 = \|f\|_s^2 + \|f\|_{L_r^2}^2, \quad f \in J_{r,s}$$

where $H^s(R)$ is the usual real Sobolev space and $L_r^2(R)$ is the collection of all measurable functions $f : R \to R$ such that

$$\|f\|_{L_r^2}^2 = \int (1+x^2)^r |f|^2 dx < +\infty$$

For simplicity, we write $J_r = J_{r,r}$. Obviously, $J_{r,s} \subset J_r$ and $J_{s,s} \subset H^s$ if $r \le s$.

Using the main result of Appendix A in [1] we can easily prove the following

Theorem 1.1 *If $f \in J_{r,s}, r \leq s$, then $x^\beta \partial_x^\alpha f \in L^2$ for all nonnegative integers α, β, $0 \leq \alpha + \beta \leq r$; and there exists a constant $C_{\alpha,\beta} > 0$ such that*

$$\left\| x^\beta \partial_x^\alpha f(x) \right\|_0 \leq C_{\alpha,\beta} \|f\|_{J_r} \leq C_{\alpha,\beta} \|f\|_{J_{r,s}}$$

Since the ideas and methods involved here are the same as those of [2], we shall only indicate the main points without going into much details in establishing our results. For further information on the BO equation of high order and symbols used here, we refer the reader to [2]. For convenience, we would like to list the following frequently used symbols in this paper as follows:

$$E_\mu(\xi, t) = \exp[-(\mu\xi^4 + 4i\xi^3)t]; \widehat{S_\mu(t)}\nu = E_\mu(\xi, t)\hat{\nu}(\xi)$$

The following two are the initial value problems for the BO equation of high order and its parabolic regularization.

$$\begin{aligned} &\partial_t u = -\partial_x(u^3 + 3uHu_x + 3H(uu_x) - 4u_{2x}) \\ &u(x,0) = \varphi(x) && (1.1) \\ &\partial_t u = -\mu\partial_x^4 u - \partial_x(u^3 + 3uHu_x + 3H(uu_x) - 4u_{2x}), \quad 0 < \mu < 1 \\ &u(x,0) = \varphi(x) && (1.2) \end{aligned}$$

2. Main Results and Their Concise Proof

In this part we establish an Iorio's type result for problems (1.1), (1.2). The results show that the presence of higher order local dispersive term (and even higher order local dissipative terms) are not necessarily capable of taming the slow decay of solutions. In this case, the slow decay would be the effect of the non-local nonlinear terms.

Theorem 2.1 *Let $\mu \in (0,1)$ be fixed, $\varphi \in J_{2,4}$. Then there exist a constant $T' > 0$ (depending only on $\|\varphi\|_{J_{2,4}}$), and a unique function $u_\mu \in C([0,T']; J_{2,4})$ such that*

$$u_\mu = S_\mu(t)\varphi - \int_0^t S_\mu(t-\tau)\partial_x \left(u_\mu^3 + 3u_\mu H(\partial_x u_\mu) + 3H(u_\mu \partial_x u_\mu) \right) d\tau$$

Proof First consider the following complete metric space

$$I_{2,4}(T) = \{f; f \in C([0,T]; J_{2,4}), \|f(t) - S_\mu(t)\varphi\|_{J_{2,4}} \leq \|\varphi\|_{J_{2,4}}, \ t \in [0,T]\}$$

The topology of $I_{2,4}(T)$ is induced by $\operatorname*{Sup}_{[0,T]} \|f(t) - g(t)\|_{J_{2,4}}$ for $f(t), g(t) \in I_{2,4}(T)$

Define mapping A as follows:

$$Af(t) = S_\mu(t)\varphi - \int_0^t S_\mu(t-\tau)\partial_x \left(f^3 + 3fHf_x + 3H(ff_x) \right) d\tau$$

Up to now, what we need is to prove that there exists a small positive constant T' such that for this T', mapping A is contractive in $I_{2,4}(T)$ with T replaced by T'. The proof is carried out in the following three steps.

