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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A CLASS OF SINGULAR PARABOLIC EQUATIONS

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Abstract We will establish an existence and regularity theory for weak solutions of a class of singular parabolic equations associated with Dirichlet data, whose prototype is

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad (1 < p < 2)$$

Key Words Existence; regularity; singular; weak solutions; parabolic equations.
Classification 35D05, 35D10, 35K65.

0. Introduction and Statement of Results

The aim of this paper is to obtain $C^{1+\alpha}$–weak solutions (in a sense to be made precise) for the first boundary value problems of a class of singular parabolic equations

$$u_t - \operatorname{div}\vec{a}(x,t,u,\nabla u) + b(x,t,u,\nabla u) = 0 \text{ in } \Omega_T \tag{0.1}$$

$$u = \varphi \text{ on } \partial^*\Omega_T \tag{0.2}$$

where $\Omega_T = \Omega \times (0,T]$, $0 < T < \infty$, $\Omega \subset \boldsymbol{R}^N$ is a bounded domain with $\partial\Omega \in C^2$, $\partial^*\Omega_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$, ∇u denotes the gradient with respect only to the spatial variable $x = (x_1, x_2, \cdots, x_N)$.

Throughout this paper, we make the following assumptions on $\vec{a} = (a_1, a_2, \cdots, a_N)$ and b

(A_1) $\quad a_j \in C(\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N)$, $a_j(x,t,z,0) = 0$; $\dfrac{\partial a_j}{\partial x_i}$ and $\dfrac{\partial a_j}{\partial z}$, $\quad \dfrac{\partial a_j}{\partial p_i}$

exist, respectively, in $\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N$ and in $\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}$; $b(x,t,z,\eta)$ is measurable in $(x,t) \in \Omega_T$ and continuous in (z,η) in $\boldsymbol{R} \times \boldsymbol{R}^N$.

(A_2) $$\lambda|\eta|^{p-2}|\xi|^2 \le \frac{\partial a_j(x,t,z,\eta)}{\partial p_i}\xi_i\xi_j \le \Lambda|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \boldsymbol{R}^N,\ \eta \in \boldsymbol{R}^n \setminus \{0\}$$

(A_3) $$|\vec{a}(x,t,z,\eta)|, \quad \left|\frac{\partial a_j}{\partial x_i}(x,t,z,\eta)\right| \le \gamma_0(|\eta|^{p-1} + |z| + 1)$$

$$\left|\frac{\partial a_j(x,t,z,\eta)}{\partial z}\right| \le \gamma_0 \begin{cases} (|\eta|^{p-1-\delta_0}+1) & \text{if } |\eta| \ge 1 \\ (|\eta|^{\frac{p-2}{2}}+1) & \text{if } 0<|\eta|<1 \end{cases}$$

$$|b(x,t,z,\eta)| \le \gamma_0(|\eta|^{p-\delta_0}+|z|+1)$$

Here $\lambda, \Lambda, \gamma_0, \delta_0$ and p are positive constants and

$$1<p<2, \quad 0<\delta_0 \le \frac{p-1}{2} \tag{0.3}$$

In this class of equations, the well-known equation is embraced

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{0.4}$$

Existence theory for (0.1) and (0.2) was developed in [9] via Galerkin method under the following extra condition

$$p > \max\left\{1, \frac{2N}{N+2}\right\} \tag{0.5}$$

The interior regularity (∇u locally Hölder continuous) for weak solutions of (0.4) (or system in the same form) was established, respectively, by Chen Ya-zhe [2], DiBenedetto & Friedman [6] under the restriction (0.5). This restriction is critical in their arguments.

We will establish, without the restriction (0.5) the existence of weak solutions with this regularity property by solving classically approximation problems and by deriving some uniform estimates.

Definition 0.1 *By a weak solution of* (0.1), (0.2) *we mean a function u from $V_{2,p}(\Omega_T) = C(0,T;L^2(\Omega)) \cap W_p^{1,0}(\Omega_T)$ satisfying*

$$\int_\Omega u(x,t)\xi(x,t) - \int_\Omega \varphi(x,0)\xi(x,0) + \int_0^t \int_\Omega [-u\xi_t + \vec{a}(x,\tau,u,\nabla u)\cdot\nabla\xi + b(x,\tau,u,\nabla u)\xi] = 0 \tag{0.6}$$

for all $\xi \in \overset{\circ}{W}{}_p^{1,0}(\Omega_T)$ with $\xi_t \in L^2(\Omega_T)$; $0<t\le T$,

$$u(x,t) - \varphi(x,t) \in \overset{\circ}{W}{}_p^{1,0}(\Omega_T) \tag{0.7}$$

For the sake of approximation, we add more assumptions on $\vec{a}$ and b as follows

(B_1) $\exists a_j^\varepsilon \in C^2(\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N) \cap C^\infty(\Omega_T \times \boldsymbol{R} \times \boldsymbol{R}^N)$ such that $a_j^\varepsilon(x,t,z,\eta)$, $j=1,\cdots,N$, converge uniformly to $a_j(x,t,z,\eta)$ in any compact set of $\overline{\Omega}_T \times \boldsymbol{R} \times \boldsymbol{R}^N$ as $\varepsilon \to 0^+$; $a_j^\varepsilon(x,t,z,0)=0$.

(B_2) $$\frac{\lambda}{k_0}[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2 \le \frac{\partial a_j^\varepsilon(x,t,z,\eta)}{\partial p_i}\xi_i\xi_j$$

$$\le k_0\Lambda[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2 \text{ for all } \xi,\eta \in \boldsymbol{R}^N$$

(B_3) $|\vec{a}^\varepsilon(x,t,z,\eta)| \le k_0\gamma_0(\varepsilon|\eta| + |\eta|^{p-1} + |z| + 1)$

$$\left|\frac{\partial a_j^\varepsilon(x,t,z,\eta)}{\partial x_i}\right| \le k_0\gamma_0(|\eta|^{p-1} + |z| + 1)$$

$$\left|\frac{\partial a_j^\varepsilon(x,t,z,\eta)}{\partial z}\right| \le k_0\gamma_0 \begin{cases} ((|\eta| + \varepsilon)^{p-1-\delta_0} + 1) & \text{if } |\eta| \ge \frac{1}{2} \\ ((|\eta| + \varepsilon)^{\frac{p-2}{2}} + 1) & \text{if } |\eta| \le \frac{1}{2} \end{cases}$$

Here constant $k_0 = k_0(p, N, \lambda, \Lambda, \gamma_0)$.

A more general example than (0.4) is the following type

$$u_t - \operatorname{div}(a(x,t,u,|\nabla u|)\nabla u) = 0 \tag{0.8}$$

where $a(x,t,z,r) = \tilde{a}(r) + f(x,t,z)g(x,t,r)$, f is continuous in $(x,t,z) \in \overline{\Omega}_T \times \boldsymbol{R}$, g is continuous in $(x,t,r) \in \overline{\Omega}_T \times \boldsymbol{R}^+$, and

$$\begin{cases} a_0 r^{p-2} \le \tilde{a}(r) \le a_1 r^{p-2} \\ |a'(r)| \le a_0(1-\varepsilon_0) r^{p-3} \end{cases} \quad \text{for } r > 0 \tag{0.9}$$

$$\begin{cases} |f_z|, \ |f_{x_j}|, |f| \le C_0 \\ |g| \le C_1, \ |g_r| \le C_1 C_2, \ |g_{x_j}| \le C_3 \\ g(x,t,r) = 0 \text{ if } 0 \le r \le a_2 \text{ or } r \ge a_3 \end{cases} \tag{0.10}$$

where $a_i, C_i (i = 0, \cdots, 3)$ and ε_0 are positive numbers with $0 < \varepsilon_0 < 1$ and

$$\begin{cases} a_0 - C_0 C_1 C_2 a_3^{2-p}(a_3 - a_2) \overset{\Delta}{=} \bar{a}_0 > 0, \\ [(1-\varepsilon_0)a_0 + C_0 C_1 C_2 a_3^{3-p}]/\bar{a}_0 \overset{\Delta}{=} \bar{\varepsilon}_0 < 1 \end{cases} \tag{0.11}$$

At the end of this section we will give the outline of the proof of $\vec{a}(x,t,z,\eta) = a(x,t,z,|\eta|) \cdot \eta$ satisfying (A_1)-(A_3) and (B_1)-(B_3).

