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BOUNDARY VALUE PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS IN CYLINDRICAL DOMAINS

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In this paper we consider weak solutions of the equation

$$L(u) \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i} = |u|^{p-1}u \tag{1}$$

in a domain

$$S(0,\infty) = \{x : \hat{x} \in \omega, 0 < x_n < \infty\}$$

with the boundary condition

$$\frac{\partial u}{\partial \gamma} \equiv \sum_{i,j=1}^{n} a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\nu_i = 0 \quad \text{on} \quad \sigma(0,\infty) \tag{2}$$

where $x = (x_1, \cdots, x_n)$, $\hat{x} = (x_1, \cdots, x_{n-1})$, ω is a bounded domain in $R_{\hat{x}}^{n-1}$ with a smooth boundary $\partial\omega$,

$\sigma(0,\infty) = \{x : \hat{x} \in \partial\omega,\ 0 < x_n < \infty\}$, $\nu = (\nu_1, \cdots, \nu_n)$ is the exterior unit normal to $\sigma(0,\infty)$, $a_{ij}(\hat{x})$, $a_i(\hat{x})$ are measurable bounded functions,

$$m|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(\hat{x})\xi_i\xi_j \le M|\xi|^2, \quad \xi \in R^n,\ \hat{x} \in \omega$$

$$a_{in}(\hat{x}) \equiv 0 \text{ for } i < n,\ a_{nn}(\hat{x}) \equiv 1;\ m, M = \text{const } > 0,\ p = \text{const } > 1$$

We denote

$$S(a,b) = \{x : \hat{x} \in \omega, a < x_n < b\},\ \sigma(a,b) = \{x : \hat{x} \in \partial\omega, a < x_n < b\}$$

The function $u(x)$ is called a weak solution of problem (1), (2), if for any $T > 0$ function $u(x) \in H^1(S(0,T))$, $u(x)$ is bounded in $S(0,T)$ and

$$-\int_{S(0,T)} \sum_{i,j=1}^{n} a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}dx + \int_{S(0,T)} \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i}\varphi dx = \int_{S(0,T)} |u|^{p-1}u\varphi dx \tag{3}$$

for any function $\varphi \in H^1(S(0,T))$, $\varphi(\hat{x},0) = 0$, $\varphi(\hat{x},T) = 0$.

Many problems of mathematical physics lead one to consider the solutions of problem (1), (2) and to study the behaviour of the solutions at infinity (stationary states, travelling waves, homogenization, boundary layer problems, Saint-Venant's principle and so on). These problems are considered in many papers (see, for example, [1]-[5]). For linear equations such problems are investigated in [4].

We shall use the following propositions for linear second order equations.

1. Consider the problem

$$L(u) + q(x)u = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} + f_0 \quad \text{in } S(-\infty,+\infty), \ \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \sigma(-\infty,+\infty) \tag{4}$$

under condition

$$J_h(f_i) \equiv \int_{S(-\infty,+\infty)} e^{2hx_n}|f_i|^2 dx < \infty, \ i = 0,1,\cdots,n \tag{5}$$

with constant h such that the eigenvalue problem

$$\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\Big(a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i} - \lambda^2 u = 0 \quad \text{in } \omega \tag{6}$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial\omega \tag{7}$$

has no eigenvalues λ with $J_m h = J_m \lambda$. Then there exists a constant $\varepsilon > 0$ such that if

$$|q(x)| < \varepsilon \quad \text{in } S(-\infty,+\infty) \tag{8}$$

there exists a unique solution $u(x)$ of problem (4) and

$$T_h(u) \equiv \sum_{i=1}^{n} \int_{S(-\infty,+\infty)} \Big(\frac{\partial u}{\partial x_i}\Big)^2 e^{2hx_n} + \int_{S(-\infty,+\infty)} |u|^2 e^{2hx_n} dx \le C_h \sum_{i=1}^{n} J_h(f_i) \tag{9}$$

(see [6]).