a) First we show that $Af \in C([0,T];J_{2,4})$. According to [2], $Af \in C([0,T];H^4)$. So it suffices to point out $Af \in C([0,T];L_2^2)$. To this end, consider

$$\widehat{Af}(\xi,t) = E_\mu(\xi,t)\hat{\varphi} - \int_0^t E_\mu(\xi+\tau)i\xi(\widehat{f^3+3fHf_x+3H(ff_x)})d\tau \tag{2.1}$$

By using Leibniz formula we get

$$\begin{aligned}\partial_\xi^2\widehat{Af}(\xi,t) = &\left[p_6(\xi)t^2+p_2(\xi)t\right]E_\mu(\xi,t)\hat{\varphi} + p_3(\xi)tE_\mu(\xi,t)\partial_\xi\hat{\varphi}(\xi) + E_\mu(\xi,t)\partial_\xi^2\hat{\varphi}(\xi)\\ &-\int_0^t\left[p_6(\xi)(t-\tau)^2+p_2(\xi)(t-\tau)\right]E_\mu(\xi,t-\tau)i\xi(\widehat{f^3}+3\widehat{fHf_x}\\ &+3\widehat{H(ff_x)})d\tau - \int_0^t p_3(\xi)(t-\tau)E_\mu(\xi,t-\tau)\partial_\xi(i\xi(\widehat{f^3}+3\widehat{fHf_x}\\ &+3\widehat{H(ff_x)})d\tau - \int_0^t E_\mu(\xi,t-\tau)\partial_\xi^2[i\xi(\widehat{f^3}+3\widehat{fHf_x}+3\widehat{H(ff_x)})]d\tau\end{aligned}$$

where $p_j(\xi)$ are polynomials in ξ of degree j. Using Theorem 2.1 of [2] we get

1) $\left\|p_6(\xi)t^2E_\mu(\xi,t)\hat{\varphi}\right\|_0 \le Ct^2\sum_{j=0}^6\left(1+(4\mu t)^{-\frac{j}{4}}\right)\|\varphi\|_{J_{2,4}}$,

2) $\|p_3(\xi)tE_\mu(\xi,t)\partial_\xi\hat{\varphi}+p_2(\xi)tE_\mu(\xi,t)\hat{\varphi}\|_0 \le Ct\sum_{j=0}^3\left(1+(4\mu t)^{-\frac{j}{4}}\right)\|\varphi\|_{J_{2,4}}$,

3) $\left\|E_\mu(\xi,t)\partial_\xi^2\hat{\varphi}\right\|_0 \le \|\varphi\|_{J_{2,4}}$,

4) $$\begin{aligned}&\left\|\int_0^t p_6(\xi)(t-\tau)^2E_\mu(\xi,t-\tau)i\xi[\widehat{f^3}+3\widehat{fHf_x}+3\widehat{H(ff_x)}]d\tau\right\|_0\\ &\le C\int_0^t(t-\tau)^2\sum_{j=0}^6\left(1+(4\mu(t-\tau))^{-\frac{j}{4}}\right)\left(\|f\|_{J_{2,4}}^3+\|f\|_{J_{2,4}}^2\right)d\tau,\end{aligned}$$

5) $$\begin{aligned}&\left\|\int_0^t E_\mu(\xi,t-\tau)(t-\tau)[p_3(\xi)\partial_\xi(\xi\widehat{f^3})+p_2(\xi)\xi\widehat{f^3}]d\tau\right\|_0\\ &\le C\int_0^t(t-\tau)\sum_{j=0}^4\left(1+(4\mu(t-\tau))^{-\frac{j}{4}}\right)\|f\|_{J_{2,4}}^2d\tau,\end{aligned}$$