The main results of this paper:

Theorem 0.1 *Let (A_1)-(A_3), (B_1)-(B_3) hold, and let $\partial\Omega \in C^2$, $\varphi \in W_2^{1,1}(\Omega_T) \cap L^\infty(\Omega_T)$. Then problem (0.1, (0.2) has a weak solution $u(x,t)$. Moreover, there exists $\alpha_0 \in (0,1)$ depending only on $N, p, \lambda, \Lambda, \gamma_0$ and δ_0 such that ∇u is locally C^{α_0}-continuous in Ω_T, that is,*

$$\|\nabla u\|_{C^{\alpha_0, \frac{\alpha_0}{2}}(\Omega_{T,\delta})} \le C_\delta$$

where $\Omega_{T,\delta} = \Omega_\delta \times (\delta^2, T]$, $\Omega_\delta = \{x \in \Omega; dist(x, \partial\Omega) > \delta\}$, constant C_δ depends only upon $N, p, \lambda, \Lambda, \gamma_0, \delta_0, T, \delta, \|\varphi\|_{L^\infty}, \|\varphi\|_{W_2^{1,1}}$.

If (A_3) is replaced by

$$(A_3)' \qquad |\vec{a}|, \left|\frac{\partial a_j}{\partial x_i}\right| \le \gamma_0(|\eta|^{p-1} + |z| + 1)$$

$$\left|\frac{\partial a_j}{\partial z}\right|, \left|\frac{\partial b}{\partial p_i}\right| \le \gamma_0|\eta|^{\frac{p-2}{2}}, \forall \eta \in \mathbf{R}^N \setminus \{0\}$$

$$|b|, |b_z|(1 + |z| + |\eta|^{\frac{p}{2}}) \le \gamma_0(1 + |z| + |\eta|^{\frac{p}{2}})$$

we have

Theorem 0.2 *Let (A_1), (A_2) and $(A_3)'$ hold, and let $\varphi \in W_p^{1,0}(\Omega_T)$ with $\varphi_t \in L^2(\Omega_T)$. Then problem* (0.1), (0.2) *cannot have two distinct functions from $V_{2,p}(\Omega_T)$ satisfying* (0.6) *and* (0.7).

As a direct result of Theorems 0.1 and 0.2, we have

Corollary 0.1 *Let $\varphi \in W_2^{1,1}(\Omega_T) \cap L^\infty(\Omega_T)$ and $\partial\Omega_T \in C^2$. Then the following problem* $(1 < p < 2)$

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \ in \ \Omega_T \\ u = \varphi \ on \ \partial^*\Omega_T \end{cases}$$

has a unique weak solution $u \in V_{2,p}(\Omega_T)$. Moreover ∇u is locally C^{α_0}-continuous in Ω_T.

We now show that example (0.8) satisfies our structure conditions. It is easy to obtain, from (0.9)–(0.11), the following inequalities

$$\begin{aligned} a(x,t,z,r) &\ge (a_0 - C_0C_1C_2a_3^{2-p}(a_3 - a_2))r^{p-2} = \bar{a}_0 r^{p-2} \\ a(x,t,z,r) &\le (a_1 + C_0C_1C_2a_3^{3-p})r^{p-2} \triangleq \bar{a}_1 r^{p-2} \end{aligned} \tag{0.12}$$

$$|a_r(x,t,z,r)| \le [(1-\varepsilon_0)a_0 + C_0C_1C_2a_3^{3-p}]r^{p-3} = \bar{\varepsilon}_0\bar{a}_0 r^{p-3} \tag{0.13}$$

$$\begin{aligned} |\vec{a}(x,t,z,\eta)| &= |a(x,t,z,|\eta|)\eta| \le \gamma_0|\eta|^{p-1} \\ \left|\frac{\partial a_j}{\partial x_i}\right|, &\quad \left|\frac{\partial a_j}{\partial z}\right| \le \gamma_0 \end{aligned} \tag{0.14}$$

for some $\gamma_0 = \gamma_0(N, p, a_1, a_3, C_0, C_1)$.

(0.14) implies that (A_3) holds. From (0.12), (0.13),

$$(1-\bar{\varepsilon}_0)\bar{a}_0|\eta|^{p-2}|\xi|^2 \le \frac{\partial a_j(x,t,z,\eta)}{\partial p_i}\xi_i\xi_j \le (\bar{a}_1 + \bar{\varepsilon}_0\bar{a}_0)|\eta|^{p-2}|\xi|^2$$

(i.e. (A_2) holds). For $\eta_1, \eta_2 \in \mathbf{R}^N$,

a) if $|\eta_2| \le 2|\eta_1 - \eta_2|$, then from (0.14)

$$|\vec{a}(\eta_1) - \vec{a}(\eta_2)| \le \gamma_0(|\eta_1|^{p-1} + |\eta_2|^{p-1}) \le 5\gamma_0|\eta_1 - \eta_2|^{p-1}$$

b) if $|\eta_2| > 2|\eta_1 - \eta_2|$ (this implies $|\tau\eta_1 + (1-\tau)\eta_2| \geq |\eta_1 - \eta_2|$ for all $\tau(0,1)$), then

$$|\vec{a}(\eta_1) - \vec{a}(\eta_2)| = \Big|\int_0^1 \frac{d\vec{a}}{d\tau}(\tau\eta_1 + (1-\tau)\eta_2)d\tau\Big|$$
$$\leq C\int_0^1 |\tau\eta_1 + (1-\tau)\eta_2|^{p-2}d\tau|\eta_1 - \eta_2| \leq C|\eta_1 - \eta_2|^{p-1}$$

that is, $\vec{a}(x,t,z,\eta)$ is continuous in η. Obviously, $\vec{a}(x,t,z,\eta)$ is continous in (x,t,z), i.e. (A_1) holds.

As for (B_1)–(B_3), we consider, for simplicity, the case of $\vec{a}(x,t,z,\eta) = a(|\eta|)\eta$. Take $a_j^\varepsilon(\eta) = (a_\varepsilon(\sqrt{|\eta|^2+\varepsilon^2}) + \varepsilon)\eta_j$, where $a_\varepsilon(r) = \int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}} a(r - y + \varepsilon^2)\rho_{\frac{\varepsilon^2}{2}}(y)dy$, $\rho_{\frac{\varepsilon^2}{2}}(y)$ is a usual mollifier.

$$|\vec{a}^\varepsilon(\eta) - \vec{a}(\eta)| \leq \varepsilon|\eta| + \Big|\int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}} [a(\sqrt{|\eta|^2+\varepsilon^2} - y + \varepsilon^2) - a(|\eta|)]\eta\rho_{\frac{\varepsilon^2}{2}}(y)dy\Big|$$
$$= \varepsilon|\eta| + \Big|\int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}}\int_0^1 \frac{d}{d\tau}a(\sqrt{|\eta|^2+\tau\varepsilon^2} + \tau(\varepsilon^2 - y))\eta d\tau\rho_{\frac{\varepsilon^2}{2}}(y)dy\Big|$$
$$\leq \varepsilon|\eta| + \int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}}\int_0^1 |a_r(\cdots)|(\varepsilon^2 - y + \frac{\varepsilon^2}{\sqrt{|\eta|^2+\tau\varepsilon^2}})|\eta|d\tau\rho_{\frac{\varepsilon^2}{2}}(y)dy$$
$$\leq \varepsilon|\eta| + C\int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}}\int_0^1 (\sqrt{|\eta|^2+\tau\varepsilon^2} + \tau(\varepsilon^2 - y))^{p-3}.$$
$$\cdot(\varepsilon^2 + \frac{\varepsilon^2}{\sqrt{|\eta|^2+\tau\varepsilon^2}})d\tau|\eta|\rho_{\frac{\varepsilon^2}{2}}(y)dy$$
$$\leq \varepsilon|\eta| + C\varepsilon^{p-1}\int_0^1 \tau^{\frac{p-3}{2}}d\tau|\eta| + C\varepsilon^{p-1}\int_0^1 \tau^{\frac{p-3}{2}}d\tau$$