2. Assume that $q(x) \equiv 0$, $u(x)$ is a solution of problem (4), satisfying condition $T_{h_2}(u) < \infty$,

$$J_{h_1}(f_i) < \infty, \ J_{h_2}(f_i) < \infty, \ i = 0,1,\cdots,n$$

and, in addition, in the strip $h_1 \le J_m\lambda \le h_2$ there is only one eigenvalue λ_0 of the problem (6), (7), $h_1 < J_m\lambda_0 < h_2$. Then

$$u(x) = \sum_{j=0}^{k} C_j x_n^j \phi_j(\hat{x}) e^{i\lambda_0 x_n} + u_1(x)$$

$$T_{h_1}(u_1) \le C \sum_{i=0}^{n} J_{h_1}(f_i), \ C, C_j = \text{const} \tag{10}$$

where $\phi_0, \phi_1, \cdots, \phi_k$ is a chain of eigenfunctions and adjoint functions of problem (6), (7), corresponding to the eigenvalue $\lambda = \lambda_0$ (see [7]).

Let us prove some auxiliary results.

Lemma 1 *Any Solution $u(x)$ of problem (1), (2) satisfies the inequality*

$$|u(x)| \le M_p x_n^{2/(1-p)}, \ M_p = \Big[\frac{2(1+p)}{(1-p)^2}\Big]^{1/(p-1)} \tag{11}$$

Proof Let us prove firstly that

$$\lim_{x_n \to \infty} u(x) = 0 \tag{12}$$

we consider the function $v_t(x_n)$ such that

$$v_t'' - |v_t|^{p-1} v_t = 0, \ v_t(t) = \varepsilon > 0, \ v_t'(t) = 0, \ x_n > t \tag{13}$$

For a solution of the problem (13) we have

$$\Big(\frac{p+1}{2}\Big)^{1/2} \int_\varepsilon^{v_t(x_n)} (w^{p+1} - \varepsilon^{p+1})^{-1/2} dw = x_n - t \tag{14}$$

From (14) it follows that

$$\lim v_t(x_n) = +\infty \quad \text{as } x_n \to t + K(\varepsilon)$$

where $K(\varepsilon)$ does not depend on t. We extend the function $v_t(x_n)$ for $t - K(\varepsilon) \le x_n < t$, seting $v_t(2t - x_n) = v_t(x_n)$. If (12) is not valid, then there exists a sequence $t_m \to \infty$ such that

$$|u(\hat{x}_m, t_m)| > \varepsilon, \ \varepsilon = \text{const } > 0 \tag{15}$$

From (15) it follows, if $u \in C^2(S(0,\infty)) \cap C^1(\overline{S}(0,\infty))$, then for $u_m(x) = u(x) - u_{t_m}(x_n)$ we have

$$\frac{\partial u_m}{\partial \gamma} = 0 \quad \text{on} \quad \sigma(t_m - K(\varepsilon), t_m + K(\varepsilon)), u_m(\hat{x}_m, t_m) > 0$$

$$\text{if} \quad u(\hat{x}_m, t_m) > \varepsilon \tag{16}$$

$$\lim u_m(x) = -\infty \text{ as } x_n \to t_m + K(\varepsilon) \text{ and } x_n \to t_m - K(\varepsilon) \tag{17}$$

$$L(u_m) - (|u|^{p-1}u - |u_{t_m}|^{p-1} u_{t_m})(u - u_{t_m})^{-1} u_m = 0 \tag{18}$$

From (16) and (17) we get that u_m must attain a positive maximum in $S(t_m - K(\varepsilon), t_m + K(\varepsilon))$. It contradicts (18). Thus, we have (12). From (12) and maximum principle we obtain (11).

For a weak solution of problem (1), (2) we can get (11) approximating $u(x)$ by $u^k(x)$, where $u^k(x)$ is a solution of problem (1), (2) with smooth coefficients a_{ij}^k, $a_i^k(x)$, which approximate $a_{ij}(x)$, $a_i(x)$.

Remark Using the maximum principle and approximation of coefficients of (1), we get for $u(x) \geq 0$ the estimate

$$M_p(x_n + \theta)^{2/(1-p)} \leq u(x) \leq M_p x_n^{2/(1-p)} \tag{19}$$

where θ is a constant, independent of u.

Theorem 1 *If a solution $u(x)$ of problem (1), (2) changes sign is $S(0,\infty)$ (it means that $u(x)$ takes positive and negative values in $S(k,\infty)$ for any $k > 0$), then*

$$|u(x)| \leq C_1 \exp\{-hx_n\} \tag{20}$$

where C_1, h are some positive constants, h does not depond on u.