6) $$\begin{aligned}&\left\|\int_0^t E_\mu(\xi,t-\tau)(t-\tau)[p_3(\xi)\partial_\xi(\xi\widehat{fHf_x})+p_2(\xi)\xi\widehat{fHf_x}]d\tau\right\|_0\\ &\le C\int_0^t(t-\tau)\sum_{j=0}^4\left(1+(4\mu(t-\tau))^{-\frac{j}{4}}\right)\|f\|_{J_{2,4}}^2d\tau,\end{aligned}$$

7) For $\partial_\xi(i\xi\widehat{H(ff_x)}) = i\widehat{H(ff_x)} - 2\xi\delta(\xi)\widehat{ff_x} + i\xi\widehat{H(xff_x)}, \xi\delta(\xi)\widehat{ff_x} = 0$,

and $\|H(ff_x)\|_0 \le \|xf\|_0\|f\|_4$, we have

$$\left\| \int_0^t E_\mu(\xi, t-\tau)(t-\tau)[p_3(\xi)\partial_\xi(\xi \widehat{H(ff_x)}) + p_2(\xi)\xi \widehat{H(ff_x)}]d\tau \right\|_0$$

$$\le C\int_0^t (t-\tau)\sum_{j=0}^4 \left(1 + (4\mu(t-\tau))^{-\frac{j}{4}}\right)\|f\|^2_{J_{2,4}}d\tau,$$

8) $$\left\| \int_0^t E_\mu(\xi, t-\tau)\partial_\xi^2(\xi \widehat{f^3} + 3\xi \widehat{H(ff_x)} + 3\xi \widehat{fHf_x})d\tau \right\|_0$$

$$\le C\int_0^t \sum_{j=0}^1 \left(1 + (4\mu(t-\tau))^{-\frac{j}{4}}\right)\left(\|f\|^2_{J_{2,4}} + \|f\|^3_{J_{2,4}}\right)d\tau$$

From the estimates obtained above we see that for every fixed $\mu > 0$ there exists a $T_1 > 0$ such that

$$\|S_\mu\varphi\|_{J_{2,4}} \le \|\varphi\|_{J_{2,4}}$$

and

$$Af(t) \in C([0, T_1]; J_{2,4})$$

b) For T_1 described above we have for $f \in I_{2,4}(T_1)$

$$\|f(t)\|_{J_{2,4}} = \|f(t) - S_\mu(t)\varphi + S_\mu(t)\varphi\|_{J_{2,4}} \le \|f(t) - S_\mu(t)\varphi\|_{J_{2,4}} + \|S_\mu(t)\varphi\|_{J_{2,4}} \le 2\|\varphi\|_{J_{2,4}}$$

c) Define

$$Bf(t) = \int_0^t S_\mu(t-\tau)\partial_x(f^3 + 3fHf_x + 3H(ff_x)d\tau$$

For $f, g \in I_{2,4}(T_1)$ we can obtain

$$\|Bf(t)\|_{J_{2,4}} \le p_3(\|\varphi\|_{J_{2,4}})\int_0^t \left(1 + (4\mu(t-\tau))^{-\frac{1}{2}}\right)d\tau$$

and

$$\|Bf - Bg\|_{J_{2,4}} \le p_2(\|\varphi\|_{J_{2,4}})\|f - g\|_{J_{2,4}}\int_0^t \left(1 + (4\mu(t-\tau))^{-\frac{1}{2}}\right)d\tau$$

From c) we know that there exists $T' > 0$ such that the mapping A is strictly contractive in the complete metric space $I_{2,4}(T')$. This concludes the proof.

We are now ready to state the following conclusions.