that is, (B_1) holds. From (0.12) and (0.13),

$$\frac{\partial a_j^\varepsilon}{\partial p_i}\xi_i\xi_j = \int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}}\Big[a(\sqrt{|\eta|^2+\varepsilon^2} - y + \varepsilon^2)|\xi|^2$$
$$+ a_r(\sqrt{|\eta|^2+\varepsilon^2} - y + \varepsilon^2)\frac{(\eta,\xi)^2}{\sqrt{|\eta|^2+\varepsilon^2}}\Big]\rho_{\frac{\varepsilon^2}{2}}(y) + \varepsilon|\xi|^2$$
$$\geq \Big(\frac{5}{2}\Big)^{p-2}(1-\bar{\varepsilon}_0)\bar{a}_0((|\eta|+\varepsilon)^{p-2} + \varepsilon)|\xi|^2$$
$$\frac{\partial a_j^\varepsilon}{\partial p_i}\xi_i\xi_j \leq C[(|\eta|+\varepsilon)^{p-2} + \varepsilon]|\xi|^2 \quad (C = C(\bar{a}_0, \bar{a}_1, N, p))$$

that is, (B_2) holds. Obviously, $\vec{a}^\varepsilon$ satisfies (B_3).

In what follows we say that a constant C depends only upon the data means that C can be calculated only in terms of $N, p, \lambda, \Lambda, \gamma_0, \delta_0$ and k_0.

1. Estimates on Approximation Solutions

In this section we first solve classically approximation problems, then by Moser's iteration and integration by parts, we derive the boundedness of the gradients of the approximation solutions, and finally by splitting up method, we give C^α-estimate of the gradients of the solutions.

Consider

$$u_t - \mathrm{div}\vec{a}^\varepsilon(x,t,u,\nabla u) + b^\varepsilon(x,t,u,\nabla u) = 0 \text{ in } \Omega_T \tag{1.1}$$

$$u = \varphi^\varepsilon \text{ on } \partial^*\Omega_T \tag{1.2}$$

where $\vec{a}^\varepsilon(x,t,z,\eta) = (a_1^\varepsilon(x,t,z,\eta),\cdots,a_N^\varepsilon(x,t,z,\eta))$ satisfies (B_1)–(B_3); $b^\varepsilon(x,t,z,\eta)$ and $\varphi^\varepsilon(x,t)$ are the usual smoothing approximations of $b(x,t,z,\eta)$ and $\varphi(x,t)$ satisfying

$$\begin{cases} |b^\varepsilon(x,t,z,\eta)| \le 2\gamma_0(|\eta|^{p-\delta_0} + |z| + 1) \\ \|\varphi^\varepsilon\|_{L^\infty(\Omega_T)} \le \|\varphi\|_{L^\infty(\Omega_T)}, \|\varphi^\varepsilon\|_{W_2^{1,1}(\Omega_T)} \le 2\|\varphi\|_{W_2^{1,1}(\Omega_T)} \end{cases} \tag{1.3}$$

From well-known results, see [9, Chapter V], problem (1.1) (1.2) has a unique solution

$$u^\varepsilon(x,t) \in C(\overline{\Omega}_T) \cap C^\infty(\Omega_T) \cap W_2^{2,1}(\Omega_T)$$

Theorem 1.1 *Let $u^\varepsilon(x,t)$ be a solution of (1.1)(1.2), and let (B_1)–(B_3) and (1.3) hold. Then we have*

$$\max_{\Omega_T} |u^\varepsilon| \le M_0 \tag{1.4}$$

and

$$\iint_{\Omega_T} |\nabla u^\varepsilon|^p + \varepsilon \iint_{\Omega_T} |\nabla u^\varepsilon|^2 \le \tilde{M}_0 \tag{1.5}$$

where M_0 depends only upon T, $\|\varphi\|_{L^\infty}$ and the data, $\tilde{M}_0$ only upon $\|\varphi\|_{L^\infty}$, $\|\varphi\|_{W_2^{1,1}}$, $|\Omega_T|$ and the data.

Proof (1.4) follows from [9, II. Theorem 2.9, p.23].

Denote $u^\varepsilon - \varphi^\varepsilon = w$, then from (1.1) (1.2) we have

$$\iint_{\Omega_T} [u_t^\varepsilon w + \vec{a}^\varepsilon \cdot \nabla w + b^\varepsilon w] = 0$$

Straightforward computations reveal that:

$$\iint_{\Omega_T} u_t^\varepsilon w \ge -4|\Omega_T|^{\frac{1}{2}}(\|\varphi\|_{L^\infty} + M_0) \cdot (\|\varphi\|_{W_2^{1,1}})^{\frac{1}{2}}$$

$$\iint_{\Omega_T}\vec{a}^\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\cdot\nabla w=\iint_{\Omega_T}[\vec{a}^\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)-\vec{a}^\varepsilon(x,t,u^\varepsilon,0)]\cdot\nabla u^\varepsilon$$
$$-\iint_{\Omega_T}\vec{a}^\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\cdot\nabla\varphi^\varepsilon$$
$$\geq\frac{\lambda}{k_0}\iint_{\Omega_T}\int_0^1[(s|\nabla u^\varepsilon|+\varepsilon)^{p-2}+\varepsilon]ds|\nabla u^\varepsilon|^2$$
$$-k_0\gamma_0\iint_{\Omega_T}|\nabla\varphi^\varepsilon|[(|\nabla u^\varepsilon|+\varepsilon)^{p-1}+\varepsilon|\nabla u^\varepsilon|+|u^\varepsilon|+1]$$
$$\geq\frac{\lambda}{4k_0}\iint_{\Omega_T}[(|\nabla u^\varepsilon|+\varepsilon)^p+\varepsilon|\nabla u^\varepsilon|^2]-C$$

and

$$\iint_{\Omega_T}b^\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)w\geq-\frac{\lambda}{8k_0}\iint_{\Omega_T}|\nabla u^\varepsilon|^p-C$$

where C depends only upon $\|\varphi\|_{L^\infty}$, $\|\varphi\|_{W_2^{1,1}}$, $|\Omega_T|$ and the data. Estimate (1.5) easily follows from these computations.

Denote

$$\Omega_\delta=\{x\in\Omega;\operatorname{dist}(x,\partial\Omega)>\delta\},\quad \Omega_{T,\delta}=\Omega_\delta\times(\delta^2,T]$$
$$B(R)=\{x\in\boldsymbol{R}^N;|x|<R\},\qquad Q(R)=B(R)\times(-R^2,0]$$
$$\hat{Q}(R)=B(2R)\times(-R^2,0]$$

Theorem 1.2 *Let u^ε be as in Theorem 1.1, and let (B_1)–(B_3) and (1.3) hold. Then there exists M_1 depending only on $\|\varphi\|_{L^\infty}$, $\|\varphi\|_{W_2^{1,1}}$ Ω,T,δ and the data such that*

$$\max_{\Omega_{T,\delta}}|\nabla u^\varepsilon|\leq M_1$$

This theorem is a direct consequence of Theorem 1.1 and the following lemma.

Lemma 1.1 *Let $v\in C(\overline{Q(R)})\cap C^\infty(Q(R))$ be a solution of (1.1) with $\|v\|_{L^\infty}\leq\Gamma_0$, and let (B_1)–(B_3) and (1.3) hold. Then*

$$\max_{Q(\frac{R}{2})}|\nabla v|^2\leq\hat{C}_1(1+R^{-2(N+2)}\iint_{Q(R)}(1+|\nabla v|^p+\varepsilon|\nabla v|^2))$$

where $\hat{C}_1$ depends only upon the data and Γ_0.