Proof It is known that there exists $h > 0$ such that the strip $-h \leq J_m\lambda \leq h$ contains only eigenvalue $\lambda = 0$ of the problem (6), (7) [8]. Since according to Lemma 1 $u(x) \to 0$ as $x_n \to \infty$, there exists t_0 such that $|u|^{p-1} < \varepsilon$ for $x_n > t_0$, where ε is a constant defined in (8).

Let $\theta(t)$ be a function such that $\theta(t) = 1$ for $t > t_0 + 1$, $\theta(t) = 0$ for $t < t_0$, $\theta \in C^\infty(R^1)$, $0 \leq \theta(t) \leq 1$. Consider function $v(x) = \theta(x_n)u(x)$. This function satisfies the linear equation

$$\begin{aligned} L_1(v) &\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(\hat{x})\frac{\partial v}{\partial x_j}\Big) + \sum_{j=1}^{n-1} a_i(\hat{x})\frac{\partial v}{\partial x_j} - q(x)v \\ &= 2\frac{\partial \theta}{\partial x_n}\frac{\partial u}{\partial x_n} + \frac{\partial}{\partial x_n}\Big(u\frac{\partial \theta}{\partial x_n}\Big) \equiv F_0 + \frac{\partial F_1}{\partial x_n} \end{aligned} \tag{21}$$

in $S(-\infty,+\infty)$, where $q(x) = |u|^{p-1}$ for $x_n > t_0 + 1$, $q(x) = 0$ for $x_n < t_0$, and the boundary condition

$$\frac{\partial v}{\partial \gamma} = 0 \quad \text{on } \sigma(-\infty,+\infty) \tag{22}$$

Functions F_0 and F_1 have compact supports. From Proposition 1 it follows that there exists a function $v_1(x)$ which is a solution of the problem (21), (22) and

$$T_h(v_1) < \infty \tag{23}$$

It is easy to see that the estimate (23) provides the estimate

$$|v_1(x)| \leq C\exp\{-hx_n\},\ C = \text{const} \tag{24}$$

Indeed, according to E.De Giorgi theorem [9] and (9)

$$\begin{aligned} \max_{S(T-1,T+1)} |v_1(x)|^2 &\leq C_1 \int_{S(T-2,T+2)} |v_1|^2 dx \\ &\leq C_2 e^{-2hT} \int_{S(T-2,T+2)} |v_1|^2 e^{2hx_n} dx \leq C_3 e^{-2hT} \end{aligned}$$

and therefore we get (24). The function $v_2(x) = \theta_1(x_n)v_1(x)$, where $\theta_1(x_n) = 1$ for $x_n < t_0 - 1$, $\theta_1(x_n) = 0$ for $x_n > t_0$, $0 \le \theta_1(x_n) \le 1$, $\theta_1 \in C^\infty(R)$, satisfies the equation of the form (21) with $q(x) \equiv 0$ is $S(-\infty, +\infty)$ and the boundary condition $\dfrac{\partial v_2}{\partial \gamma} = 0$ on $\sigma(-\infty, +\infty)$. According to Proposition 2, since $T_h(v_2) < \infty$, we have

$$v_2(x) = a + bx_n + v_0(x) \tag{25}$$

where $a, b = \text{const}$, $T_{-h}(v_0) < \infty$. From the last inequality it follows that

$$|v_0(x)| \le C_4 e^{hx_n}, \quad C_4 = \text{const} \tag{26}$$

for $x_n < 0$. Let us note that $v_2(x) = v_1(x)$ for $x_n < t_0 - 1$. The eigenvalue problem (6), (7) has an eigenfunction $\phi_0 = \text{const}$ and an adjoint function $\phi_1 = \text{const}$, corresponding to eigenvalue $\lambda = 0$. The equality (25) is a consequence of (10).

Consider $w = v_1 - v$. The function w satisfies the equation

$$L_1(w) = 0 \quad \text{in } S(-\infty, +\infty)$$

and the boundary condition

$$\frac{\partial w}{\partial \gamma} = 0 \quad \text{on } \sigma(-\infty, +\infty)$$

Since $v = 0$ for $x_n < t_0$, $v_2(x) = v_1(x)$ for $x_n < t_0 - 1$ and (25) and (26) are valid, we have

$$w(x) = a + bx_n + v_0(x) \tag{27}$$

for $x_n < t_0 - 1$. Let us prove that $w \equiv 0$ in $S(-\infty, +\infty)$. It means that $v(x) = v_1(x)$. For $v_1(x)$ the estimate (24) is valid and since $v(x) = u(x)$ for $x_n > t_0 + 1$, we get (20). First we prove that $b = 0$ in (27). Suppose that $b < 0$. Then $w(x) > 0$ for $x_n < T_0$, where T_0 is negative and sufficiently large by modulus. From the maximum principle it follows that $w > 0$ in $S(T_0, +\infty)$, since $w(x) \to 0$ as $x_n \to \infty$.