Theorem 2.2 *Let $\mu \in (0,1)$ be fixed. The solution u_μ of the integro-differential equation in Theorem 2.1 is the unique solution satisfying (1.2) such that $u_\mu \in C([0, T'];$ $J_{2,4})$ and $\partial_t u_\mu \in C([0, T']; L^2)$.*

Theorem 2.3 *Let $\mu \in (0,1)$ be fixed, $\varphi \in J_{2,4}$, and $u_\mu \in C([0,T]; J_{2,4})$ be the solution of (1.2). Then $x^2u_\mu, xu_\mu \in C((0,T]; H^s)$ for $s \ge 0$, $u_\mu \in C((0,T]; H^r)$ for $r \ge 4$ and $\partial_t u_\mu \in C((0,T]; H^q)$ for $q \ge 0$.*

Corollary 2.1 *Let $\mu \in (0,1)$ be fixed, $\varphi \in J_{2,4}$, $u_\mu \in C([0,T]; J_{2,4})$ be the solution of problem (1.2). Then for $t \in (0,T]$, $x^j\partial_x^m u_\mu(t)$ is continuous and tends to zero as $|x| \to \infty$, where $j, m = 0, 1, 2, \cdots$*

Theorem 2.4 *Suppose that $\mu \in (0,1)$, $\varphi \in J_{2,4}$ and $u_\mu \in C([0,T]; J_{2,4})$ solves problem (1.2). Then*

$$\|u_\mu(t)\|_{L_2^2}^2 \le C, \quad \forall t \in [0,T]$$

where C depends only on $\|\varphi\|_{J_{2,4}}$ and T.

Proof For simplicity, write $u = u_\mu$. From [2] we know that for every $T > 0$, $\|u\|_4$ is bounded on $[0,T]$, uniformly for $\mu \in (0,1)$ and its majoration on $[0,T]$ depends only on $\|\varphi\|_4$. We now consider $t > 0$

$$\frac{d}{dt}\|u(t)\|_{L_2^2}^2 = 2\int (1+x^2)^2 u\left\{-\mu\partial_x^4 u - \partial_x\left(u^3 + 3uHu_x + 3H(uu_x) - 4\partial_x^2 u\right)\right\}dx$$

a) $2\int u\left\{-\mu\partial_x^4 u - \partial_x\left(u^3 + 3uHu_x + 3H(uu_x) - 4\partial_x^2 u\right)\right\}dx \le C,\ \mu \in (0,1),$

b) $4\int x^2 u\left\{-\mu\partial_x^4 u - \partial_x\left(u^3 + 3uHu_x + 3H(uu_x) - 4\partial_x^2 u\right)\right\}dx$

$\le C\left(\|u\|_{L_2^2}^2 + 1\right),\ \mu \in (0,1).$

In order to estimate $R = \int x^4 u\left\{-\mu\partial_x^4 u - \partial_x\left(u^3 + 3uHu_x + 3H(uu_x) - 4\partial_x^2 u\right)\right\}dx$, we need the following two statements:

i) Under the assumptions of Theorem 2.4 we have

$$\|xu_{2x}\|_0^2 \le C(\|u\|_{L_2^2} + 1)$$

In fact,

$$\begin{aligned}\int x^2 u_{2x}^2 dx &= \int u\partial_x^2\left(x^2 u_{2x}\right)dx = 2\int uu_{2x}dx + 4\int xuu_{2x}dx + \int x^2 u\partial_x^4 u dx \\ &\le C(\|u\|_{L_2^2} + 1),\ \mu \in (0,1)\end{aligned}$$

Similarly

$$\|xu_x\|_{L_2^2}^2 \le C(\|u\|_{L_2^2} + 1), \quad \mu \in (0,1)$$

ii) Under the assumptions of Theorem 2.4 we have

$$x^2 H\left(\partial_x^2 p(u)\right) = H\left(x^2\partial_x^2 p(u)\right),\ xH(\partial_x p(u)) = H(x\partial_x p(u))$$

where $p(u)$ is a polynomial of u. In fact,

$$\begin{aligned}x^2 H\left(\partial_x^2 p(u)\right) &= \frac{1}{n}\int \frac{x^2\partial_y^2 p(u)}{y-x}dy = \frac{1}{\pi}\left\{\int -x\partial_y^2 p(u)dy + \int \frac{xy\partial_y^2 p(u)}{y-x}dy\right\} \\ &= -x\partial_y p[u(y)]\,|_{-\infty}^{+\infty} - y\partial_y p[u(y)]\,|_{-\infty}^{+\infty} + p[u(y)]\,|_{-\infty}^{+\infty} + H\left(x^2\partial_x^2 p(u)\right) \\ &= H\left(x^2\partial_x^2 p(u)\right)\end{aligned}$$

The second identity can be proved similarly.