Proof For $h=0,1,\cdots$, denote

$$R_h=R\Big(\frac{1}{2}+\frac{1}{2^{h+1}}\Big),\qquad B_h=B(R_h)$$
$$Q_h=Q(R_h),\qquad Q_h^\tau=Q_h\cap\{t\leq\tau\}$$

and take cutoff functions $\xi_h(x,t)\in C_0^{1,1}(Q_h)$ with $0\leq\xi_h\leq1$;

$$\xi_h=1\text{ in }Q_{h+1};\ |\nabla\xi_h|\leq C\frac{2^h}{R},\ |(\xi_h)_t|\leq C\frac{4^h}{R^2}$$

Differentiating (1.1) with respect to x_j, multiplying the both sides of the resulting equation by $\psi_h = v_{x_j} V^\alpha \xi_h^2 (\alpha \geq 0,\ V = |\nabla v|^2 + \varepsilon^2)$, and then integrating and summing over j from 1 to N, we obtain

$$\iint_{Q_h^\tau} \left[(v_{x_j})_t \psi_h + \Big(\frac{\partial a_r^\varepsilon}{\partial p_s} v_{x_s x_j} + \frac{\partial a_r^\varepsilon}{\partial z} v_{x_j} + \frac{\partial a_r^\varepsilon}{\partial x_j} \Big) (\psi_h)_{x_r} - b^\varepsilon (\psi_h)_{x_j} \right] = 0 \tag{1.6}$$

Using Holder's inequality and the elementary inequality: If $A \geq 0,\ 0 \leq a_0 \leq a_1 \leq a_2$, then

$$A^{a_1} \leq A^{a_0} + A^{a_1} \leq 2(1 + A^{a_2})$$

we compute:

$$\iint_{Q_h^\tau} (v_{x_j})_t \psi_h = \frac{1}{2(\alpha+1)} \int_{B_h} V^{\alpha+1} \xi_h^2(x,t) - \frac{1}{\alpha+1} \iint_{Q_h^\tau} V^{\alpha+1} \xi_h (\xi_h)_t \tag{1.7}$$

$$\begin{aligned} &\iint_{Q_h^\tau} \Big(\frac{\partial a_r^\varepsilon}{\partial p_s} v_{x_s x_j} + \frac{\partial a_r^\varepsilon}{\partial z} v_{x_j} + \frac{\partial a_r^\varepsilon}{\partial x_j} \Big) v_{x_r x_j} V^\alpha \xi_h \\ &\geq \frac{\lambda}{k_0} \iint_{Q_h^\tau} [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |D^2 v|^2 V^\alpha \xi_h^2 - C \iint_{Q_h^\tau} (|\nabla v|^{p-\delta_0} + 1) |D^2 v| V^\alpha \xi_h^2 \\ &\geq \Big(\frac{\lambda}{k_0} - \eta_1 \Big) \iint_{Q_h^\tau} [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |D^2 v|^2 V^\alpha \xi_h^2 \\ &\quad - C(\eta_1) \iint_{Q_h^\tau} (|\nabla v|^{2+p-2\delta_0} + 1) V^\alpha \xi_h^2 (\forall \eta_1 > 0) \end{aligned} \tag{1.8}$$

$$\begin{aligned} &\iint_{Q_h^\tau} \Big(\frac{\partial a_r^\varepsilon}{\partial p_s} v_{x_s x_j} + \frac{\partial a_r^\varepsilon}{\partial z} v_{x_j} + \frac{\partial a_r^\varepsilon}{\partial x_j} \Big) \xi_h (\xi_h)_{x_r} V^\alpha v_{x_j} \\ &\geq -C \iint_{Q_h^\tau} \{ [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |D^2 v| + (|\nabla v| + \varepsilon)^{p-\delta_0} + 1 \} \xi_h |\nabla \xi_h| V^\alpha |\nabla v| \\ &\geq -\eta_1 \iint_{Q_h^\tau} [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |D^2 v|^2 V^\alpha \xi_h^2 - C(\eta_1) \iint_{Q_h^\tau} \{ \varepsilon |\nabla v|^2 V^\alpha |\nabla \xi_h|^2 \\ &\quad + |\nabla v|^2 V^{\alpha + \frac{p-1-\delta_0}{2}} \xi_h |\nabla \xi_h| + V^{\alpha + \frac{p}{2}} (|\nabla \xi_h|^2 + 1) + |\nabla \xi_h| \} \end{aligned} \tag{1.9}$$

$$\begin{aligned} \alpha &\iint_{Q_h^\tau} \Big(\frac{\partial a_r^\varepsilon}{\partial p_s} v_{x_s x_j} + \frac{\partial a_r^\varepsilon}{\partial z} v_{x_j} + \frac{\partial a_r^\varepsilon}{\partial x_j} \Big) v_{x_j} V_{x_r} V^{\alpha-1} \xi_h^2 \\ &\geq \frac{\partial \lambda}{2k_0} \iint_{Q_h^\tau} [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |\nabla V|^2 V^{\alpha-1} \xi_h^2 \\ &\quad - \alpha C \iint_{Q_h^\tau} (|\nabla v|^{p-\delta_0} + 1) |\nabla v| |\nabla V| V^{\alpha-1} \xi_h^2 \\ &\geq \alpha \Big(\frac{\lambda}{2k_0} - \eta_1 \Big) \iint_{Q_h^\tau} [(|\nabla v| + \varepsilon)^{p-2} + \varepsilon] |\nabla V|^2 V^{\alpha-1} \xi_h^2 \\ &\quad - \alpha C(\eta_1) \iint_{Q_h^\tau} (|\nabla v|^{2+p-2\delta_0} V^\alpha \xi_h^2 + V^\alpha \xi_h^2) \end{aligned} \tag{1.10}$$

$$
\begin{aligned}
&-\iint_{Q_h^\tau} b^\varepsilon(v_{x_ix_j}V^\alpha\xi_h^2+\alpha v_{x_j}V_{x_j}V^{\alpha-1}\xi_h^2+2v_{x_j}(\xi_h)_{x_j}\xi_hV^\alpha)\\
&\geq -C\iint_{Q_h^\tau}(|\nabla v|^{p-\delta_0}+1)(|D^2v|V^\alpha\xi_h^2+\alpha|\nabla v||\nabla V|V^{\alpha-1}\xi_h^2+|\nabla v||\nabla\xi_h|V^\alpha\xi_h)\\
&\geq -\eta_1\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}|D^2v|^2V^\alpha\xi_h^2-\alpha\eta_1\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}|\nabla V|^2V^{\alpha-1}\xi_h^2\\
&\quad -C(\eta_1)\iint_{Q_h^\tau}\{|\nabla v|^{2+p-2\delta_0}V^\alpha\xi_h^2+|\nabla\xi_h|^2V^{\alpha+\frac{p}{2}}+(1+|\nabla\xi_h|)V^\alpha\\
&\quad +\alpha|\nabla v|^{2+p-2\delta_0}V^\alpha\xi_h^2+\alpha V^\alpha\}
\end{aligned}
\tag{1.11}
$$

It follows, from (1.6)–(1.11) and fixing η_1 suitably small, that

$$
\begin{aligned}
&\frac{1}{2(\alpha+1)}\int_{B_h}V^{\alpha+1}\xi_h(x,\tau)+\frac{\lambda}{2k_0}\iint_{Q_h^\tau}[(|\nabla v|+\varepsilon)^{P-2}+\varepsilon]|D^2v|^2V^\alpha\xi_h^2\\
&\quad+\frac{\alpha\lambda}{4k_0}\iint_{Q_h^\tau}[(|\nabla v|+\varepsilon)^{p-2}+\varepsilon]V^{\alpha-1}|\nabla V|^2\xi_h^2\\
&\leq C\iint_{Q_h^\tau}\Big\{(1+\alpha)|\nabla v|^{2+p-2\delta_0}V^\alpha\xi_h^2+|\nabla v|^2V^{\alpha+\frac{p-1-\delta_0}{2}}\xi_h|\nabla\xi_h|\\
&\quad+\varepsilon|\nabla v|^2V^\alpha|\nabla\xi_h|^2+(1+\alpha)V^\alpha+(1+|\nabla\xi_h|^2)V^{\alpha+\frac{p}{2}}\\
&\quad+\frac{1}{\alpha+1}V^{\alpha+1}\xi_h|(\xi_h)_t|+|\nabla\xi_h|\Big\}
\end{aligned}
\tag{1.12}
$$

where C depends only upon the data and Γ_0.