It is easy to see that for $x_n > t_0 + 1$ and $x_n > 0$ we have $L_1(M_p x_n^{2/(1-p)}) \ge 0$. Let us take t_1 such that $t_1 > 0$ and $t_1 > t_0 + 1$. There exists a constant $\varepsilon > 0$ such that $\varepsilon H \le w$ for $x_n = t_1$, $\hat{x} \in \omega$, where $H = M_p x_n^{2/(1-p)}$. Since $L_1(\varepsilon H - w) \ge 0$, $q(x) \ge 0$ in $S(t_1, +\infty)$, according to the maximum principle we have $\varepsilon H - w \le 0$ in $S(t_1, +\infty)$. Consider the set z of points x, where $v = u > 0$. For the points of z we have

$$\varepsilon H(x) \le w(x) \le v_1(x)$$

This inequality contradicts (24). In the same way we prove that b can not be positive. If $b = 0$, then similarly we prove that $a = 0$. Therefore, $w(x) \to 0$ as $x_n \to -\infty$ according to (27) and (26). Since $w(x) \to 0$ as $x_n \to +\infty$ by the maximum principle we have that $w(x) \equiv 0$. The theorem is proved.

The case, when a solution of the problem (1), (2) preserves sign in $S(0, \infty)$, is considered in [3].

Equation (1) has no term with $\dfrac{\partial u}{\partial x_n}$. This term in contrast to terms $\sum\limits_{i=1}^{n-1} a_i(\hat{x})\dfrac{\partial u}{\partial x_i}$, has an essential influence on the asymptotic behaviour of the solutions of problem (1), (2). Consider the equation

$$\Delta(u) - \frac{\partial u}{\partial x_n} - u^3 = 0 \quad \text{in } S(0,\infty) \tag{28}$$

and its solutions satisfying the boundary condition (2). Suppose that $u(x) \geq z(x_n)$ for $x_n = 1$, $u(x) \to 0$ as $x_n \to \infty$, where $z(x_n) = (2x_n)^{-1/2}$. Then $\Delta(u - z) - \dfrac{\partial}{\partial x_n}(u - z) - u^3 + z^3 = -\Delta z = -\dfrac{3}{4\sqrt{2}}x_n^{-5/2} < 0$. By the maximum principle we have that $u(x) \geq z(x_n)$ in $S(1,\infty), u(x) \geq (2x_n)^{-1/2}$. It means that (19) is not valid for Equation (28). It can be proved that, if $u(x) > 0$ and u is a solution of Equation (28) with the boundary condition (2), then $u(x) \geq \alpha(x_n + \beta)^{-1/2}$ in $S(0,\infty)$, $\alpha, \beta = \text{const} > 0$.

Equation (28) has positive solution in $S(0,\infty)$. Such solution can be obtained as a limit of the solutions of the problem

$$\Delta u - \frac{\partial u}{\partial x_n} - u^3 = 0 \quad \text{in } S(0,T)$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \sigma(0,T), u = 0 \text{ for } x_n = T, \quad u(x) = \varphi(\hat{x}) \text{ for } x_n = 0$$

as $T \to \infty$, for any smooth function $\varphi(\hat{x})$.

Similar considerations can be done for the equation

$$\Delta u + \frac{\partial u}{\partial x_n} - u^3 = 0 \tag{29}$$

For any solution of the problem (29), (2) we can get the estimate

$$|u(x)| \leq C_1 \exp\{-hx_n\}, \ C_1, h = \text{const} > 0 \tag{30}$$

For a positive solution $u(x)$ we have

$$C_2 \exp\{-hx_n\} \leq u(x) \leq C_3 \exp\{-hx_n\}$$

In order to prove (30) we use a function $z(x_n) = \beta e^{-\alpha x_n}$, $\alpha, \beta = \text{const} > 0$. For α, β sufficiently small $\Delta z + \dfrac{\partial z}{\partial x_n} - z^3 < 0$.

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