Now we turn to the estimate for R as follows:

R$_1$) $-\mu\int x^4 u\partial_x^4 u dx \le 12\mu\left|\int x^2u_x^2dx\right| + 12\mu\left|\int x^2uu_{2x}dx\right| \le C\mu\left(\|u\|_{L_2^2}^2+1\right)$,

R$_2$) $\left|\int x^4 u\partial_x(u^3)dx\right| = 3\left|\int x^4u^3u_xdx\right| \le \|u\|_{L_2^2}$,

R$_3$) $\left|\int x^4u(uHu_x)_xdx\right| \le C\|u\|_{L_2^2}^2$,

R$_4$) Using statements i), ii) we get

$$\left|\int x^4u(H(u^2))_{xx}dx\right| \le \left|\int x^2uH\left(x^2\partial_x^2(u^2)\right)dx\right| + 2\left|\int x^2uH\left(x^2u_x^2\right)dx\right|$$
$$\le 2\|u_{2x}\|_{L^\infty}\|u\|_{L_2^2}^2 + 2\|xu_x\|_0^{\frac{1}{2}}(\|u_x\|_0+\|xu_{2x}\|_0)^{\frac{1}{2}}\|x^2u\|_0\|xu_x\|_0$$
$$\le C\left(\|u\|_{L_2^2}^2+1\right),$$

R$_5$) $\left|\int x^4u\partial_x^3udx\right| \le C\left(\|u\|_{L_2^2}^2+1\right)$.

From R$_1$–R$_5$) we obtain

$$\int x^4u\left[-\mu\partial_x^4u - \partial_x(u^3+3uHu_x+3H(uu_x)) - 4\partial_x^2u\right]dx \le C\left(\|u\|_{L_2^2}^2+1\right),\ \mu\in(0,1)$$

Therefore

$$\frac{d}{dt}\|u(t)\|_{L_2^2}^2 \le C\left(\|u\|_{L_2^2}^2+1\right),\quad t>0,\ \mu\in(0,1) \tag{2.2}$$

Applying Gronwall's inequality to (2.2) yields the result of the theorem.

Corollary 2.2 *Let $\mu\in(0,1)$, $\varphi\in J_{2,4}$. Then there exists a unique $u_\mu\in C([0,\infty);J_{2,4})$ such that $\partial_tu_\mu\in C([0,\infty);H^0)$ and u_μ satisfies problem (1.2).*

Employing the same arguments as used in [1, 2] we can easily establish

Theorem 2.5 *Let $\varphi\in J_{2,4}$ with $\|\varphi\|_0<\sqrt{2}/3$. Then there exists a unique $u_0\in C([0,\infty);J_{2,4})$ such that $\partial_tu_0\in C([0,\infty);H^1)$ and u_0 satisfies problem (1.1).*

Corollary 2.3 *Let $\varphi\in J_{2,s}$, $s\ge3$, $\|\varphi\|_0<\sqrt{2}/3$. Then there exists a unique $u_0\in C([0,\infty);J_{2,3})$ such that $\partial_tu_0\in C([0,\infty);H^{s-3})$ and u_0 satisfies problem (1.1).*

The final result of this section shows that there exists an upper limit for the rate of decay of the solutions of the μ-BO equation of high order, which is similar to that of the μ-BO equation [1]. Precisely speaking, we have

Theorem 2.6 *Let $\mu\ge0$ be fixed and $u\in([0,T];J_{3,4})$ solve problem (1.2). Then $u(t)=0$, $\forall t\in(0,T]$.*