By virtue of integration by parts, we have

$$
\begin{aligned}
\iint_{Q_h^\tau}V^{\alpha+1}\xi_h|(\xi_h)_t|&\leq C\frac{4^h}{R^2}\iint_{Q_h^\tau}(|\nabla v|^2+\varepsilon^2)V^\alpha\xi_h\\
&=C\frac{4^h}{R^2}\iint_{Q_h^\tau}v_{x_j}v_{x_j}V^\alpha\xi_h+C\varepsilon^2\frac{4^h}{R^2}\iint_{Q_h^\tau}V^\alpha\xi_h\\
&\leq C\frac{4^h}{R^2}\iint_{Q_h^\tau}\{|v||D^2v|V^\alpha\xi_h+|v||\nabla v||\nabla\xi_h|V^\alpha\\
&\quad+\alpha|v||\nabla v||\nabla V|V^{\alpha-1}\Big\}+C\frac{4^h}{R^2}\iint_{Q_h^\tau}V^\alpha\\
&\leq\bar\eta\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}\xi_h^2(|D^2v|^2V^\alpha+\alpha|\nabla V|^2V^{\alpha-1})\\
&\quad+C(\bar\eta)\frac{4^{2h}}{R^4}(\alpha+1)\iint_{Q_h^\tau}(V^{\alpha+\frac{p}{2}}+1)\quad(\forall\bar\eta>0)
\end{aligned}
\tag{1.13}
$$

$$\iint_{Q_h^\tau} |\nabla v|^2 V^{\alpha+\frac{p-1-\delta_0}{2}} \xi_h |\nabla \xi_h| \le C\frac{2^h}{R}\iint_{Q_h^\tau} v_{x_j} v_{x_j} V^{\alpha+\frac{p-1-\delta_0}{2}} \xi_h$$

$$\le C\frac{2^h}{R}\iint_{Q_h^\tau} \{|v||D^2 v| V^{\alpha+\frac{p-1-\delta_0}{2}} \xi_h + |v||\nabla v||\nabla \xi_h| V^{\alpha+\frac{p-1-\delta_0}{2}}$$

$$+\alpha|v||\nabla v||\nabla V| V^{\alpha+\frac{p-1-\delta_0}{2}-1}\xi_h\}$$

$$\le \tilde{\eta}\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}\xi_h^2(|D^2 v|^2 V^\alpha + \alpha|\nabla V|^2 V^{\alpha-1})$$

$$+C(\tilde{\eta})\frac{4^h}{R^2}(\alpha+1)\iint_{Q_h^\tau}(V^{\alpha+\frac{p}{2}}+1) \tag{1.14}$$

Take a positive integer m such that

$$2^m\delta_0 \le p < 2^{m+1}\delta_0 \tag{1.15}$$

By integration by parts and Holder's inequality, we deduce

$$(1+\alpha)\iint_{Q_h^\tau}|\nabla v|^{2+p-2\delta_0} V^\alpha \xi_h^2 = (1+\alpha)\iint_{Q_h^\tau} v_{x_j} v_{x_j}|\nabla v|^{p-2\delta_0} V^\alpha \xi_h^2$$

$$\le (1+\alpha)C\iint_{Q_h^\tau}\{|D^2 v||\nabla v|^{p-2\delta_0} V^\alpha \xi_h^2 + |\nabla v|^{p+1-2\delta_0} V^\alpha \xi_h |\nabla \xi_h|$$

$$+\alpha|\nabla v|^{p+1-2\delta_0}|\nabla V| V^{\alpha-1}\xi_h^2\}$$

$$\le \tilde{\eta}\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}\xi_h^2(|D^2 v|^2 V^\alpha + \alpha|\nabla V|^2 V^{\alpha-1})$$

$$+C(\tilde{\eta})(1+\alpha)^2\iint_{Q_h^\tau}\{(|\nabla v|^{2+p-2^2\delta_0}+1)V^\alpha\xi_h^2 + V^{\alpha+\frac{p}{2}}|\nabla \xi_h|^2\}$$

$$\le \tilde{\eta}\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}\xi_h^2(|D^2 v|^2 V^\alpha + \alpha|\nabla V|^2 V^{\alpha-1})$$

$$+\frac{1}{2}\iint_{Q_h^\tau}|\nabla v|^{2+p-2\delta_0} V^\alpha \xi_h^2 + C(\tilde{\eta})(1+\alpha)^{2^{(m+3)}}\iint_{Q_h^\tau} V^\alpha$$

$$+C(\tilde{\eta})(1+\alpha)^2\iint_{Q_h^\tau} V^{\alpha+\frac{p}{2}}|\nabla \xi_h|^2 \left(\text{since } \frac{2+p-2\delta_0}{\delta_0} \le 2^{m+3}\right)$$

Therefore

$$(1+\alpha)\iint_{Q_h^\tau}|\nabla v|^{2+p-2\delta_0} V^\alpha \xi_h^2$$

$$\le 2\tilde{\eta}\iint_{Q_h^\tau}(|\nabla v|+\varepsilon)^{p-2}\xi_h^2(|D^2 v|^2 V^\alpha + \alpha|\nabla V|^2 V^{\alpha-1})$$

$$+C(\tilde{\eta})(1+\alpha)^{2^{m+3}}\iint_{Q_h^\tau}(V^{\alpha+\frac{p}{2}}+1)(1+|\nabla \xi_h|^2) \tag{1.16}$$

(1.12)-(1.14) and (1.16) reveal that (taking $\tilde{\eta}$ suitably small)

$$\frac{1}{\alpha+1}\max_{-R_h^2<t<0}\int_{B_h}V^{\alpha+1}\xi_h^2(x,t)+\iint_{Q_h^\tau}|\nabla(V^{\frac{p+2\alpha}{4}}\xi_h)|^2+\varepsilon\iint_{Q_h^\tau}|\nabla(V^{\frac{\alpha+1}{2}}\xi_h)|^2$$
$$\leq C(1+\alpha)^{2^{m+3}}\frac{4^{2h}}{R^4}\Big\{\iint_{Q_h^\tau}(1+V^{\alpha+\frac{p}{2}}+\varepsilon V^{\alpha+1})\Big\} \tag{1.17}$$

where C depends only upon the data and Γ_0.

By (1.17) and Sobolev's embedding inequality, we get

$$\begin{aligned}\iint_{Q_{h+1}}V^{\frac{p-2}{2}+(\alpha+1)(1+\frac{2}{N})}&\leq\iint_{Q_h}(V^{\alpha+1}\xi_h^2)^{\frac{2}{N}}(V^{\frac{p+2\alpha}{2}}\xi_h^2)\\
&\leq\max_{-R_h^2<t<0}\Big(\int_{B_h}V^{\alpha+1}\xi_h^2(x,t)\Big)^{\frac{2}{N}}\cdot\\
&\quad\cdot\int_{-R_h^2}^{0}\Big[\int_{B_h}(V^{\frac{p+2\alpha}{4}}\xi_h)^{\frac{2N}{N-2}}\Big]^{\frac{N-2}{N}}\\
&\leq C\max_{-R_h^2<t<0}\Big(\int_{B_h}V^{\alpha+1}\xi_h^2(x,t)\Big)^{\frac{2}{N}}\iint_{Q_h}|\nabla(V^{\frac{p+2\alpha}{4}}\xi_h)|^2\\
&\leq C(1+\alpha)^{2^{m+4}(1+\frac{2}{N})}\frac{16^{h(1+\frac{2}{N})}}{R^{4(1+\frac{2}{N})}}\Big\{\iint_{Q_h}(1+V^{\alpha+\frac{p}{2}}\\
&\quad+\varepsilon V^{\alpha+1})\Big\}^{1+\frac{2}{N}}\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}\varepsilon\iint_{Q_{h+1}}V^{(1+\alpha)(1+\frac{2}{N})}&\leq C\max_{-R_h^2<t<0}\Big(\int_{B_h}V^{\alpha+1}\xi_h^2(x,t)\Big)^{\frac{2}{N}}\iint_{Q_h}\varepsilon|\nabla(V^{\frac{\alpha+1}{2}}\xi_h)|^2\\
&\leq C(1+\alpha)^{2^{m+4}(1+\frac{2}{N})}\frac{16^{h(1+\frac{2}{N})}}{R^{4(1+\frac{2}{N})}}\Big\{\iint_{Q_h}(1+V^{\alpha+\frac{p}{2}}+\varepsilon V^{\alpha+1})\Big\}^{1+\frac{2}{N}}\end{aligned} \tag{1.19}$$