Proof First assume $\mu>0$. Applying ∂_ξ^3 to both sides of (2.1), we know that

$$\partial_\xi^3\hat{u}(\xi,t) = \widehat{f_1} - \int_0^t E_\mu(\xi,t-\tau)\partial_\xi^3\left(\frac{3}{2}\widehat{H(u^2)_{2x}}\right)d\tau, t>0$$

where $f_1\in L^2(R)$. We now consider $\int_0^t E_\mu(\xi,t-\tau)\partial_\xi^3\left(\widehat{H(u^2)_{2x}}\right)d\tau$. Since $\widehat{H(u^2)_{2x}} = -ih(\xi)\xi^2\widehat{u^2}$, $h(\xi)=\begin{cases}-1, \ \xi<0\\ 1, \ \xi>0\end{cases}$, we obtain

$$\partial_\xi^3(ih(\xi)\xi^2\widehat{u^2}) = i[\delta''(\xi)\xi^2\widehat{u^2} + 6\delta'(\xi)\xi\widehat{u^2} + 6\delta(\xi)\widehat{u^2}] + ih(\xi)\left[\xi^2\partial_\xi^3\widehat{u^2} + 6\xi\partial_\xi^2\widehat{u^2} + 6\partial_\xi\widehat{u^2}\right]$$

$$\equiv \widehat{f_2} + \widehat{f_3}$$

It is easy to see that $\int_0^t E_\mu(\xi, t-\tau)\widehat{f_3}d\tau \in L^2(R)$. Obviously

$$f_2 = i\left[\int (x-y)^2\partial_y^2(u^2)dy + 6\int (x-y)\partial_y(u^2)dy + 6\int u^2 dy\right]$$

According to Corollary 2.1 and the fact $J_{3,4} \subset J_{2,4}$, we know that $f_2 = 14i\int u^2 dx$. So $\widehat{f_2} = 14i\delta(\xi)\widehat{u^2} = 14i\widehat{u^2}(0)$. From the properties of Dirac distribution, we derive

$$\int_0^t E_\mu(\xi, t-\tau)\delta(\xi)\widehat{u^2}d\tau = \int_0^t \delta(\xi)\widehat{u^2}(0)d\tau$$

If $\widehat{u^2}(0) \neq 0$, then

$$\int_0^t \delta(\xi)\widehat{u^2}(0)d\tau = \int_0^t \delta(\xi)\widehat{u^2}(\xi)d\tau = \int_0^t \left(\int \widehat{u^2}dx\right)d\tau$$

Therefore

$$F^{-1}\left(\int_0^t \delta(\xi)\widehat{u^2}(0)d\tau\right) = \int_0^t \left(\int u^2 dx\right)d\tau = C(t),\ t > 0$$

where F^{-1} denotes the inverse Fourier transform and $C(t) > 0$. But $F^{-1}\Big(\partial_\xi^3\hat{u}(\xi,t) - \widehat{f_1} - \frac{3}{2}\int_0^t E_\mu(\xi, t-\tau)\widehat{f_3}(\tau)d\tau\Big) \in L^2$ and $0 < C(t) \notin L^2(R_x)$, which leads to a contradiction. This shows $C(t) = 0$, $t > 0$, i.e., $\int u^2 dx = 0$. Consequently, $u = 0$, $\forall t \in (0, T]$.

For $\mu \equiv 0$, it is easy to prove the theorem by using the fact that $1 \notin H^s$ for $s \in R$.

Theorem 2.7 *Let $\mu \geq 0$ be fixed in problem* (1.2). *Assume that $u \in C([0,T]; J_3)$ satisfies problem* (1.2). *Then $u(t) = 0$, $\forall t \in (0, T]$.*

Remark For $\varphi \in J_3$, $\varphi \neq 0$, problem (1.1) has no solution in $C([0,T]; J_3)$.

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