Take $\alpha+1=k^h\Big(k=1+\dfrac{2}{N}\Big)$ in (1.18) and (1.19) and then sum the resulting inequalities to obtain

$$\Big\{\iint_{Q_{h+1}}(V^{\frac{p-2}{2}+k^{h+1}}+\varepsilon V^{k^{h+1}})\Big\}^{\frac{1}{k^{h+1}}}\leq\frac{C^{\frac{h}{k^h}}}{R^{\frac{4}{k^h}}}\Big\{\iint_{Q_h}(1+V^{\frac{p-2}{2}+k^h}+\varepsilon V^{k^h})\Big\}^{\frac{1}{k^h}}$$
$$(h=0,1,2,\cdots) \tag{1.20}$$

Then by setting $y_j = \left\{ \iint_{Q_h} (1 + V^{\frac{p-2}{2}+k^h} + \varepsilon V^{k^h}) \right\}^{\frac{1}{k^h}}$, we have

$$y_{h+1} \le \frac{C^{(h+1)/k^h}}{R^{4/k^h}} y_h \Big(\text{due to } \iint_{Q_{h+1}} 1 \le \frac{C^k}{R^{4k}} \Big(\iint_{Q_h} 1\Big)^k\Big)$$

which implies that

$$y_h \le \frac{C}{R^{2(N+2)}} y_0$$

hence

$$\max_{Q_\infty} |\nabla v|^2 \le \max_{Q_\infty} V \le \frac{C}{R^{2(N+2)}} \iint_{Q(R)} (1 + |\nabla v|^p + \varepsilon |\nabla v|^2)$$

as claimed.

In establishing the local property of ∇u^ε, we need, instead of the estimate in Lemma 1.1, the estimate below associated with the behaviour of v on $t = -R^2$.

Lemma 1.2 *Let $v \in C(\hat{Q}(R)) \cap C^\infty(\hat{Q}(R))$ be a solution of (1.1) with $\|v\|_{L^\infty} \le \Gamma_0$, and let (B_1)–(B_3) and (1.3) hold. Then*

$$\max_{Q(R)} |\nabla v|^2 \le \hat{C}_2 \left\{ \max_{B(2R)} |\nabla v(x, -R^2)|^2 + 1 + R^{-N-2} \iint_{\hat{Q}(R)} (|\nabla v|^p + \varepsilon |\nabla v|^2) \right\}$$

where $\hat{C}_2$ depends only upon the data and Γ_0.

Sketch of Proof For $h = 0, 1, \cdots$, denote

$$R_h = R\Big(1 + \frac{1}{2^{h+1}}\Big), \quad B_h = B(R_h).$$

$$\hat{Q}_h = B_h \times (-R^2, 0], \quad \hat{Q}_h^\tau = \hat{Q}_h \cap \{t \le \tau\}$$

and take cutoff functions $\xi_h(x) \in C_0^1(B_h)$ with $0 \le \xi_h \le 1$, $\xi_h = 1$, in B_{h+1}, $|\nabla \xi_h| \le C\dfrac{2^h}{R}$.

We can get

$$\iint_{\hat{Q}_h^\tau} \left\{ (v_{x_j})_t \psi_h + (\psi_h)_{x_r} \Big(\frac{\partial a_r^\varepsilon}{\partial p_s} v_{x_s x_j} + \frac{\partial a_r^\varepsilon}{\partial z} v_{x_j} + \frac{\partial a_r^\varepsilon}{\partial x_j} \Big) - b^\varepsilon (\psi_k)_{x_j} \right\} = 0$$

$$(\psi_h = v_{x_j} V^\alpha \xi_h^2,\ \alpha \ge 0,\ V = |\nabla v|^2 + \varepsilon^2)$$

The estimate corresponding to (1.7) is as follows

$$\begin{aligned}\iint_{\hat{Q}_h^\tau} (v_{x_j})_t \psi_h &= \frac{1}{2(1+\alpha)} \int_{B_h} \left\{ V^{\alpha+1}(x,\tau)\xi_h^2(x) - V^{\alpha+1}(x, -R^2)\xi_h^2(x) \right\} \\ &\ge \frac{1}{2(1+\alpha)} \left\{ \int_{B_h} V^{\alpha+1}(x,\tau)\xi_h^2(x) - \Gamma_1^{\alpha+1}|B_h| \right\}\end{aligned}$$

$$\Big(\Gamma_1 = \max_{B(2R)} |\nabla v(x, -R^2)|^2 + 1\Big)$$

Therefore there is no the estimate of (1.13). The other terms can be estimated similarly as in Lemma 1.1. Finally, we can deduce

$$\frac{1}{\alpha+1}\max_{-R^2<t<0}\int_{B_h} V^{\alpha+1}(x,t)\xi_h^2(x) + \iint_{\hat{Q}_h}(|\nabla(V^{\frac{p+2\alpha}{4}}\xi_h)|^2 + \varepsilon|\nabla(V^{\frac{\alpha+1}{2}}\xi_h)|^2)$$

$$\leq C(1+\alpha)^{2(m+3)}\frac{4^{2h}}{R^2}\iint_{\hat{Q}_h}(\Gamma_1^{\alpha+1}+V^{\alpha+\frac{p}{2}}+\varepsilon V^{\alpha+1}) \tag{1.21}$$

The desired result easily follows from (1.21).

Theorem 1.3 *Let u^ε be a solution of (1.1) (1.2), and let (B_1)-(B_3) and (1.3) hold. Then there exists M_2 depending only on $\|\varphi\|_{L^\infty}$, $\|\varphi\|_{W_2^{1,1}}$, T, δ and the data such that*

$$\|u^\varepsilon\|_{C^{1,\frac{1}{2}}(\Omega_{T,\delta})} \leq M_2$$

Proof Theorems 1.1 and 1.2 yield that

$$\|u^\varepsilon\|_{L^\infty(\Omega_T)} \leq M_0 \quad \|\nabla u^\varepsilon\|_{L^\infty(\Omega_{T,\frac{\delta}{2}})} \leq M_1$$

For $z_1(x,t_1), z_2(x,t_2) \in \Omega_{T,\delta}$ $(t_2 > t_1)$. It is no loss of generality to assume $x = 0$. Then a simple calculation gives

$$\begin{aligned}
|u^\varepsilon(z_1) - u^\varepsilon(z_2)| &= \frac{1}{|B(R)|}\iint_{B(R)}\{[u^\varepsilon(0,t_1) - u^\varepsilon(y,t_1)] \\
&\quad + [u^\varepsilon(y,t_1) - u^\varepsilon(y,t_2)] + [u^\varepsilon(y,t_2) - u^\varepsilon(0,t_2)]\} \\
&\leq 2M_1R + \frac{1}{|B(R)|}\left|\int_{t_1}^{t_2}\int_{B(R)}\frac{\partial u^\varepsilon}{\partial t}(y,t)\right| \\
&= 2M_1R + \frac{1}{|B(R)|}\left|\int_{t_1}^{t_2}\int_{B(R)}[\mathrm{div}\vec{a}^\varepsilon(y,t,u^\varepsilon,\nabla u^\varepsilon) + b^3(y,t,u^\varepsilon,\nabla u^\varepsilon)]\right| \\
&= 2M_1R + \frac{1}{|B(R)|}\left|\int_{t_1}^{t_2}\left\{\int_{\partial B(R)}\vec{a}^\varepsilon\cdot\gamma + \int_{B(R)} b^\varepsilon\right\}\right|
\end{aligned}$$

(γ designates the outer unit normal on $\partial B(R)$)

$$\leq 2M_1R + C(M_0,M_1)\frac{|t_2-t_1|}{R}$$

Hence (taking $R = |t_2 - t_1|^{\frac{1}{2}}$)

$$|u^\varepsilon(z_1) - u^\varepsilon(z_2)| \leq C(M_0,M_1)|t_2-t_1|^{\frac{1}{2}}$$

We state a lemma which will be used as we proceed.

Lemma 1.3 *Let $v \in C(\overline{\hat{Q}(R)}) \cap C^\infty(\hat{Q}(R))$ be a solution of*

$$v_t - \mathrm{div}\vec{a}^\varepsilon(t, \nabla v) = 0 \ in\ \hat{Q}(R)$$

with $\|v\|_{L^\infty(\hat{Q}(R))} \le \Gamma_0$ and $\|\nabla v\|_{L^\infty(\hat{Q}(R))} \le \Gamma_1$, where Γ_0 and Γ_1 are independent of ε, $\vec{a}^\varepsilon(t,\eta) = (a_1^\varepsilon(t,\eta), \cdots, a_N^\varepsilon(t,\eta))$ satisfies

$$\frac{\lambda}{k_0}[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2 \le \frac{\partial a_j^\varepsilon(t,\eta)}{\partial p_i}\xi_i\xi_j \le k_0\Lambda[(|\eta|+\varepsilon)^{p-2}+\varepsilon]|\xi|^2$$

Then for small ε, we have

$$|\nabla v(z) - \nabla v(0)| \le \hat{C}_3\Big(\frac{d(z,0)}{R}\Big)^{\alpha_1} \quad \forall z = (x,t) \in Q\Big(\frac{R}{4}\Big)$$

where $\alpha_1 \in (0,1)$ depends only upon the data, $\hat{C}_3$ depends only upon Γ_0, Γ_1 and the data, $d(z,0) = (|x| + |t|^{\frac{1}{2}})$.

We can prove this lemma by using the idea of [3] and making some modification of the arguments presented in [7], see the proof of Proposition 3.1 of [10]. We omit the details.

We now state and prove the local property of ∇u^ε as follows

Theorem 1.4 *Let u^ε be a solution of (1.1) (1.2), and let (B_1)–(B_3) and (1.3) hold. Then for small ε, we have*

$$\|\nabla u^\varepsilon\|_{C^{\alpha_0,\alpha_0/2}(\Omega_{T,\delta})} \le C\delta$$

where $\alpha_0 \in (0,1)$ depends only upon the data and C_δ depends only upon $\|\varphi\|_{L^\infty}$, $\|\varphi\|_{W_2^{1,1}}$, Ω, T, δ and the data.

proof Theorems 1.1–1.3 say

$$\begin{cases} \|u^\varepsilon\|_{L^\infty(\Omega_T)} \le M_0 \\ \|\nabla u^\varepsilon\|_{L^\infty(\Omega_{T,\frac{\delta}{2}})} \le M_1 \\ \|u^\varepsilon\|_{C^{1,\frac{1}{2}}(\Omega_{T,\frac{\delta}{2}})} \le M_2 \end{cases} \tag{1.22}$$

For $z_0 = (x_0, t_0) \in \Omega_{T,\delta}$, it is no loss of generality to assume $z_0 = (0,0)$. We split u^ε as $v + w$ where v is the solution of the problem: $\Big(R \le \dfrac{\delta}{4}\Big)$

$$\begin{cases} v_t - \mathrm{div}\vec{a}^\varepsilon(0, t, u^\varepsilon(0,t), \nabla v) = 0 \text{ in } \hat{Q}(R) \\ v = u^\varepsilon \text{ on } \partial^*\hat{Q}(R) \end{cases} \tag{1.23}$$

As is known to all, there exists a unique solution v from $C(\overline{\hat{Q}(R)}) \cap C^\infty(\hat{Q}(R))$ of (1.23).

According to the comparison principle and (1.22),

$$\begin{aligned}&\|v\|_{L^\infty(\hat{Q}(R))} \le \|u^\varepsilon\|_{L^\infty(\hat{Q}(R))} \le M_0\\ &\operatorname*{osc}_{\hat{Q}(R)} v \le \operatorname*{osc}_{\hat{Q}(R)} u^\varepsilon \le M_2 R\end{aligned} \tag{1.24}$$

Lemma 1.2 and (1.22) reveal that

$$\max_{Q(R)} |\nabla v|^2 \le C(M_0)\Big\{M_1^2 + 1 + R^{-(N+2)} \iint_{\hat{Q}(R)} (|\nabla v|^p + \varepsilon|\nabla v|^2)\Big\} \tag{1.25}$$

By (B_1)–(B_3) and (1.3), we have

$$\begin{aligned}0 = &\frac{1}{2}\int_{B(R)} W^2(x,0) + \iint_{\hat{Q}(R)} [\vec{a}^\varepsilon(x,t,u^\varepsilon(x,t),\nabla u^\varepsilon(x,t))\\ &-\vec{a}^\varepsilon(0,t,u^\varepsilon(0,t),v(x,t))]\cdot\nabla w + \iint_{\hat{Q}(R)} b^\varepsilon w\\ \ge &\iint_{\hat{Q}(R)}\int_0^1\Big\{\frac{\partial a_i^\varepsilon}{\partial p_j}(\psi(s))w_{x_i}w_{x_j} + \frac{\partial a_i^\varepsilon}{\partial z}(\psi(s))w_{x_i}(u^\varepsilon(x,t)-u^\varepsilon(0,t))\\ &+\frac{\partial a_i^\varepsilon}{\partial x_j}(\psi(s))w_{x_i}x_j\Big\}ds + \iint_{\hat{Q}(R)} b^\varepsilon w\\ \ge &\frac{1}{\hat{C}}\iint_{\hat{Q}(R)}\Big[\int_0^1(|s\nabla u^\varepsilon + (1-s)\nabla v| + \varepsilon)^{p-2}ds + \varepsilon\Big]|\nabla w|^2\\ &-\hat{C}\iint_{\hat{Q}(R)}(R + \operatorname*{osc}_{\hat{Q}(R)} v + \operatorname*{osc}_{\hat{Q}(R)} u^\varepsilon)(1 + |\nabla u^\varepsilon|^p + |\nabla v|^p)\\ \ge &\frac{1}{\hat{C}}\iint_{\hat{Q}(R)}[(|\nabla u^\varepsilon| + |\nabla v| + \varepsilon)^{p-2} + \varepsilon]|\nabla w|^2\\ &-(1+2M_2)\hat{C}R\iint_{\hat{Q}(R)}(1 + |\nabla u^\varepsilon|^p + |\nabla v|^p)\end{aligned}$$

where $\psi(s) = (sx, t, su^\varepsilon(x,t) + (1-s)u^\varepsilon(0,t), s\nabla u^\varepsilon(x,t) + (1-s)\nabla v(x,t))$, $\hat{C}$ depends only upon M_0 and the data. Therefore we have

$$\begin{aligned}&\iint_{\hat{Q}(R)}[(|\nabla u^\varepsilon| + |\nabla v| + \varepsilon)^{p-2} + \varepsilon]|\nabla w|^2\\ &\qquad\le (1+2M_2)\hat{C}^2 R\iint_{\hat{Q}(R)}(1 + |\nabla u^\varepsilon|^p + |\nabla v|^p)\end{aligned} \tag{1.26}$$

Taking into account that $|\nabla v| \ge 2(1+M_1)$ implies $|\nabla w|^2 \ge \frac{1}{4}|\nabla v|^2$ and $2|\nabla v| \ge$

$\varepsilon + |\nabla u^\varepsilon| + |\nabla v|$, we deduce

$$\begin{aligned}\iint_{\hat{Q}(R)\cap\{|\nabla v|\geq 2(1+M_1)\}} (|\nabla v|^p + \varepsilon|\nabla v|^2) &\leq 4\iint_{\hat{Q}(R)} [(|\nabla u^\varepsilon| + |\nabla v| + \varepsilon)^{p-2} + \varepsilon]\cdot|\nabla w|^2 \\ &\leq 4\hat{C}^2(1+2M_2)R\iint_{\hat{Q}(R)} (1 + |\nabla u^\varepsilon|^p + |\nabla v|^p) \\ &\leq 4\hat{C}^2(1+2M_2)R\iint_{\hat{Q}(R)\cap\{|\nabla v|\geq 2(M_1+1)\}} |\nabla v|^p \\ &\quad + 2^{N+6}\hat{C}^2(1+2M_2)(1+M_1^p)R^{N+3} \\ &\leq C(1+M_1^p)R^{N+2}\end{aligned}$$

where $R \leq R_0$, R_0 is so small that $4\hat{C}^2(1+2M_2)R_0 \leq \frac{1}{2}$; constant C depends only upon M_0 and the data. Therefore we have

$$\begin{aligned}\iint_{\hat{Q}(R)} (|\nabla v|^p + \varepsilon|\nabla v|^2) &= \iint_{\hat{Q}(R)\cap\{|\nabla v|\geq 2(M_1+1)\}} (\cdots) + \iint_{\hat{Q}(R)\cap\{|\nabla v|<2(M_1+1)\}} \\ &\leq \hat{C}_4(1+M_1^2)R^{N+2}\end{aligned} \tag{1.27}$$

where $\hat{C}_4$ depends only upon M_0 and the data.

(1.25) and (1.27) yield

$$\max_{Q(R)} |\nabla v|^2 \leq C(M_0)(1+\hat{C}_4)(1+M_1^2) \triangleq \Gamma_1 \tag{1.28}$$

By (1.28) and Lemma 1.3, we have (for small ε)

$$\sup_{\substack{z_1,z_2\in Q(\frac{R}{8})\\ d(z_1,z_2)\leq\rho\leq\frac{R}{8}}} |\nabla v(z_1) - \nabla v(z_2)| \leq C(M_0,\Gamma_1)\Big(\frac{\rho}{R}\Big)^{\alpha_1} \tag{1.29}$$

From (1.26), (1.27) and (1.28),

$$\begin{aligned}\iint_{Q(R)} |\nabla w|^2 &\leq 2(1+\Gamma_1)\iint_{Q(R)} (|\nabla u^\varepsilon| + |\nabla v| + \varepsilon)^{p-2}|\nabla w|^2 \\ &\leq C(M_0,M_1,M_2)R^{N+3}\end{aligned} \tag{1.30}$$

Fixing $R_0 = \min\Big\{\frac{\delta}{4}, \Big(\frac{1}{8}\Big)^{\frac{1}{\theta}}, \frac{1}{8\hat{C}^2(1+2M_2)}\Big\}$ $\Big(\theta = \frac{1}{N+3}\Big)$ and letting $\rho = R^{1+\theta}$, then by (1.29) and (1.30) we get

$$\begin{aligned}\iint_{Q(\rho)} |\nabla u^\varepsilon - (\nabla u^\varepsilon)_\rho|^2 &\leq \iint_{Q(\rho)} |\nabla u^\varepsilon - (\nabla v)_\rho|^2 \\ &\leq 2\iint_{Q(\rho)} |\nabla w|^2 + 2\iint_{Q(\rho)} |\nabla v - (\nabla v)_\rho|^2 \\ &\leq C(M_0,M_1,M_2)\Big(R^{N+3} + \rho^{N+2}\Big(\frac{\rho}{R}\Big)^{2\alpha_1}\Big) \\ &\leq C(M_0,M_1,M_2)\rho^{N+2+2\alpha_1\frac{\theta}{1+\theta}}\end{aligned} \tag{1.31}$$

where

$$(f)_\rho = \frac{1}{|Q(\rho)|}\iint_{Q(\rho)} f$$

It follows that (see [4] or [8])

$$\|\nabla u^\varepsilon\|_{C^{\alpha_0,\alpha_0/2}(\Omega_{T,\delta})} \le C_\delta \tag{1.32}$$

where $\alpha_0 = \alpha_1 \dfrac{\theta}{1+\theta}$.

2. Proofs of Results

It is trivial, from Theorems 1.1–1.4, to prove Theorem 0.1.

Proof of Theorem 0.2 Let $u_1(x,t)$, $u_2(x,t) \in V_{2,p}(\Omega_T)$ be any two weak solutions of (0.1) (0.2). Denote $w(x,t) = u_1(x,t) - u_2(x,t)$. Without loss of generality, we may assume $(u_1)_t, (u_2)_t \in L^2(\Omega_T)$ (otherwise, we may take Steklov averagings of u_1, u_2 with respect to t, see [9, Chapter III]). By (0.7) we have

$$\int_0^t\int_\Omega w_t\xi + \int_0^t\int_\Omega \{[\vec{a}(x,\tau,u_1,\nabla u_1) - \vec{a}(x,\tau,u_2,\nabla u_2)]\cdot\nabla\xi + [b(x,\tau,u_1,\nabla u_1) - b(x,\tau,u_2,\nabla u_2)]\cdot\xi\} = 0 \tag{2.1}$$

for all $t \in (0,T]$, $\xi \in \overset{\circ}{W}{}_p^{1,0}(\Omega_T)$ with $\xi_t \in L^2(\Omega_T)$. Take $\xi = w$ in the above equation to obtain

$$0 = \frac{1}{2}\int_\Omega w^2(x,t) + \int_0^t\int_\Omega\int_0^1 \left\{\frac{\partial a_i}{\partial p_j}(\psi(s))w_{x_i}w_{x_j} + \frac{\partial a_j}{\partial z}(\psi(s))w_{x_i}w + \frac{\partial b}{\partial p_j}(\psi(s))w_{x_i}w + \frac{\partial b}{\partial z}(\psi(s))w^2\right\}ds$$

where $\psi(s) = (x, t, su_1 + (1-s)u_2, s\nabla u_1 + (1-s)\nabla u_2)$.

By (A_1), (A_2) and $(A_3)'$ we deduce

$$\int_\Omega w^2(x,t) \le C\int_0^t\int_\Omega w^2(x,\tau) \quad t \in (0,T]$$

Obviously, $F(t) \triangleq \int_0^t\int_\Omega w^2(x,\tau)$ is absolutely continuous in $[0,T]$. Then by the above inequality and the well-known Gronwall's lemma, we have

$$F(t) = 0 \quad t \in [0,T]$$

that is,

$$u_1(x,t) = u_2(x,t) \quad \text{a.e. on } \Omega_T$$

References

[1] Alikakos N. D. & Evans L. C., Continuity of the gradient of the solutions of certain degenerate parabolic equations, *J. Math. pures et appl*, **62** (1983), 253–268.

[2] Chen Ya-zhe, Hölder continuity of the gradient of solutions of nonlinear degenerate parabolic systems, *Acta Mat. Sinica.* New Series 2.4 (1986), 309–331.

[3] Chen Ya-zhe, Hölder continuity of the gradient of the solutions of certain degenerate parabolic equations, *Chin. Ann. of Math.*, **8B** (3) (1987), 343–356.

[4] Da Prato G., Spazi $\mathcal{L}^{(p,\theta)}(\Omega,\delta)$ e loro proprieta, *Ann. di Mat. pura e appl.*, **69** (1965), 383–392.

[5] DiBenedetto E. & Friedman A., Regularity of solutions of nonlinear degenerate parabolic systems, *J. Reine, Angew. Math.*, **349** (1984), 84–128.

[6] DiBenedetto E. & Friedman A., Hölder estimates for nonlinear degenerate parabolic systems, *J. Reine. Angew. Math.*, **357** (1985), 1–22.

[7] Evans L. C., A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic P.D.E., *J. Diff. Eqs.*, **45** (1982), 356–373.

[8] Giaquinta M., Multiple Integrals in the Calculus and Nonlinear Elliptic Systems, *Annals of Mathematics Studies*, Princeton, New Jersery (1983).

[9] Ladyzenskaja O. A., Solonnikov V. A. and Ural'ceva N. N., Linear and Quasilinear Equations of Parabolic Type, *Transl. Math.*, **23**, (1968) A.M.S., R.I..

[10] Lee Jiun-jie, Hölder continuity of weak solutions of singular parabolic equations of the plasma type. (preprient).